Informatics 1 – Introduction to Computation

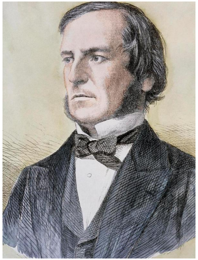
Computation and Logic

Julian Bradfield

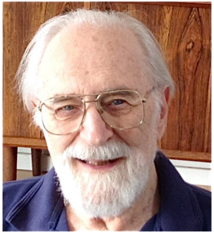
based on materials by

Michael P. Fourman

Karnaugh Maps



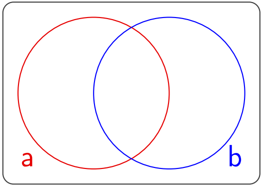
George Boole,
1815–1864



Maurice Karnaugh,
1924–

How many distinguishable universes?

Suppose we have two predicates. How many different true/false combinations are there?



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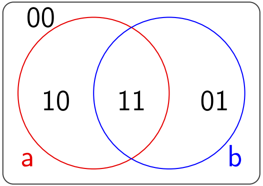

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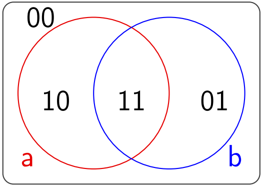

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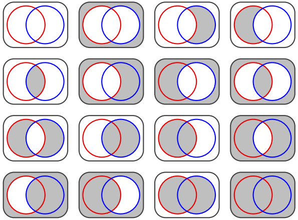


$2^4 = 16 = 2^{2^2}$

Lots of distinguishable universes

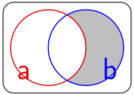
3 predicates, $2^3 = 8$ regions, $2^8 = 256$ different universes

4 predicates, $2^4 = 16$ regions, $2^{16} = 65536$ different universes

How can we characterize a universe?

By saying which regions (i.e. which boolean combinations of predicates) are inhabited/empty.

Consider



Its inhabited regions are 00,10,11, so it is described by

$$(\neg a \wedge \neg b) \vee (a \wedge \neg b) \vee (a \wedge b)$$

$$\neg b \vee (a \wedge b)$$

$$\neg(\neg a \wedge b)$$

$$a \vee \neg b$$

Binary not-SI prefixes:
Ki (Kibi) 1024 (2^{10})
Mi (Mebi) 1048576 (2^{20})
Gi (Gibi) 1073741824 (2^{30})
etc.
$65536 = 64$ Ki

From Truth Tables to Karnaugh Maps

There are algorithmic ways to simplify boolean formulae.

But Karnaugh Maps are a human way, exploiting our pattern-matching abilities.

We start with tables of values:

What formula describes

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{b} \\ & & 0 & 1 \\ \hline \multirow{2}{*}{a} & 0 & 1 & 0 \\ & 1 & 1 & 1 \end{array}$$

?

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Highlight the 1 cells and look for the largest **even rectangles** that cover them.

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What formula describes

a \ b	0	1
0	1	0
1	1	1

?

Highlight the 1 cells and look for the largest **even rectangles** that cover them. Thus we see

$$a \lor \neg b$$

(By *even rectangle*, we mean rectangles with width and height powers of two.)

Things get interesting when we have three or four variables. Write tables so:

		cd			
		00	01	11	10
ab	00				
	01				
	11				
	10				

This order of values is a Gray code. The point is that only one variable changes as you go one step up/down/left/right.

Four-variable Maps

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01				
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cd

		00	01	11	10
	00				
	01				
ab	11				
	10				

This order of values is a Gray code. The point is that only one variable changes as you go one step up/down/left/right.

The order of entries means that the 1-values of each variable occupy adjacent rows (c, d) or columns (a, b). What about the 0-values?

The Karnaugh Map as a torus

cd

ab		00	01	11	10
	00				
	01				
	11				
	10				

The Karnaugh Map as a torus

ab \ cd	00	01	11	10
00				
01				
11				
10				

The Karnaugh Map as a torus

		cd		
ab	00	01	11	10
00				
01				
11				
10				

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cd

ab	00	01	11	10
00				
01				
11				
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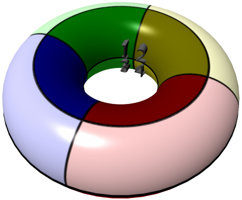
		cd		
ab	00	01	11	10
00				
01				
11				
10				

The Karnaugh Map as a torus

		cd			
		00	01	11	10
ab	00				
	01				
	11				
	10				

The 0-values for each variable occupy adjacent rows/columns *if we view the table as wrapping round bottom to top and right to left*.



via pngwing.com

KM example 1

	cd	00	01	11	10
ab	00	0	0	0	0
	01	0	1	1	0
	11	0	1	1	0
	10	0	0	0	0

KM example 1

ab \ cd	00	01	11	10
00	0	0	0	0
01	0	1	1	0
11	0	1	1	0
10	0	0	0	0

KM example 1

ab \ cd	00	01	11	10
00	0	0	0	0
01	0	1	1	0
11	0	1	1	0
10	0	0	0	0

$$b \wedge d$$

KM example 2

$ab \backslash cd$	00	01	11	10
00	0	0	0	0
01	1	1	0	0
11	1	1	0	0
10	0	0	0	0

KM example 2

ab \ cd	00	01	11	10
00	0	0	0	0
01	1	1	0	0
11	1	1	0	0
10	0	0	0	0

KM example 2

$ab \backslash cd$	00	01	11	10
00	0	0	0	0
01	1	1	0	0
11	1	1	0	0
10	0	0	0	0

$$b \land \neg c$$

KM example 3

ab \ cd	00	01	11	10
00	1	1	1	1
01	0	0	0	0
11	0	0	0	0
10	0	0	0	0

KM example 3

$ab \backslash cd$	00	01	11	10
00	1	1	1	1
01	0	0	0	0
11	0	0	0	0
10	0	0	0	0

KM example 3

ab \ cd	00	01	11	10
00	1	1	1	1
01	0	0	0	0
11	0	0	0	0
10	0	0	0	0

$$\neg a \wedge \neg b$$

KM example 4

ab \ cd	00	01	11	10
00	1	0	0	1
01	1	0	0	1
11	0	0	0	0
10	0	0	0	0

KM example 4

ab \ cd	00	01	11	10
00	1	0	0	1
01	1	0	0	1
11	0	0	0	0
10	0	0	0	0

KM example 4

ab \ cd	00	01	11	10
00	1	0	0	1
01	1	0	0	1
11	0	0	0	0
10	0	0	0	0

$$\neg a \wedge \neg d$$

KM example 5

$ab \backslash cd$	00	01	11	10
00	1	1	0	0
01	1	1	0	0
11	0	0	1	1
10	0	0	1	1

KM example 5

		cd			
		00	01	11	10
ab	00	1	1	0	0
	01	1	1	0	0
	11	0	0	1	1
	10	0	0	1	1

KM example 5

$ab \backslash cd$	00	01	11	10
00	1	1	0	0
01	1	1	0	0
11	0	0	1	1
10	0	0	1	1

$$(\neg a \land \neg c) \lor (a \land c)$$

KM example 6

ab \ cd	00	01	11	10
00	1	0	0	1
01	0	0	0	0
11	0	0	0	0
10	1	0	0	1

KM example 6

$ab \backslash cd$	00	01	11	10
00	1	0	0	1
01	0	0	0	0
11	0	0	0	0
10	1	0	0	1

KM example 6

$ab \backslash cd$	00	01	11	10
00	1	0	0	1
01	0	0	0	0
11	0	0	0	0
10	1	0	0	1

$$\neg b \wedge \neg d$$

KM example 7

ab \ cd	00	01	11	10
00	1	0	0	0
01	1	1	0	0
11	0	1	1	1
10	0	0	1	1

KM example 7

ab \ *cd*	00	01	11	10
00	1	0	0	0
01	1	1	0	0
11	0	1	1	1
10	0	0	1	1

KM example 7

$ab \backslash cd$	00	01	11	10
00	1	0	0	0
01	1	1	0	0
11	0	1	1	1
10	0	0	1	1

$$(\neg a \wedge \neg c \wedge \neg d) \vee (b \wedge \neg c \wedge d) \vee (a \wedge c)$$

KM example 8

$ab \backslash cd$	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	0	1	1	1
10	0	0	1	1

KM example 8

$ab \backslash cd$	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	0	1	1	1
10	0	0	1	1

KM example 8

		cd			
		00	01	11	10
	00	0	0	0	0
	01	0	0	0	0
ab	11	0	1	1	1
	10	0	0	1	1

$$(a \wedge b \wedge \neg c \wedge d) \vee (a \wedge c)$$

KM example 8

$ab \backslash cd$	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	0	1	1	1
10	0	0	1	1

$$(a \wedge b \wedge d) \vee (a \wedge c)$$

Disjunctive Normal Form

The descriptions we've built from KMs all have the form

$$(\cdots \wedge \cdots) \vee (\cdots \wedge \cdots) \vee \cdots$$

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This is disjunctive normal form (DNF). Formally, we say a formula is in DNF iff has the form

$$\bigvee_i \Big(\bigwedge_j p_{ij} \Big)$$

where each p_{ij} is either a literal (a boolean variable/predicate $a, b, \ldots$) or a negated literal ($\neg a, \neg b, \ldots$).

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Later we will see more mechanistic ways of converting to DNF.

KMs: looking at zeros

Sometimes it's a bit easier to look at the zeros.

Looking at the ones:

ab \ cd	00	01	11	10
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$$\neg(b \wedge d)$$

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Another example:

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		cd			
		00	01	11	10
ab	00	0	0	1	1
	01	1	1	1	1
	11	1	1	1	1
	10	1	1	0	0

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$$(\neg a \land c) \lor (a \land \neg c) \lor b$$

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		00	01	11	10
	00	0	0	1	1
	01	1	1	1	1
ab	11	1	1	1	1
	10	1	1	0	0

$$\neg((\neg a \wedge \neg b \wedge \neg c) \vee (a \wedge \neg b \wedge c))$$

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01	1	1	1	1
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$$(a \vee b \vee c) \wedge (\neg a \vee b \vee \neg c)$$

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Any boolean expression can be put into either DNF or CNF – each is better for some applications.

If I give you a formula in DNF, can you convert it (*not* its negation) to CNF? How big might the result be?