# Introduction to Algorithms and Data Structures

# Lecture 10: Divide-conquer-combine and the Master Theorem

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19 October 2023

# Data structures: reflection

We've looked at ...

- some classic abstract datatypes (lists, stacks, queues, sets, dictionaries)
- various concrete implementations of them (via extensible arrays, linked lists, hash tables, red-black trees ... )

We've analysed their pros/cons in terms of asymptotic runtimes for operations. (Measured as number of line executions, paying attention to what's allowed as a $\Theta(1)$ time basic memory operation.)

The above datatypes are used frequently in programming – and many other algorithms build on them.

Most of these data structures already provided in standard libraries (e.g. Java API classes).

But understanding of runtime characteristics can help in

- writing efficient programs
- constructing efficient database queries.

# Recursion: a recurring theme

As we've seen, many algorithms can be presented as recursive: i.e. they involve subcall to (one or more instances of) same problem.

Examples:

- **Expmod**(a,n,m) involves call to **Expmod**(a,$\lfloor$n/2$\rfloor$,m).
- **Mergesort**(A,m,n) calls **Mergesort**(A,m,p) and **Mergesort**(A,p,n).
- **Insert**(x,k) (for binary trees) may call **Insert**(x.left,k) or **Insert**(x.right,k).

Common pattern:

- 'Simple' (e.g. small) instances can be dealt with directly.
- For larger instances, may do work before/during/after the recursive call(s): we divide into subproblems, conquer these, combine results.

E.g. for Mergesort:

- divide is simply checking $n - m > 1$ and computing $\lfloor (m + n)/2 \rfloor$.
- combine is *merging* the two lists returned by the recursive calls.

# Recurrence relations

**How can we calculate the (asymptotic) runtime for a recursive algorithm?**

E.g. write $T(n)$ for the worst-case runtime for Mergesort on array segments of size $n$.

```
MergeSort (A,m,n):
    if n−m = 1
        return [ A(m) ]
    else
        p = ⌊(m+n)/2⌋
        B = MergeSort (A,m,p)
        C = MergeSort (A,p,n)
        D = Merge (B,C)
        return D
```

Whatever the function $T$ is, it will satisfy

$$T(n) \;=\; T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + F(n) \quad \text{for all } n > 1\ ,$$

where $F(n)$ is the worst-case time for the divide and combine phases on inputs of size $n$. Can also say $T(1)$ is a constant $C$.

# Recurrence relations, continued

$$T(n) \;=\; \begin{cases} C & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + F(n) & \text{otherwise} \end{cases}$$

This is an example of a recurrence relation.
If we know $C$ and $F$, can compute $T(n)$ for a specific $n$, e.g.

$$T(4) \;=\; 2T(2) + F(4) \;=\; 2(2T(1) + F(2)) + F(4) \;=\; 4C + 2F(2) + F(4)$$

But can we 'solve' the rec. rel. to find an explicit formula for $T(n)$?
Or at least, for its asymptotic growth rate?

# Recurrence relations for growth rates

$$T(n) \;=\; \begin{cases} C & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + F(n) & \text{otherwise} \end{cases}$$

Actually, if we only want the growth rate of $T$, don't need to know $F$ precisely — knowing its growth rate is enough.

E.g. in Mergesort example, have $F(n) = \Theta(n)$ (time for **Merge** on lists of length $n/2$).

Leads to the concept of an asymptotic recurrence relation. E.g.

$$T(n) \;=\; \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

Solution we're seeking isn't a precise function, but a growth rate.

(Omission of $\lfloor - \rfloor$ and $\lceil - \rceil$ a bit sloppy . . . but can be shown these 'don't affect asymptotic solution' in cases like this.)

# Recurrence relations ctd.

Asymp. rec. relation for Mergesort again:

$$T(n) \; = \; \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

In Lecture 5 we saw informally that in this case $T(n) = \Theta(n \lg n)$.

Other examples:

- Runtime of **Expmod**(a,n,m) for fixed a,m:

$$T(n) \; = \; T(n/2) + \Theta(1) \quad \text{for n>1}$$

- Runtime of **Exp**(a,n) for fixed a (**Expmod** without the **mod**):

$$T(n) \; = \; T(n/2) + \Theta(n^2) \quad \text{for n>1}$$

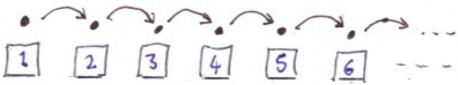
⋆ Can we solve such recurrences systematically?
Is there a general pattern here?

# How do we come up with solutions?

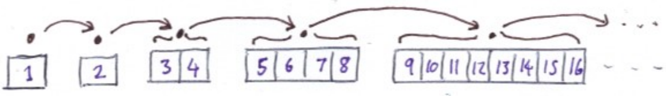
Approach 1: Use intuition/experience/numerical data to 'guess' a solution, then verify it using induction.

Usual concept of induction may need extending a bit.
E.g. for MergeSort:

Ordinary induction:




'Log induction':




Note: log induction on array size $n$ $\simeq$
ordinary induction on MergeSort recursion depth.

# The Master Theorem

Approach 2: If our recurrence just happens to be of the form ...

$$T(n) \;=\; \begin{cases} \Theta(1) & \text{if } n \le n_0 \\ aT(n/b) + \Theta(n^k) & \text{if } n > n_0 \end{cases}$$

... then there's a Master Theorem that simply gives us the answer.
(Also works with 'floors and ceilings' around.)

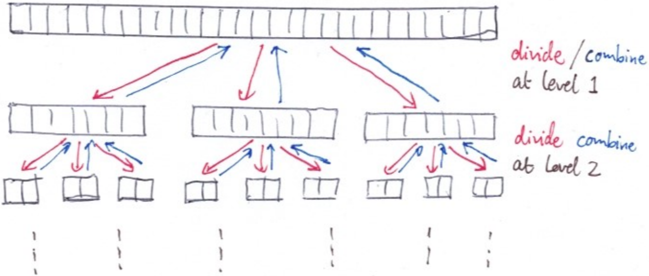
The answer depends on how $a$ compares with $b^k$ (will explain!).
Equivalently, how $e = \log_b a$ compares with $k$.

$$T(n) \;=\; \begin{cases} \Theta(n^e) & \text{if } e > k \\ \Theta(n^k \lg n) & \text{if } e = k \\ \Theta(n^k) & \text{if } e < k \end{cases}$$

This applies in many (not all) commonly arising situations.
(CLRS 4.5 gives a more general version of the theorem.)

# Master Theorem: informal intuition

Think about **total** work done by all divide / combine phases at each recursion level. Does this increase or decrease as we go down?




- Larger $a$ (no. of subproblems) means more work as we descend.
- But larger $b$ means each subproblem is smaller. If divide/combine work is $F(n) = \Theta(n^k)$, then reducing problem size by factor $b$ will reduce this work by $b^k$.
- So break-even point is when $a = b^k$. In this case, amount of work is 'essentially the same' for all levels.

# **Optional slide** (not examinable)

A bit more mathematical detail for those interested ...

- If $a < b^k$, then the most work is done at the top level. Thereafter, amount of work roughly decreases in geometric progression, by factor $r = a/b^k < 1$.
  So total work will be roughly top-level work $(\Theta(n^k))$ times $1 + r + r^2 + \cdots \leq 1/(1-r)$ (constant). Still $\Theta(n^k)$.
- If $a > b^k$, work increases by $r = a/b^k > 1$ as we descend. Around $\log_b(n)$ levels. So bottom-level exceeds top-level by

$$r^{\log_b(n)} \;=\; b^{\log_b(r).\log_b(n)} \;=\; b^{\log_b(n).\log_b(a/b^k)} \;=\; n^{e-k}$$

  So total work comes out as $\Theta(n^k).\Theta(n^{e-k}) = \Theta(n^e)$.
- If $a = b^k$, all levels are 'essentially the same'. So work is roughly (top-level work $\times$ number of levels), i.e. $\Theta(n^k \lg n)$.

# Master Theorem in action

- Mergesort recurrence again:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

  Here $a = 2$, $b = 2$, $k = 1$. So $e = \log_b a = 1$ and $e = k$.
  So we're in the middle case: $\Theta(n \log n)$.

- Exp(a,n) for fixed a:

$$T(n) = T(n/2) + \Theta(n^2) \quad \text{if } n > 1$$

  Here $a = 1$, $b = 2$, $k = 2$. So $e = \log_b a = 0$ and $e < k$.
  Work at top-level dominates: solution is $\Theta(n^2)$.

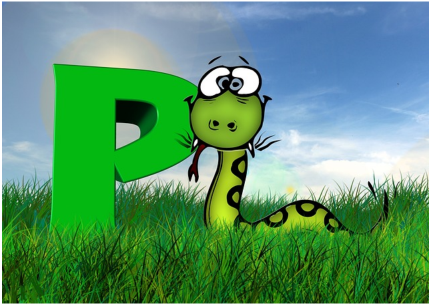
- Karatsuba algorithm for multiplying two $n$-digit numbers:

$$T(n) = 3T(n/2) + \Theta(n) \quad \text{if } n > 1$$

  Here $a = 3$, $b = 2$, $k = 1$. So $e = \log_2 3$ and $e > k$.
  Solution is $\Theta(n^{1.584\cdots})$ (cf. $\Theta(n^2)$ for school method)

# Thanks for listening!

Enjoy Aris's lectures, and see you again in Sem 2 for some language processing and computability theory.



Reading for today's lecture:

Roughgarden Chapter 4 (recommended)

CLRS Chapter 4, especially 4.5

KT Chapter 5 (many good examples, doesn't explicitly state MT)

GTG 4.2 (relevant, again doesn't explicitly state MT)