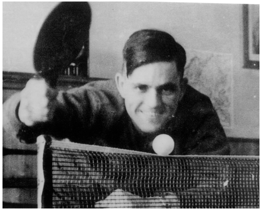
Informatics 1 – Introduction to Computation
Computation and Logic

Julian Bradfield

Sequent Calculus, Completeness, and Incompleteness



Gerhard Gentzen
1909–1945

Kurt Gödel
1906–1978

Abandoning Meaning

Recall that we been working with $\vDash$, which is *semantic* – it talks about meaning of formulae in universes. We have also been rather sloppy with notation! To be more precise, we should have said:

- In universe $\mathfrak{X}$,

  $$\Gamma \vDash_{\mathfrak{X}} \Delta \quad \text{iff} \quad \forall x, y, z, \ldots . \bigwedge \Gamma(x, y, z, \ldots) \to \bigvee \Delta(x, y, z, \ldots)$$

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'Proof theory' looks at logical proof just with the *syntax* – we formulate rules of reasoning we believe to be correct.

Then we use 'model theory' to connect proofs to meaning, and we *prove* (by mathematics) that if we 'prove' a formula valid, then it is *semantically* valid too.

Syntactic entailment

We introduce the symbol $\vdash$ for syntactic entailment.

Now the sequent calculus is no longer statements about how $\vDash$ works, it's just a bunch of *stipulated* rules about how $\vdash$ is *defined* to work.

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We saw completeness ($\Gamma \vDash \Delta \Longrightarrow \Gamma \vdash \Delta$) intuitively: we can mechanically build a proof of any valid sequent. It is possible to prove it formally.

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Rules for quantifiers ($\forall$)

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$$\frac{\Gamma \vdash \phi[y/x], \Delta}{\Gamma \vdash \forall x.\phi, \Delta}\ \forall R$$

where y *does not occur in* Γ, ϕ, Δ and $\phi[y/x]$ means the result of substituting y for x in ϕ.

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A term is a variable or functions applied to variables, such as x or $y + dbl(z)$. We haven't discussed this, but the language of logic usually includes function symbols, as in Haskell, as well as predicate symbols. (Plain logic does not have types, though.)

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t will come from elsewhere in the proof, or from an assumption in Γ.

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Rules for quantifiers: left side

We know that swapping sides is negation, and exists is the dual of forall. So the left side rules are just the duals of the right side rules:

$$\frac{\Gamma, \phi[t/x] \vdash \Delta}{\Gamma, \forall x.\phi \vdash \Delta}\ \forall L \qquad \frac{\Gamma, \phi[y/x] \vdash \Delta}{\Gamma, \exists x.\phi \vdash \Delta}\ \exists L$$

with the same restrictions on y and t.

Example

We should be able to prove $\exists x.p(x), \forall x.p(x) \to q(x) \vdash \exists x.q(x)$
(Exercise: rewrite this in syllogism terms.)

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The proof of this theorem, even in modern notation, is quite long and detailed, although not difficult in a deep way.

Death by 10^{1000} cuts

The standard sequent calculus includes the rule:

$$\frac{\Gamma \vdash \phi, \Delta \qquad \Gamma', \phi \vdash, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \; Cut$$

that is, if one sequent needs assumption ϕ, and another sequent shows ϕ, then you can ‘cut out’ ϕ. Obviously sound (right?), but why do we want it?

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Hauptsatz is simply German for 'main theorem'.

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(Actually, it can be much worse than that. See the final 'fun lecture' of the course for an idea of what a *really* big proof might be.)

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The solution to this paradox is that first-order logic is not strong enough to *fully* describe the natural numbers. If $\mathbb{N}$ satisfies N, then there are other universes satisfying N, and in some ϕ_N is false.

The Incompleteness Theorem, and the closely connected Undecidability Theorems of Church and Turing, shattered the hope expressed by David Hilbert in 1901 that maths might one day be reduced to mechanical procedures.

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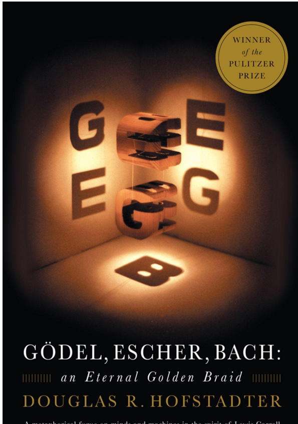
For a fully detailed proof, get Douglas R. Hofstadter, *Gödel, Escher, Bach: An Eternal Golden Braid*



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The proof goes like this:

- Turing showed that no program can compute whether arbitrary other programs ever finish. (The proof is very similar to the incompleteness proof.)
- If we have enough symbols, we can express the execution of a program in logic and write a formula that is valid iff a program halts.