Informatics 1 – Introduction to Computation
Computation and Logic
Julian Bradfield
based on materials by
Michael P. Fourman
Satisfying Assignments
Boolean Algebra, Tseytin, Counting
Boolean operators – recap

The basic operators are $\neg$, $\wedge$, and $\vee$. 
Boolean operators – recap

The basic operators are \( \neg \), \( \land \), and \( \lor \).

By now you have several times seen \( \rightarrow \) and \( \leftrightarrow \), and perhaps \( \oplus \).
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The basic operators are ¬, ∧, and ∨.
By now you have several times seen → and ↔, and perhaps ⊕.
How many binary boolean operators are there?
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Using boolean operators

Everything we’ve done with boolean operators can be extended to use →, ↔ and others.

In the optional question of tutorial 5, you were asked for sequent calculus rules for →. They are:

\[
\begin{align*}
\frac{\Gamma, b \models \Delta \quad \Gamma \models a, \Delta}{\Gamma, a \rightarrow b \models \Delta} \quad (\rightarrow L) \\
\frac{\Gamma, a \models b, \Delta}{\Gamma \models a \rightarrow b, \Delta} \quad (\rightarrow R)
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In the optional question of tutorial 5, you were asked for sequent calculus rules for \( \rightarrow \). They are:

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Rules for \( \leftrightarrow \) are even more obvious:

\[
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\Gamma, a \rightarrow b, b \rightarrow a &\models \Delta & (\leftrightarrow L) \\
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Boring exercise: take all the stuff you’ve done in Haskell on WFFs etc., and extend it for these operators, if you haven’t already.
Using boolean operators

Everything we’ve done with boolean operators can be extended to use $\to$, $\leftrightarrow$ and others.

In the optional question of tutorial 5, you were asked for sequent calculus rules for $\to$. They are:

$$
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$$

Rules for $\leftrightarrow$ are even more obvious:

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\frac{\Gamma, a \to b, b \to a \models \Delta}{\Gamma, a \leftrightarrow b \models \Delta} \quad \text{(\leftrightarrow L)} \quad \frac{\Gamma \models a \to b, \Delta \quad \Gamma \models b \to a, \Delta}{\Gamma \models a \leftrightarrow b, \Delta} \quad \text{(\leftrightarrow R)}
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Boolean algebra

Algebra is about equations between things. These equations characterize boolean logic and operators:

- **Associativity:** \((a \lor b) \lor c = a \lor (b \lor c)\) and sim. for \(\land\)

- **Commutativity:** \(a \lor b = b \lor a\) and sim. for \(\land\)

- **Absorption:** \(a \lor (a \land b) = a\) and vice versa

- **Identity:** \(a \lor 0 = a\) and \(a \land 1 = a\)

- **Distributivity:** \(a \lor (b \land c) = (a \lor b) \land (a \lor c)\) and vice versa

- **Complements:** \(a \lor \neg a = 1\) and \(a \land \neg a = 0\)

Other convenient derived equations include:

- **Negation cancellation:** \(\neg \neg a = a\)

- **Zero/One:** \(\neg 1 = 0\) and \(\neg 0 = 1\)

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This set of axioms is far from minimal. Astonishingly, this single axiom suffices:
\[
\neg(\neg(\neg(a \lor b) \lor c) \lor \neg(a \lor \neg(\neg c \lor \neg(c \lor d)))) = c
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https://doi.org/10.1023/A:1020542009983
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An example of proofs using boolean algebra: prove
\( \neg(a \lor b) = \neg a \land \neg b \) from the six axioms.
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\( \neg(a \lor b) = \neg a \land \neg b \) from the six axioms.

**Lemma** If \( x \lor y = 1 \) and \( x \land y = 0 \), then \( x = \neg y \) (and \( y = \neg x \)).

**Proof** \( \neg y = 1 \land \neg y = (x \lor y) \land \neg y = (x \land \neg y) \lor (y \land \neg y) = (x \land \neg y) \lor 0 = (x \land \neg y) \lor (x \land y) = x \land (\neg y \lor y) = x \lor 1 = x \).
An example of proofs using boolean algebra: prove \( \neg(a \lor b) = \neg a \land \neg b \) from the six axioms.

**Lemma** If \( x \lor y = 1 \) and \( x \land y = 0 \), then \( x = \neg y \) (and \( y = \neg x \)).

**Proof**
\[
\neg y = 1 \land \neg y = (x \lor y) \land \neg y = (x \land \neg y) \lor (y \land \neg y) = (x \land \neg y) \lor 0 = (x \land \neg y) \lor (x \land y) = x \land (\neg y \lor y) = x \land 1 = x.
\]

By the lemma, to prove \( \neg(a \lor b) = \neg a \land \neg b \), it suffices to prove \( (a \lor b) \lor (\neg a \land \neg b) = 1 \) and \( (a \lor b) \land (\neg a \land \neg b) = 0 \). Both these follow easily by distributivity, complement, associativity and commutativity: e.g. the first is
\[
(a \lor b) \lor (\neg a \land \neg b) = ((a \lor b) \lor \neg a) \land ((a \lor b) \lor \neg b) = \cdots = 1 \land 1 = 1.
\]
We can add derived rules, as we did in sequent calculus:

- **Bi-implication:** $a \leftrightarrow b = (a \rightarrow b) \land (b \rightarrow a)$
- **Implication:** $a \rightarrow b = \neg a \lor b$
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Programming this was in FP tutorial 6! If you didn't try the optional and challenge parts, go back and try them now.

Doing this by hand tends to be boring: see textbook chapter 22 for worked examples.
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Ultimately, logic is implemented in silicon via transistors, referred to as logic gates. Circuit designers draw gates like this:

- **AND**
- **OR**
- **NOT**
- **NAND**
- **NOR**
- **XOR**

Gates (boolean operators) are connected by drawing wires:

```
  a
  b
  c
```

is the circuit for \((a \land b) \lor \neg c\).
Circuits can duplicate expressions

A circuit can use the same output more than once:

\[ \phi = \neg(a \land b) \lor \neg((a \land b) \lor \neg c) \]

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How can we simulate re-use using only logic?

\[ \psi = (\neg v \lor \neg(v \lor \neg c)) \land (v \leftrightarrow a \land b) \text{ (or think: } \neg v \lor \neg(v \lor \neg c) \text{ where } v = a \land b) \]
Circuits can duplicate expressions

A circuit can use the same output more than once:

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\psi &= (\neg v \lor \neg(v \lor \neg c)) \land (v \leftrightarrow a \land b) \quad \text{(or think: } \neg v \lor \neg(v \lor \neg c) \text{ where } v = a \land b) \\
\phi \text{ and } \psi \text{ are not equal, but they are equisatisfiable: } \phi \text{ has a satisfying assignment iff } \psi \text{ does, because any sat. asst. for } \phi \text{ gives one for } \psi \text{ and vice versa.}
\end{align*}
\]
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\[ a \quad b \quad c \]
\[ v \quad x \quad y \quad z \quad r \]

is \( \phi = \neg(a \land b) \lor \neg((a \land b) \lor \neg c) \)

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We can do this for all the intermediate values, and forget the original formula.
The Tseytin transformation

does with formulae what we’ve just done with gates.
Introduce a new variable $x$ for every subformula $\phi$, and add a clause saying $x \leftrightarrow \phi$. For example:
(see live demo)
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Tseytin is an $O(n)$ conversion to an equisatisfiable CNF formula.
Unfortunately CNF-SAT can still be exponential – no free lunch.
Final question for you: how long does it take to check satisfiability of a DNF formula?
2-CNF-SAT (or just 2-SAT) is the special case where *every clause has at most two literals*, such as:

\[ (-A \lor -C) \land (-B \lor -C) \land (B \lor A) \land (-C \lor D) \land (-D \lor -B) \]
2-CNF-SAT (or just 2-SAT) is the special case where every clause has at most two literals, such as:

\[ (\neg A \lor \neg C) \land (\neg B \lor C) \land (B \lor A) \land (\neg C \lor D) \land (\neg D \lor \neg B) \]

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Any 2-SAT problem can be solved in linear time. They arise naturally in problems involving may/must/must not relations between things: e.g. which courses you are able to take. Sometimes unfortunate consequences arise from simple rules . . .
2-variable clauses

Any two-variable clause can be written in terms of $\lor$ and $\neg$, and vice versa.

Rewriting the previous out of CNF gives:

$$\neg(A \land C) \land (B \to C) \land (A \lor B) \land (C \to D) \land \neg(D \land B)$$

which might represent the following rules:

1. You may not take both Astrology and Chiromancy
2. If you take Belomancy, you must take Chiromancy
3. You must take Astrology or Belomancy
4. If you take Chiromancy, you must take Dream Interpretation
5. You may not take both Dream Interpretation and Belomancy

What can you take?
Implication clauses

Any two-variable clause can also be written in terms of $\rightarrow$ and $\neg$: 

$$(A \rightarrow \neg C) \land (B \rightarrow C) \land (\neg A \rightarrow B) \land (C \rightarrow D) \land (D \rightarrow \neg B)$$

$\neg A \lor \neg C$ is symmetrical. Is $A \rightarrow \neg C$ symmetrical? (Remember back to sequents and contraposition...)
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This is useful because *implication is transitive*:

if $B \rightarrow C$ and $C \rightarrow D$, then $B \rightarrow D$

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Remember *barbara*!
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This is useful because implication is transitive:
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We can build a partial graph of implication between literals:

\[
0 \rightarrow B \rightarrow C \rightarrow D \rightarrow \neg B \rightarrow 1
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0 → anything, and anything → 1
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- If \( C \) is true, so is \( D \)

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Remember *barbara*!

0 \( \rightarrow \) anything, and anything \( \rightarrow 1 \)

What should I do with \( A \), \( \neg A \), and \( \neg C \)?
Implication clauses

Any two-variable clause can also be written in terms of \( \rightarrow \) and \( \neg \):

\[(A \rightarrow \neg C) \land (B \rightarrow C) \land (\neg A \rightarrow B) \land (C \rightarrow D) \land (D \rightarrow \neg B)\]

This is useful because implication is transitive:

if \( B \rightarrow C \) and \( C \rightarrow D \), then \( B \rightarrow D \)

We can build a partial graph of implication between literals:

\[0 \rightarrow B \rightarrow C \rightarrow D \rightarrow \neg B \rightarrow 1\]

This tells us a lot about satisfying assignments:

- If a literal is true, everything to the right must be true
- If it’s false, everything to the left must be false
- \( B \) must be false
- if \( C \) is true, so is \( D \)

Satisfying assignments are got from cutting the line somewhere, which must be right of \( B \). (And then dealing with the rest.)
The Arrow Rule

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\[(\neg R \lor Q) \land (\neg R \lor S) \text{ equiv } (R \rightarrow Q) \land (R \rightarrow S)\]

\[
\begin{array}{c}
1 \\
\downarrow \\
R \\
\uparrow \\
0 \\
\end{array}
\]

\[
\begin{array}{ccc}
& Q & S \\
\downarrow & & \downarrow \\
R & & \end{array}
\]

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There are five satisfying assignments, one for each valid cut.
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\begin{center}
\begin{tikzpicture}
  \node (q) at (0,0) {$Q$};
  \node (s) at (1,0) {$S$};
  \node (r) at (0.5,-1) {$R$};
  \node (0) at (-0.5,-2) {$0$};
  \node (1) at (0.5,-2) {$1$};

  \draw (q) -- (r);
  \draw (s) -- (r);
  \draw (0) -- (r);
  \draw (1) -- (r);

  \draw (q) -- (0);
  \draw (s) -- (0);
  \draw (q) -- (1);
  \draw (s) -- (1);

  \end{tikzpicture}
\end{center}

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Counting assignments

A more complex example:

\[(A \rightarrow B) \land (B \rightarrow C) \land (C \rightarrow D) \land (A \rightarrow E) \land (E \rightarrow D)\]
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There are eight ways to cut this.
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We can count cuts thus:

- one cut above \(D\)
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We can count cuts thus:

- one cut above \(D\)
- cuts across the pentagon: 2 ways to cut the right side, 3 ways to cut the left, so 6

For an even more complicated example, see the textbook (Chapter 23, p. 252).
Counting assignments

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Complementary literals

What happens with formulae that have $A$ and $\neg A$ (like the very first one)? Such as:

$$(A \rightarrow B) \land (B \rightarrow C) \land (C \rightarrow D) \land (A \rightarrow \neg B) \land (\neg B \rightarrow D)$$

A valid cut must separate complementary literals, so only 3 cuts survive. Note $A \rightarrow \neg B$ is the same as $B \rightarrow \neg A$ (contraposition), so sometimes you can remove complementary literals. This makes thing easier!
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Cycles in the graph

It’s quite possible for the implication graph to contain cycles. For example:

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Sometimes cycles can be removed by taking the contrapositive. Go back to the first example (slide 12) and complete it both with and without a cycle.
Summary

Drawing the implication graph and counting valid cuts lets us count satisfying assignments of 2-SAT formulae.
A valid cut must:
  ➤ separate 0 and 1
  ➤ separate complementary literals
  ➤ not cut a cycle

Why do we care? It turns out that #2-SAT (as it is known) has application in statistical physics and artificial intelligence. It is also of theoretical interest in several ways. (There is one quirk we haven't considered. What if the implication graph is non-planar? See the book for how to deal with that.)
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