Informatics 1 – Introduction to Computation
Computation and Logic
Julian Bradfield
based on materials by
Michael P. Fourman
Finite State Machines

Stephen Kleene,
1909–1994
Photo: Harold Hone
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Applications

FA have countless applications:

- washing machine/central heating/etc. controllers
- traffic light controllers
- parsing programming languages
- CPU controllers
- natural language processing
- ...

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We often think about FAs by drawing them:

The circles are the states, with their names: the set of states is \{0, 1\}. The connecting arrows are the transitions, with the input letter that activates them: the input alphabet is \{a, b\}. The short arrow marks the initial or start state. This machine reads b until it reads an a, after which it reads a or b for ever.
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This machine reads \(b\) until it reads an \(a\), after which it reads \(a\) or \(b\) for ever.
If we feed the machine $ab$:

1. In state 0, $a$ fires the right transition and the state changes to 1.
2. Then from state 1, $b$ fires the top transition and the state remains 1.
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FAs with accepting states

On input *babaaba* we see:

We say that the automaton has accepted the string *babaaba*. If the automaton ends in a non-accepting state, it has rejected the string. Verify for yourself that this automaton rejects *babbaa*. 

![Diagram of an automaton with states 0, 1, and 2, labeled with transitions for 'a' and 'b'.](image)
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![Automaton Diagram]
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What input strings does this automaton accept?

Any number of $b$s then an $a$ and any number of $b$s, then optionally an $a$ and any number of $b$s, then two $a$s followed by everything all over again.

That's a little hard to understand: we will see later how to turn this into a precise description.

If instead we think about it, we see: the state labels 0, 1, 2 count how many $a$s we have seen, modulo 3. The automaton accepts any string of $a$s and $b$s where the number of $a$s is not a multiple of 3.
Automata accepting languages

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The **black hole convention** says that if you don’t write a transition for letter $a$ from state $q$, there is an implicit $a$-transition from $q$ to a non-accepting black hole state.
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Determinism

So far, our automata have had

- a single start state
- exactly one transition from each state for each input letter

Such automata are called deterministic, because their next move is fully determined by the input letter. Later, we’ll see non-deterministic automata, but for now we stick with DFAs.

at most one transition, if we use the black hole convention
There are several ways to mathematize DFAs. Here’s one:

A DFA comprises:

- A finite set $Q$ of states
- A finite alphabet $\Sigma$ of input letters
- A transition function $\delta : Q \times \Sigma \rightarrow Q$
- A starting state $q_0 \in Q$
- A subset $F \subseteq Q$ of accepting (or final) states

The use of $F$ for ‘final’ states is traditional.
The states of a DFA are its memory. Using this fact is the easiest way to construct a DFA: first think about the states, then the transitions. Example:

Build a DFA over the alphabet \{0, 1\} that accepts strings with an even number of zeros and an odd number of ones. We need to track two bits of information: have we seen even/odd numbers of zeros/ones. One bit needs two states, two bits need four states. So:

\[ Q = \{ E_0, E_1, O_0, O_1 \} \]

Initially, we've read nothing: 

\[ q_0 = E_0 E_1 \]

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\[ \begin{array}{c|c}
0 & 1 \\
\hline
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The book shows how bad things can get if you just try to work from an initial state by following your nose!
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More formalism

We need a few notations and terms to talk more about DFAs:

- for any set $\Sigma$, $\Sigma^*$ is the set of *strings* over $\Sigma$. The empty string is written $\varepsilon$. If $s \in \Sigma^*$ and $x \in \Sigma$, then $xs$ is the string comprising $x$ followed by $s$.

We’ll use some lazy conventions on slides: $M$ is $Q, \Sigma, \delta, q_0, F$, and $M'$ is $Q', \Sigma', \delta', q'_0, F'$ unless otherwise stated. Similarly $M'', M_1, M_2$ etc.
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▶ If \( \delta : Q \times \Sigma \to Q \) is the transition function, then \( \delta^* : Q \times \Sigma^* \to Q \) is the string transition function defined by \( \delta^*(q, \epsilon) = q \) and \( \delta^*(q, xs) = \delta^*(\delta(q, x), s) \)

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- If $\Sigma^* \ni s = a_1 \ldots a_n$, the **trace** of $s$ is the sequence $q_0 \ldots q_n$ where $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} q_n$.

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- The language accepted by $M$ is $L(M) = \{ s \in \Sigma^* : \delta^*(q_0, s) \in F \}$.
- A language $L \subseteq \Sigma^*$ is regular iff there is some DFA $M$ over $\Sigma$ that accepts $L$.

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We can use automata as building blocks in others (as we build formulae out of formulae...).

Start with *complement*: if $M$ accepts $L$, how do we build a machine that accepts $\overline{L} = \Sigma^* - L$?

There are two common notations for set difference: $A - B$ and $A \setminus B$. They mean $\{x \in A : x \notin B\}$.
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Easy: swap accepting and rejecting states:

- The complement of $M = (Q, \Sigma, \delta, q_0, F)$ is $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$.

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Hence we know that the set of regular languages is closed under complement.

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The term closed under ... is common in algebra. Be sure to understand it.
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Building up automata: product

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![Diagram showing automata product]

Even number of 0s
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Building up automata: product

If $M, M'$ accept $L, L'$, can we make something accepting $L \cap L'$?
It's a bit trickier, but yes: we need somehow to feed input to $M$ and $M'$ at the same time.

- Let $M = (Q, \Sigma, \delta, q_0, F)$, and $M' = (Q', \Sigma, \delta', q_0', F')$.
  The product $M \times M'$ is $(Q \times Q', \Sigma, \delta'', (q_0, q_0'), F \times F')$ where
  $\delta''((q, q'), a) = (\delta(q, a), \delta'(q', a))$.

Notice that we can run $M$ and $M'$ in parallel without ever constructing all of $M \times M'$. This is on the fly construction. Unfortunately, many things do need the whole automaton.

What about black hole states?

Even number of 0s
Odd number of 1s
Even 0s and odd 1s

What about black hole states?
Building up automata: sum

If $M, M'$ accept $L, L'$, can we make something accepting $L \cup L'$?
Building up automata: sum

If $M, M'$ accept $L, L'$, can we make something accepting $L \cup L'$?
Yes, with almost the same construction:

- Let $M = (Q, \Sigma, \delta, q_0, F)$, and $M' = (Q', \Sigma, \delta', q'_0, F')$.
  The sum $M +_d M'$ is $(Q \times Q', \Sigma, \delta'', (q_0, q'_0), Q \times F' \cup F \times Q')$
  where $\delta''((q, q'), a) = (\delta(q, a), \delta'(q', a))$.

The difference is the accepting states: we accept if either component accepts. Hence $L(M +_d M') = L(M) \cup L(M')$. Later we will see a different sum for other automata. I'll write this one as $+_d$ (d for deterministic).
Building up automata: sum

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$Q \times F' \cup F \times Q'$ can also be written as $(Q \times Q') - ((Q - F) \times (Q' - F'))$ which we can notate $\overline{F} \times \overline{F'}$.

Does this remind you of something?

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So now we know regular languages are closed under complement, intersection, and union.

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