Introduction to Algorithms and Data Structures

Lecture 7: Classic datatypes: lists, stacks, queues

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'Lists' in general ...

We've seen that 'lists' can be implemented in several ways, e.g. via arrays or linked lists. How might we compare these?

Start by listing the operations we'd like any impl to support.

E.g. for (unsorted) lists of items of type X, might want operations

get	: $\texttt{int} \to X$	# read item at given pos
set	: $\texttt{int} * X \to \texttt{void}$	# write item at given pos
cons	: $X \to \texttt{void}$	# add item at start
append	: $X \to \texttt{void}$	# add item at end
insert	: $\texttt{int} * X \to \texttt{void}$	
delete	: $\texttt{int} \to \texttt{void}$	
length	: $\texttt{void} \to \texttt{int}$	

Much like an interface in Java.

Abstract interfaces, concrete implementations

As in Java, we can consider various concrete implementations of this abstract interface.

Further points:

- For some purposes, could consider an interface with fewer operations, or with more: e.g. **reverse** : $\mathtt{void} \to \mathtt{void}$.
- May be other operations that make sense for specific impl's. E.g. for linked lists, 'insert/delete at current position' is useful.
- Some of our operations will be definable from others: e.g.

$$\textbf{append}(x) \ \equiv \ \textbf{insert}(\textbf{length}(), x)$$

  But may want to include **append** in its own right: could be implementable more efficiently than general **insert**.

Implementation 1: Fixed-size arrays

Use an array A of some fixed size m.
Can store a list $L = x_0, \ldots, x_{n-1}$ (where $n \leq m$)
in the first n cells of A (so A[i] $= x_i$ for each i $< n$).
Also want an integer variable n to store the value of n.

List operations are easy to implement. E.g.

```
get(i):
    return A[i]

append(x):
    A[n] = x
    n = n+1

insert (i,x):
    for j = n−1 downto i
        A[j+1] = A[j]
    A[i] = x
    n = n+1
```

- **length**, **get**, **set** and **append** (when it works) take $\Theta(1)$ time.
- **cons**, **insert**, **delete** require $\Theta(n)$ time in worst case.

Lists via fixed-size arrays, ctd.

Fixed-size arrays have some strengths . . .

- Fast **get** and **set** operations – especially if we can keep the array on the stack!
- Fixed, predictable size good for memory management. (If on stack, can reclaim space immediately on expiry.)

. . . but a major weakness . . .

- Can't cope with lists longer than pre-set limit m.
- If a computation involves a lot of lists, of unpredictable sizes, very likely we'll either under-cater (some array will overflow) or over-cater (many arrays will contain a lot of wasted space).

So not a good choice for 'general-purpose' lists.

Implementation 2: Extensible arrays

Idea is simple: if array A overflows, replace it by a bigger one!

- If memory space 'after' A happens to be free, cheap to do.
- But if not, may have to allocate a fresh array B, and copy contents of A into it. E.g. for some real number $r > 1$:

```
append (x):
    if n = |A|
        B = new array (⌈ n × r ⌉)
        copy contents of A into B (n items)
        A = B
    # Now do ordinary append:
    A[n] = x
    n = n+1
```

So a 'normal' append takes $\Theta(1)$ time – but occasionally we may get a bad one, taking $\Theta(n)$.

Might seem 'dirty', but widely used in practice.
Runtime analysis is interesting . . .

Amortized cost

Perhaps in some apps, even one bad **append** day could be fatal.

But often, we're happy if over any long run of **append**s, the *average* time is reasonable. A bad one may be acceptable if we regard its cost as amortized ('spread out') over the next 100 good ones – i.e. if invested effort 'pays for itself' over time.

Does it?

Suppose array has initial capacity a, and starting from **nil** we do m **append**s in succession, expanding by factor $r > 1$ when need be.

Array size grows as $a, ar, ar^2, ar^3, \ldots$. How many steps to reach m? Solving '$ar^s = m$' yields $s = \log_r(m/a)$ for the number of steps. An item may get copied this many times!

Since potential number of copyings of an item grows with m, might suspect 'average cost per **append**' also grows with m . . . ??

Let's do the sums.

Calculating amortized cost of append

Example: Suppose $a = 100$, $r = 1.1$, $m = 5000$.

Note that $1.1^{41}a < m < 1.1^{42}a$. So will need 42 expansions.

Ignoring 'rounding', number of copyings (B[i] = A[i]) is basically

$$100 \times (1 + 1.1 + 1.1^2 + \cdots + 1.1^{41})$$

By 'sum of geometric progression' formula, this is

$$100 \times (1.1^{42} - 1)/(1.1 - 1) \; < \; 1.1m/0.1$$

So although some items get copied 42 times,
average no. of copyings per item stays below $1.1/0.1 = 11$.

In general, total number of copyings is basically at most $m(r/(r-1))$.

So average no. of copyings per item stays below $r/(r-1)$.

More conceptual argument

Again suppose $a = 100$, $r = 1.1$.

Imagine a copying costs 1p. Each time we do an **append**, we pay 11p into a **pension fund** to pay for future copyings.

Suppose we've just done our first expansion.
Array now has 110 cells, with 100 filled.
Next 10 **append**s pay for second expansion (110 copyings).

After second expansion, array has 121 cells, 110 filled.
Next 11 **appends** pay for third expansion (121 copyings) ...

So each **append** incurs a constant cost of 11 copyings.

Amortized cost: conclusion

So total time taken by expansion/copying is $O(m)$.

But time taken by ordinary **append**s is also clearly $O(m)$.

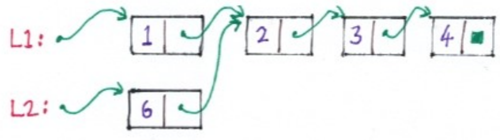
So may say the amortized cost of **append** is $O(1)$ per operation.

- Lists in Python are implemented like this, essentially with $r = 9/8$. Underlying arrays may also be shrunk if proportion in use dips below $1/2$. (For analysis, see CLRS 17.4.)
- Java class ArrayList also works like this. Precise expansion policy not prescribed, but it's required that amortized cost over a long run must be $O(1)$ per operation.

Of course, **cons**, **insert**, **delete** still take time $\Theta(n)$ in worst case (even amortized).

Implementation 3: Linked lists

We can also represent the lists over X using linked lists, where each cell contains a key of type X.



Clearly, for a list of length n:

- **get** and **set** have $\Theta(n)$ worst-case time (but with small 'C')
- **cons** takes $\Theta(1)$ time, always.
- **insert**(i,x), **delete**(i) have $\Theta(n)$ worst-case time (or $\Theta(1)$ if we've already located the cell at position $i-1$).

Linked lists also naturally allow for sharing (unlike arrays).

Offline exercise: Show how the list of all 2^n binary lists of length n can be stored in $\Theta(2^n)$ space with linked list impl. (Would take $\Theta(n.2^n)$ with arrays.)

List implementations: summary

Upper bounds on runtimes (where n is length of list):

Operation	Array impl	Linked-list impl
get	$O(1)$	$O(n)$
set	$O(1)$	$O(n)$
cons	* $O(n)$	$O(1)$
append	* $O(n)$ (amortized $O(1)$)	$O(n)$ (can make it $O(1)$)
insert	* $O(n)$	$O(n)$
delete	$O(n)$	$O(n)$

Operations marked * may fail for fixed-array implementations, or trigger expansion for extensible-array ones.

So arrays offer fast **get/set**; linked lists offer fast cons/append and insert/delete at given position, plus sharing.

?? Is there some impl of lists for which *all* the above are 'fast' **??**
Find out in Lecture 9!

Stacks and queues

Sometimes, we know that some list will only be manipulated in certain restricted ways, e.g. ...

- Elements only ever added/read/removed at front of list (stack or Last-in-first-out buffer)
- Elements added at back, read/removed at front of list (queue or First-in-first-out buffer)

Knowing this may affect our choice of implementation.
Interfaces for stacks and queues (of items of type X):

STACKS:		QUEUES:	
empty	: $\texttt{void} \to \texttt{bool}$	**empty**	: $\texttt{void} \to \texttt{bool}$
push	: $X \to \texttt{void}$	**enqueue**	: $X \to \texttt{void}$
peek	: $\texttt{void} \to X$	**peek**	: $\texttt{void} \to X$
pop	: $\texttt{void} \to X$	**dequeue**	: $\texttt{void} \to X$

Implementing stacks

In principle, any impl of lists yields an impl of stacks:

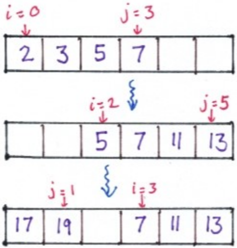
But two obvious candidates:

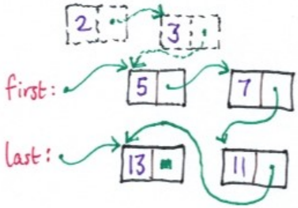
- arrays (growing at end)
- linked lists (growing at start)

Operation	Extensible array impl	Linked list impl
empty	$O(1)$	$O(1)$
push	* $O(n)$ (amortized $O(1)$)	$O(1)$
peek	$O(1)$	$O(1)$
pop	$O(1)$	$O(1)$

Implementing queues

Impl 1: Wraparound array buffer (fixed-size/extensible)
Impl 2: Linked list with references to first and last cells





Implementing queues, ctd.

How would e.g. **enqueue** look in each case?

Wraparound array:

enqueue(x):
    j = (j+1) mod |A|
    if j = i
        fail (or expand)
    else A[j] = x

Linked list:

enqueue(x):
    last.next = new Cell(x,null)
    last = last.next

For further details, see Python Lab Sheet 3.

Situation similar to stacks:

Operation	Wraparound array impl	Linked-list impl
enqueue	* $O(n)$ (amortized $O(1)$)	$O(1)$
peek	$O(1)$	$O(1)$
dequeue	$O(1)$	$O(1)$

Reading

Stacks and queues: CLRS chapter 10.
Table expansion / amortized analysis: CLRS section 17.4.