This homework runs from Thursday 3 October 2024 until 12 noon on Thursday 10 October 2023. Submission is to Gradescope Homework 3.

Questions marked with an asterisk $*$ may be a little harder than others. All are still within the course curriculum, though, and can be done using the methods taught in the study guides and textbook.

You should aim to write out solutions that someone who does not already know the answer could follow and understand.

Please remember the good scholarly practice requirements of the University regarding work for credit. You can find guidance at the School page https://web.inf.ed.ac.uk/infweb/admin/policies/academic-misconduct. This also has links to the relevant University pages.

Question 1

Prove using the element method that for any four sets A, B, C, and D, if $A \subseteq C$ and $B \subseteq D$ then $(A \cap B) \subseteq (C \cap D)$ and $(A \cup B) \subseteq (C \cup D)$.

[*4 marks*]

Question 2

Here are three functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) $f(x) = -x$.

(b) $g(x) = 2^x$.

(c) $h(x) = (x^3 - x)$.

For each of these sketch a graph of the function, say whether it is injective, and say whether it is surjective.

[*3 marks*]

$*$ Question 3

Call a function $s : X \to Y$ between two sets X and Y a *section* if there is another function (known as a *retraction*) $r : Y \to X$ such that $r \circ s = I_X$. Here I_X is the identity function on X. Suppose that $f : A \to B$ and $g : A \to C$ are both sections.

(a) Is there necessarily a section $h : A \to B \cup C$?

(b) Is there necessarily a section $j : A \to B \cap C$?

(c) Is there necessarily a section $k : A \to B \times C$?

In each case either show how to construct such a section, together with a suitable corresponding retraction, or give a counterexample.

[*3 marks*]

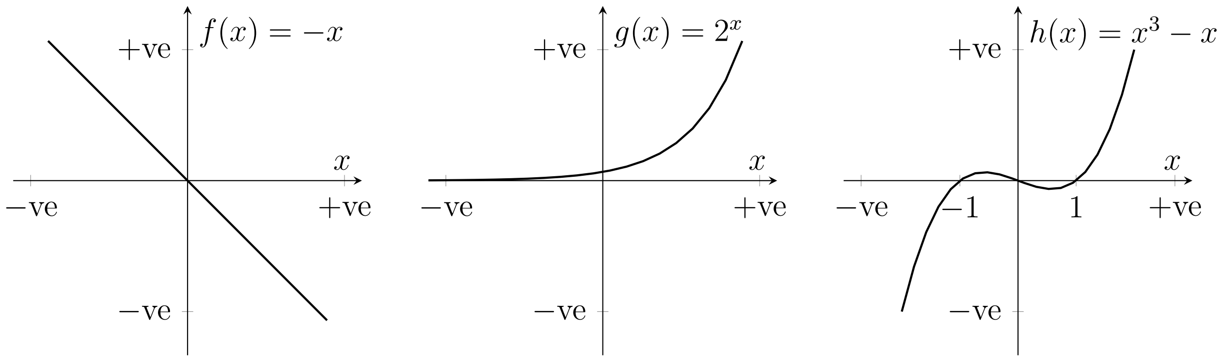
Solution 1

To show $(A \cap B) \subseteq (C \cap D)$, suppose that an element x is in $A \cap B$. Expanding definitions, this tells us that x is in both A and B. Since $A \subseteq C$ we have that x is in C; and from $B \subseteq D$ we know x is also in D. Combining these we have that x is in their intersection $C \cap D$ as required.

To show $(A \cup B) \subseteq (C \cup D)$, suppose $x \in (A \cup B)$. Then either $x \in A$ or $x \in B$ or both. For $x \in A$ since $A \subseteq C$ we have $x \in C$ and hence $x \in (C \cup D)$. For $x \in B$ since $B \subseteq D$ we know $x \in D$ and again $x \in (C \cup D)$. Since either possibility gives $x \in (C \cup D)$ we know $(A \cup B) \subseteq (C \cup D)$.

Epp does not provide enough algebraic properties of subset inclusion to show these results algebraically, in the same way Theorem 6.2.2 does for set equality.

Solution 2



(a) Function $f(x)$ is injective and surjective.

(b) Function $g(x)$ is injective but not surjective.

(c) Function $h(x)$ is surjective but not injective.

∗ Solution 3

Suppose that $p : B \to A$ and $q : C \to A$ are retractions corresponding to f and g respectively, so $p \circ f = q \circ g = I_A$.

(a) Taking $h(x) = f(x)$ for all $x \in A$ gives a section $h : A \to B \cup C$, with a corresponding retraction taking $y \in B \cup C$ to $p(y)$ if $y \in B$ and to $q(y)$ otherwise.

Using $h'(x) = g(x)$ would also serve as suitable section, with a different retraction.

Note that these constructions are still valid if set A is empty, in which case B and C are empty too. Constructions that involve picking an arbitrary element of A to use in a retraction, for example, won't work for empty A

(b) There may be no such section, for example if $A = \{0\}$, $B = \{1\}$, and $C = \{2\}$ with $f(0) = 1$ and $g(0) = 2$. In this case $B \cap C$ is empty and there is no possible function from nonempty A to $B \cap C$. More generally, there is no section possible if $B \cap C$ has fewer elements than A since a section is always an injective function.

(c) Taking $k(x) = (f(x), g(x))$ for $x \in A$ gives a section $k : A \to B \times C$, with the function taking $(y, z) \in B \times C$ to $p(y) \in A$ providing a suitable retraction.

Using the function that takes $(y, z) \in B \times C$ to $q(z) \in A$ would also serve as a retraction for the same section.