Introduction to Algorithms and Data Structures

Lecture 9: Balanced trees

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Tackling that 'worst case'

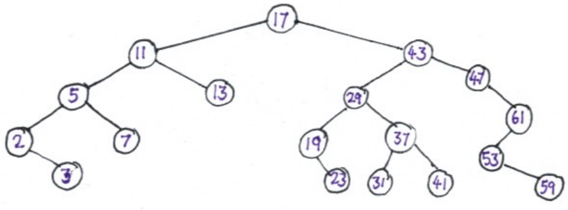
- We've considered hash table implementations of sets/dictionaries in which **lookup/insert/delete** are usually fast – but worst case time for all operations is $\Theta(n)$.
- For lists (a.k.a. vectors): some operations have worst-case time $\Theta(1)$, but **insert/delete** are $\Theta(n)$ even in average case.

??? Can we find implementations of sets/dictionaries/lists for which *all* operations have acceptable worst-case times **???**

This lecture: We'll see that 'balanced trees' (e.g. **red-black** trees) achieve this: all ops have worst-case and average time $\Theta(\lg n)$.

Will do sets/dictionaries here; ideas can also be applied to lists.

Representing sets by trees



Consider binary trees: each node x has a left and a right branch, each of which may be *null* or a pointer to a child node.
(Implementation detail: should use doubly linked tree structures.)

Write $L(x), R(x)$ for left and right subtrees at x (may be empty).

Label nodes with keys (e.g. integers or strings) in such a way that for every node x we have

$$\forall y \in L(x).\ y.key \ < \ x.key\,, \qquad \forall z \in R(x).\ x.key \ < \ z.key$$

Can use such trees to represent sets of keys.
(For dictionaries, just add value component to each node.)

Implementing contains/lookup

This is easy. Let a node x stand for the tree rooted at x.

```
contains'(x,k):
    if x = null then return False
    else if x.key = k then return True
    else if k < x.key then return contains'(x.left,k)
    else return contains'(x.right,k)

contains(k):
    return contains'(root,k)
```

Suppose the tree has n nodes and is perfectly balanced, i.e. all non-leaf nodes have 2 children, and all leaf nodes are at the same depth d. (Possible only if $n = 2^{d+1} - 1$.)
Then $d = \lfloor \lg n \rfloor$, so **contains** will take time $O(\lg n)$.

More generally, for trees that are 'not too unbalanced' (say max depth $\leq 2\lceil \lg n \rceil$), can say **contains** take $O(\lg n)$ time.

However, worst case is still $\Theta(n)$!

Insert on binary trees

This too is easy: walk down tree to find where k wants to go, and create a new leaf node for it.

```
insert'(x,k):
    if x.key = k then return KeyAlreadyPresent
    else if k < x.key then
        if x.left = null then x.left = new Node(k)
        else insert'(x.left,k)
    else
        if x.right = null then x.right = new Node(k)
        else insert'(x.right,k)

insert(k):
    if root = null then root = new Node(k)
    else return insert'(root,k)
```

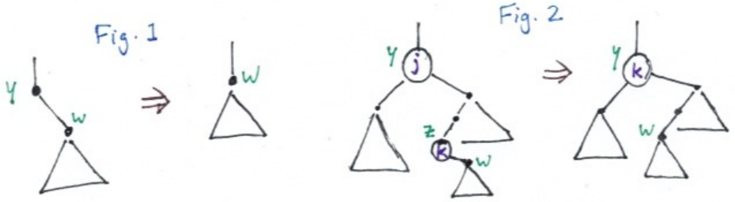
Again, $O(\lg n)$ time if tree not too unbalanced, $\Theta(n)$ in worst case.

NB. Nothing here to guard against tree *becoming* unbalanced!

Delete on binary trees

A bit more subtle. To perform **delete**(j):

- Locate the node y bearing j (assume there is one).
- If y has no children, can just delete it.
- If y has one child, easy to elide the node y (Fig. 1).
- If y has two children:
  - Locate leftmost node in $R(y)$, i.e. starting at y, turn right, then left as often as possible. This finds the node z bearing the smallest key in $R(y)$ (call it k).
  - Copy z.key to y.key.
  - If z has a right child, elide z, otherwise just delete z. (Fig. 2).



Same runtime characteristics.

Balanced tree representations

General strategy:

- Work with some special class of trees (red-black trees) that are guaranteed to be 'not too unbalanced', so that all operations will take time $O(\lg n)$.
- Whenever an **insert/delete** threatens to take us outside this class, do some 're-balancing' work to restore it.
  Clever bit: Can arrange that this re-balancing work also takes just $O(\lg n)$ time!

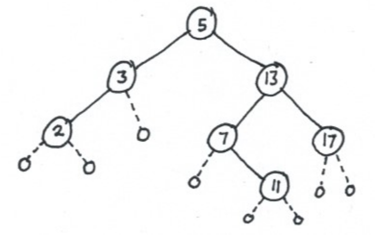
This leads to worst-case $O(\lg n)$ time for all operations.

This broad strategy works for several classes of trees:
red-black trees, AVL trees, 2-3 trees, . . .

We choose **red-black** trees as the most entertaining of these. Covered in detail in Sedgewick+Wayne and in CLRS.

Small preliminary: adding trivial nodes

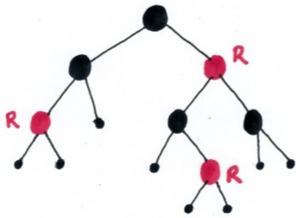
For mathematical convenience, extend our trees so that original *null* branches now point to trivial nodes, with no children and bearing no key. Original nodes are proper nodes.



Call this an extended tree.

Just makes rules easier to state.
Wouldn't need these trivial nodes in an implementation.

Red-black trees



Work with extended trees as above.

In a red-black tree, every node is coloured **red** or **black**.

- Root and all (trivial) leaves are black.
- All paths root $\rightarrow$ leaf contain same number b of blacks.
- On a path root $\rightarrow$ leaf, never have two reds in a row.

So min possible path length is b, and max is $2b - 1$.

Red-black trees are not too unbalanced.

There are $b - 1$ 'complete levels' of proper nodes, so $n \geq 2^{b-1} - 1$.

Hence $b \leq \lg(n + 1) + 1$, so all path lengths $\leq 2\lg(n + 1) + 1$.

So **contains** works as usual with worst-case time $\Theta(\lg n)$.

Insert for red-black trees

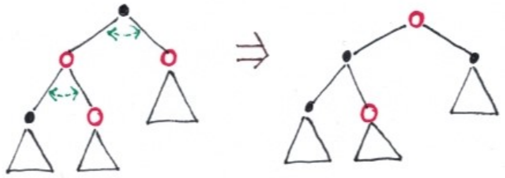
Can **insert** a key-bearing node as usual (adding two trivial leaves).
Colour it red. This all takes $O(\lg n)$ time.

Problem: Resulting tree might no longer be a legal red-black tree:

- New red node might have red parent (2 reds in succession), or
- (Trivial case) New red node might be root (should be black).

So need to apply a fix-up operation to restore red-black-ness.

Main ingredient is the red-uncle rule:



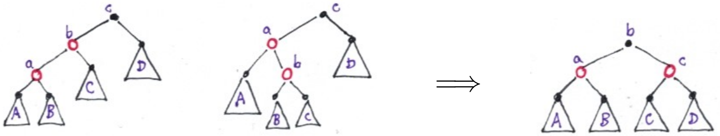
(Just colour-flipping: fast. No rewiring involved!)

Insert fix-up, continued

Applying the red-uncle rule pushes a red upward, so may result in another double-red higher up.

So we apply the red-uncle rule as often as possible (will be at most $O(\lg n)$ times). We'll then be in one of three endgame scenarios:

1. Problem cured: tree now legal.
2. Red pushed to root: **turn it black**.
   Adds 1 to all black-lengths.
3. Have some configuration involving a black with 4 'nearest black descendants'. Replace by obvious 'balanced' version:



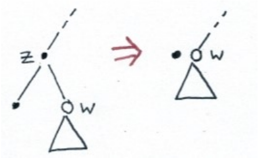
   $O(1)$ amount of rewiring.
   Note order of constituents is preserved: AaBbCcD.
   (Subtrees A,B,C,D may be empty.)

Delete for red-black trees

Just the main ideas: won't give full details.

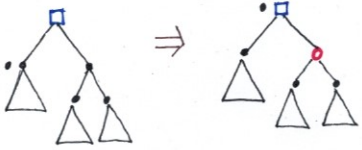
Do delete as usual: this involves removing some proper node z.

Problem: All paths must have same black-length. So if z was black, want to remove z but keep the 'blackness'.



Easy case: Node it haunts is now red: can just turn it black.

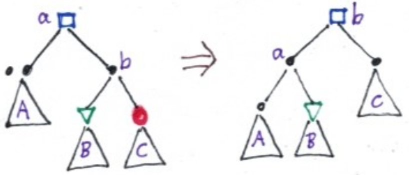
Wandering black rule: apply this as often as possible (will be $O(\lg n)$ times).



Delete for red-black trees: the endgame

Finitely many endgame scenarios, each fixable in $O(1)$ time. E.g.

- Floating black haunts a red node: **turns it black**.
- Floating black reaches root: just remove it.
- We're in some other fixable scenario, e.g.



  Blue square and green triangle are colour variables.
- 4 other scenarios like this: see CLRS 13 for full details.

Balanced trees: conclusion

- Balanced trees offer a way of implementing sets/dictionaries so that all operations have worst-case time $O(\lg n)$. (Idea can be applied to lists too.)
- Not much to choose between red-black and AVL trees. AVL are 'more balanced' (better for lookup); red-blacks possibly have faster insert/delete.
- Red-black trees used in practice:
  - Linux completely fair scheduler
  - Java 8 HashMap class: dictionary via bucket-style hash table, but each bucket is a red-black tree rather than a linked list. Retains excellent typical-case performance of hash tables, but kills off the nasty 'worst cases'.

Reading:

Sedgewick+Wayne 3.2 (first half) and 3.3 (second half)
CLRS 12.1-12.3, 13.1-13.3