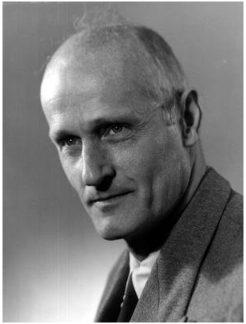
Informatics 1 – Introduction to Computation
Computation and Logic

Julian Bradfield
based on materials by
Michael P. Fourman

Finite State Machines



Stephen Kleene, 1909–1994
Photo: Harold Hone

Very Simple Computers

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These (give or take a technicality) are Finite Automata, or Finite State Machines.

Applications

FA have countless applications:

- washing machine/central heating/etc. controllers
- traffic light controllers
- parsing programming languages
- CPU controllers
- natural language processing
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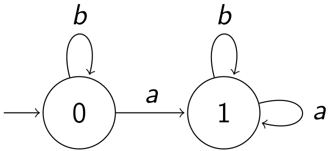
Is your laptop a finite automaton? Is anything not a finite automaton?

Drawing FAs

We often think about FAs by drawing them:



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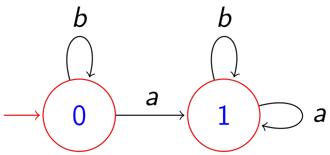

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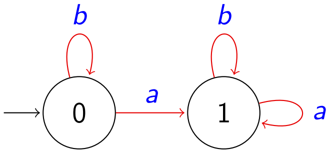

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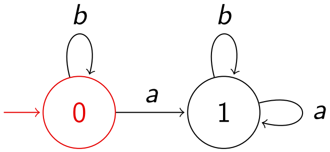

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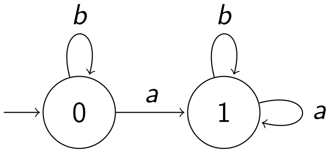

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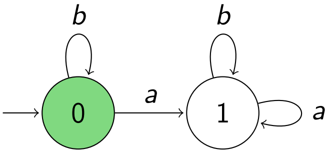
This machine reads b until it reads an a, after which it reads a or b for ever.

FA behaviour



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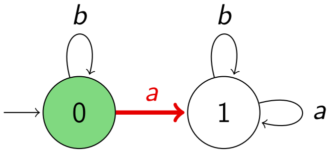

If we feed the machine *ab*:

- in state 0, *a* fires the right transition

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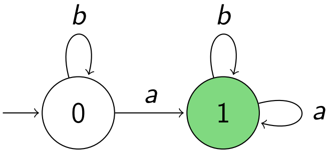

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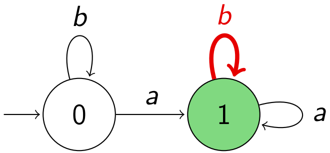

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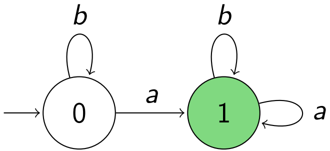

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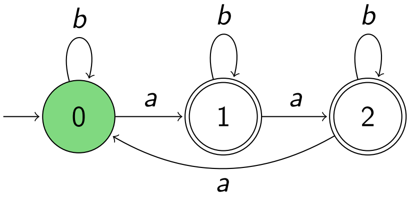
FAs with accepting states



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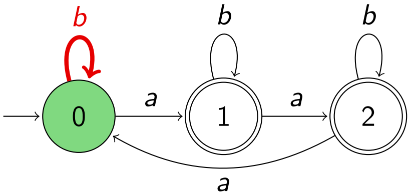

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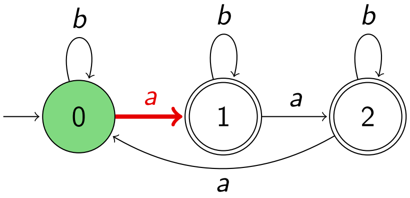

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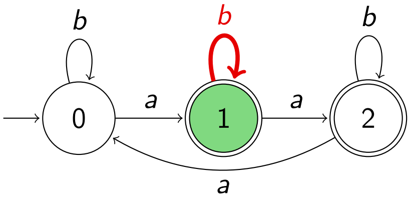

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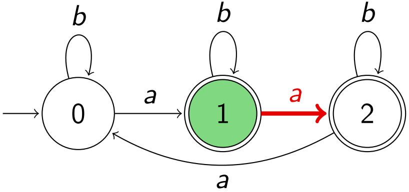

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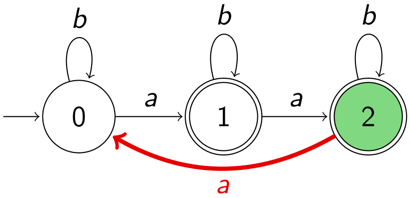

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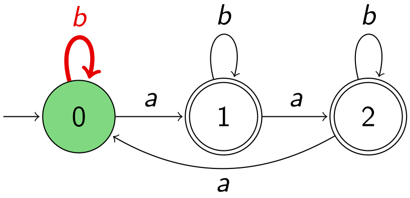

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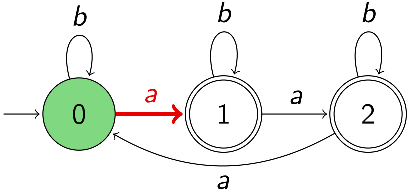

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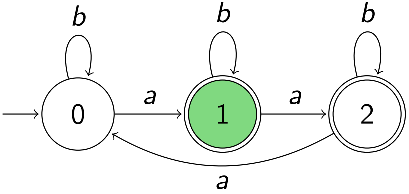


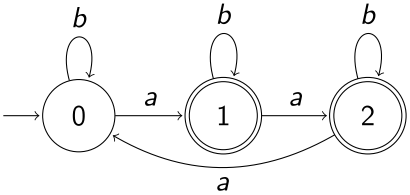
On input *babaaba* we see:

ending up in an accepting state.

We say that the automata has accepted the string *babaaba*.

If the automaton ends in a non-accepting state, it has rejected the string. Verify for yourself that this automaton rejects *babbaa*.

Automata accepting languages



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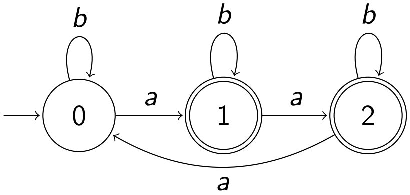

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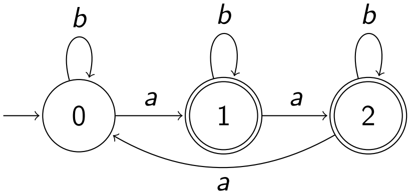

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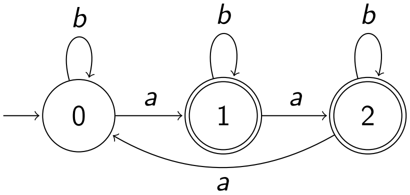

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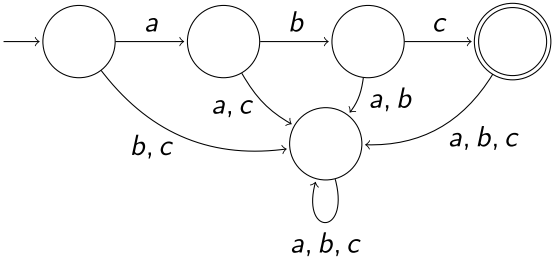
If instead we think about it, we see: the state labels $0, 1, 2$ count how many *a*s we have seen, modulo 3. The automaton accepts any string of *a*s and *b*s where the number of *a*s is not a multiple of 3.

Space saving conventions



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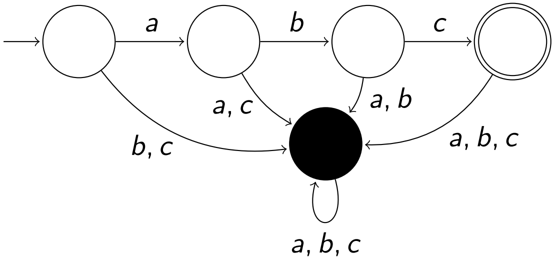

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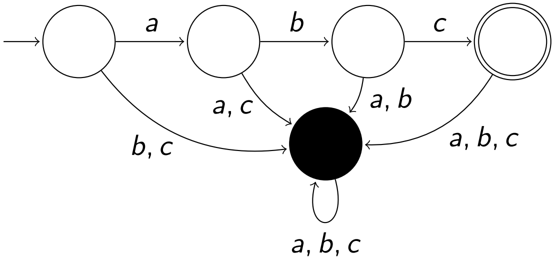

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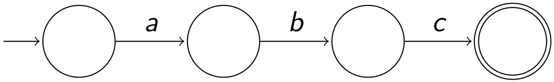

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Determinism

So far, our automata have had

- a single start state
- exactly one transition from each state for each input letter

at most one transition, if we use the black hole convention

Such automata are called deterministic, because their next move is fully determined by the input letter. Later, we'll see non-deterministic automata, but for now we stick with DFAs.

Formalism

There are several ways to mathematize DFAs. Here's one:

A DFA comprises:

- A finite set Q of states
- A finite alphabet Σ of input letters
- A transition function $\delta\colon Q \times \Sigma \to Q$
- A starting state $q_0 \in Q$
- A subset $F \subseteq Q$ of accepting (or final) states

The use of F for 'final' states is traditional.

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We need to track two *bits* of information: have we seen even/odd numbers of zeros/ones. One bit needs two states, two bits needs four states. So:

$$Q = \{ E_0 E_1, E_0 O_1, O_0 E_1, O_0 O_1 \}$$

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The accepting set is just $F = \{ E_0 O_1 \}$.

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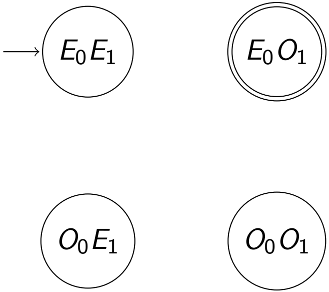
Even 0s and odd 1s



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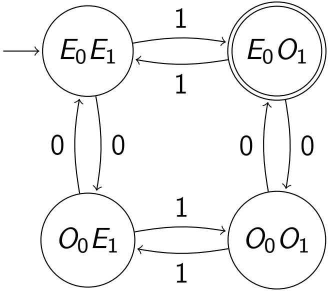

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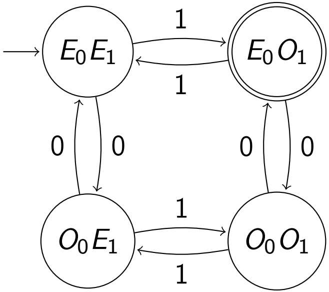

Writing it in symbols rather than diagrams:

$Q = \{\, E_0E_1, E_0O_1, O_0E_1, O_0O_1 \,\}$

$q_0 = E_0E_1$

$F = \{\, E_0O_1 \,\}$

δ is the following table:

	0	1
E_0E_1	O_0E_1	E_0O_1
E_0O_1	O_0O_1	E_0E_1
O_0E_1	E_0E_1	O_0O_1
O_0O_1	E_0O_1	O_0E_1

The book shows how bad things can get if you just try to work from an initial state by following your nose!

More formalism

We need a few notations and terms to talk more about DFAs:

- for any set Σ, Σ^* is the set of strings over Σ. The empty string is written ε. If $s \in \Sigma^*$ and $x \in \Sigma$, then xs is the string comprising x followed by s.

We'll use some lazy conventions on slides: M is $Q, \Sigma, \delta, q_0, F$, and M' is $Q', \Sigma', \delta', q_0', F'$ unless otherwise stated. Similarly M'', M_1, M_2 etc.

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- The language accepted by M is $L(M) = \{s \in \Sigma^* : \delta^*(q_0, s) \in F\}$.

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- The language accepted by M is $L(M) = \{s \in \Sigma^* : \delta^*(q_0, s) \in F\}$.
- A language $L \subseteq \Sigma^*$ is regular iff there is some DFA M over Σ that accepts L.

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Building up automata: complement

We can use automata as building blocks in others (as we build formulae out of formulae. . .).

Start with *complement*: if M accepts L, how do we build a machine that accepts $\bar{L} = \Sigma^* - L$?

There are two common notations for set difference: $A - B$ and $A \setminus B$. They mean $\{x \in A : x \notin B\}$.

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Easy: swap accepting and rejecting states:

- The complement of $M = (Q, \Sigma, \delta, q_0, F)$ is $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$.

There are two common notations for set difference: $A - B$ and $A \setminus B$. They mean $\{x \in A : x \notin B\}$.

We must remember to include any black hole states that weren't drawn!

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- The complement of $M = (Q, \Sigma, \delta, q_0, F)$ is $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$.

Hence we know that the set of regular languages is closed under complement.

There are two common notations for set difference: $A - B$ and $A \setminus B$. They mean $\{x \in A : x \notin B\}$.

We must remember to include any black hole states that weren't drawn!

The term *closed under ...* is common in algebra. Be sure to understand it.

Building up automata: product

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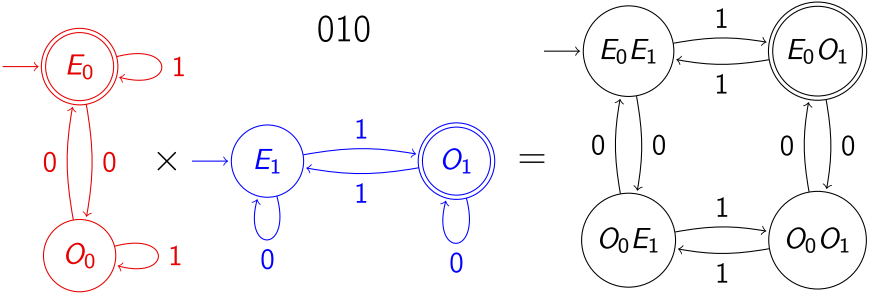

even number of 0s
odd number of 1s
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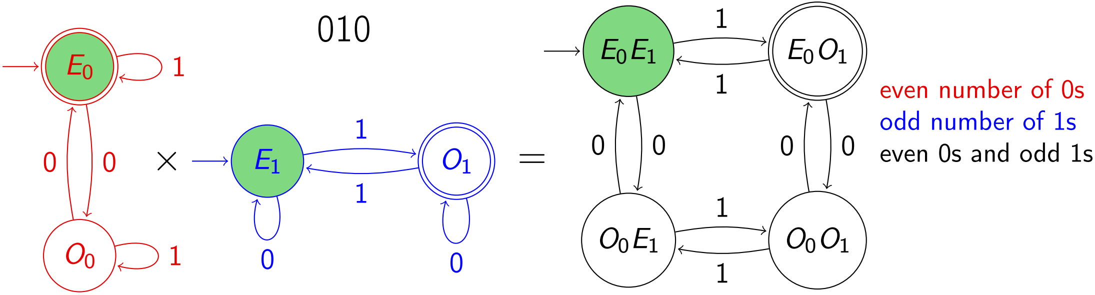

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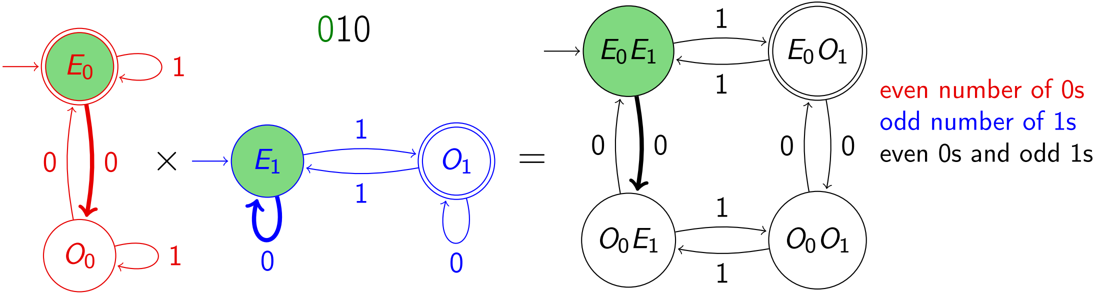

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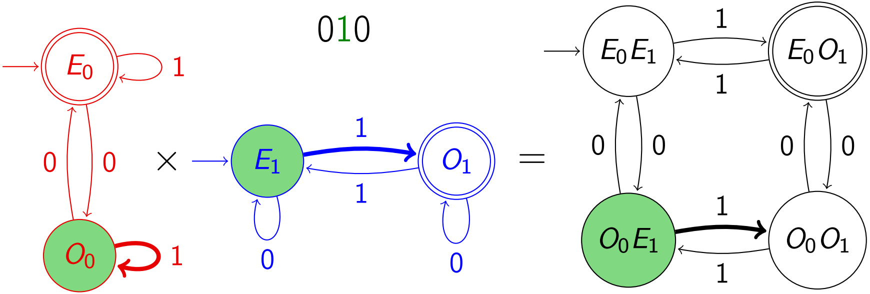

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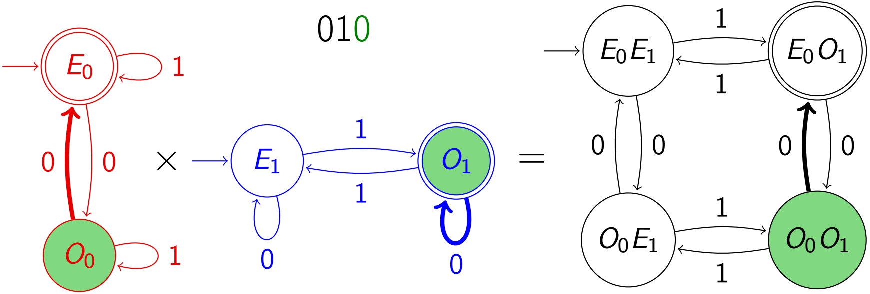

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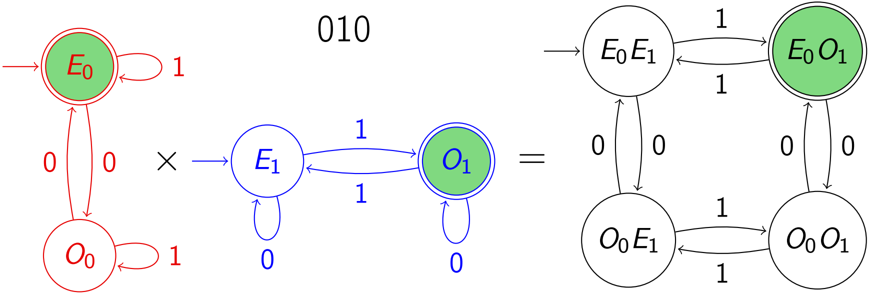

Notice that we can run M and M' in parallel without ever constructing all of $M \times M'$. This is *on the fly* construction.

Unfortunately, many things do need the whole automaton.

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What about black hole states?

Building up automata: sum

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Yes, with almost the same construction:

- Let $M = (Q, \Sigma, \delta, q_0, F)$, and $M' = (Q', \Sigma, \delta', q'_0, F')$.
  The sum $M +_{\mathrm{d}} M'$ is $(Q \times Q', \Sigma, \delta'', (q_0, q'_0), Q \times F' \cup F \times Q')$
  where $\delta''((q, q'), a) = (\delta(q, a), \delta'(q', a))$.

The difference is the accepting states: we accept if *either* component accepts. Hence $L(M +_{\mathrm{d}} M') = L(M) \cup L(M')$.

Later we will see a different sum for other automata. I'll write this one as $+_{\mathrm{d}}$ (d for deterministic).

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So now we know regular languages are closed under complement, intersection, and union.

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