Introduction to Modern Cryptography

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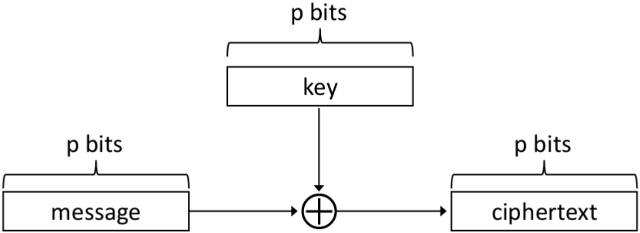
(Slides courtesy of Prof. Jonathan Katz)

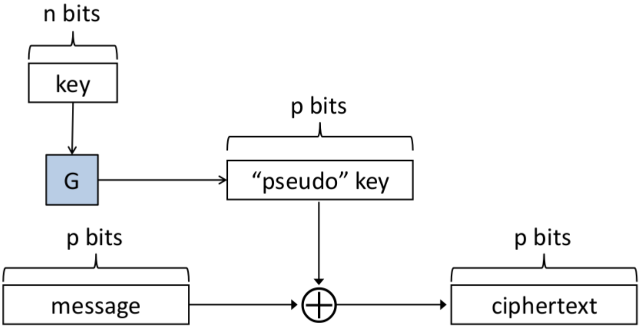
Lecture 6

Pseudo One-Time Pad

One-time Pad (recall)



Pseudo One-time Pad (POTP)



Pseudo One-time Pad

Definition

- Let G be a deterministic algorithm, with $|G(k)| = p(|k|)$
- $\mathsf{Gen}(1^n)$: output uniform n-bit key k
  - Security parameter $n \implies$ message space $\{0,1\}^{p(n)}$
- $\mathsf{Enc}_k(m)$: output $G(k) \oplus m$
- $\mathsf{Dec}_k(c)$: output $G(k) \oplus c$
- Correctness – the same as OTP

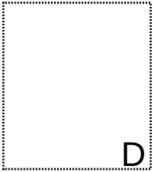
Security of POTP?

- Would like to be able to prove security
- Based on the assumption that $\boldsymbol{G}$ is a PRG

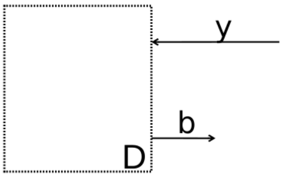
Modern Crypto = Definitions + Proofs + Assumptions

- We've **defined** computational secrecy
- Our goal is to **prove** that the pseudo OTP meets that definition
- We cannot prove this unconditionally
  - Beyond our current techniques...
  - Anyway, security clearly depends on $\boldsymbol{G}$
- Can prove security based on the **assumption** that $\boldsymbol{G}$ is a pseudorandom generator

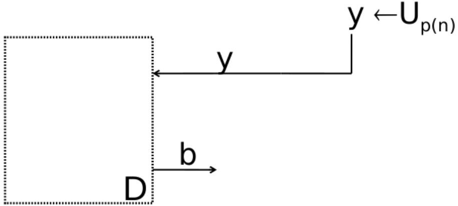
PRG Revisited



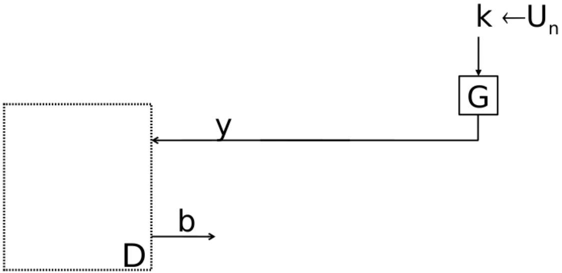
PRG Revisited



PRG Revisited

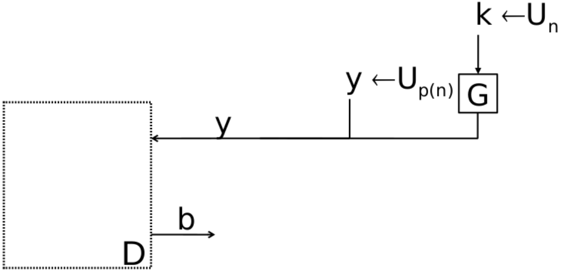


PRG Revisited



PRG Revisited

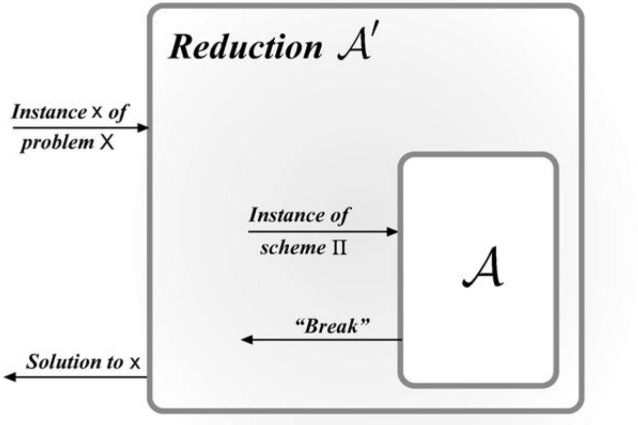
- Let G be an efficient, deterministic function with $|G(k)| = p(|k|)$
- For any efficient D, the probabilities that D outputs $\mathbf{1}$ in each case must be *close*



Proof by Reduction

- Assume $\boldsymbol{G}$ is a pseudorandom generator
- Assume toward a contradiction that there is an efficient attacker $\boldsymbol{A}$ who *breaks* POTP (as per the definition)
- Use $\boldsymbol{A}$ as a subroutine to build an efficient $\boldsymbol{D}$ that *breaks* pseudorandomness of $\boldsymbol{G}$
- By assumption, no such $\boldsymbol{D}$ exists
- $\implies$ No such $\boldsymbol{A}$ can exist

Proof by Reduction



IMC Textbook 2nd ed. CRC Press 2015

Proof by Reduction (equivalent)

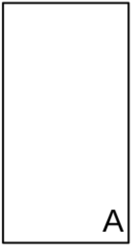
- Assume G is a pseudorandom generator
- Fix some arbitrary, efficient A attacking POTP
- Use A as a subroutine to build an efficient D attacking G
- **Relate the distinguishing gap of D to the success probability of A**
- By assumption, the distinguishing gap of D must be negligible
- $\implies$ Use this to bound the success probability of A

Security of POTP

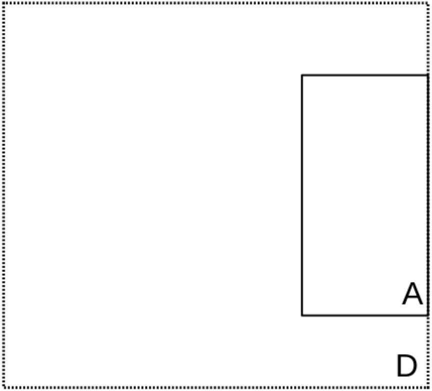
Theorem

If $\boldsymbol{G}$ is a pseudorandom generator, then the pseudo one-time pad $\mathbf{\Pi}$ is EAV-secure (i.e. computationally indistinguishable)

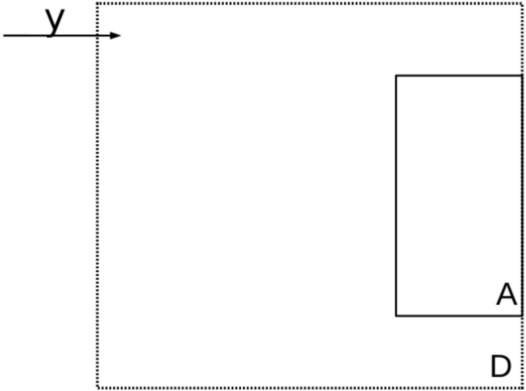
The Reduction



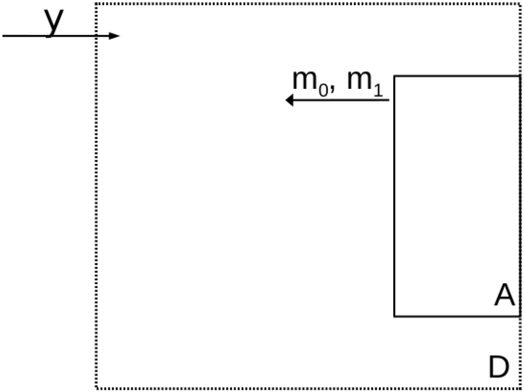
The Reduction



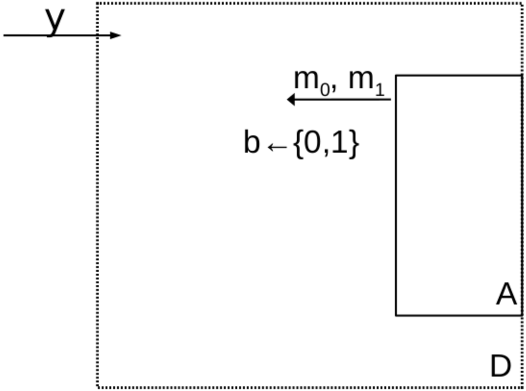
The Reduction



The Reduction



The Reduction



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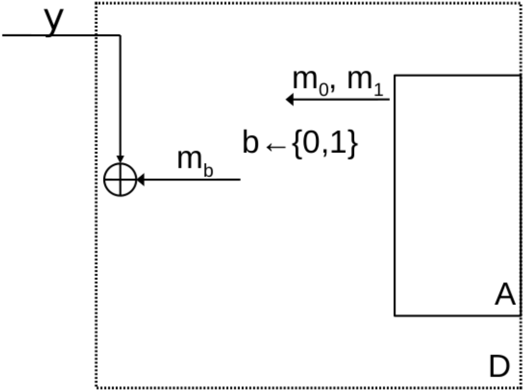

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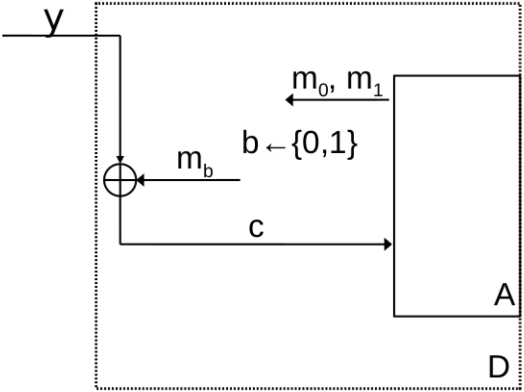
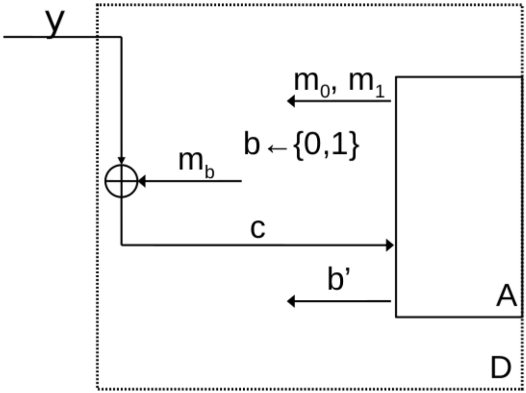


The Reduction



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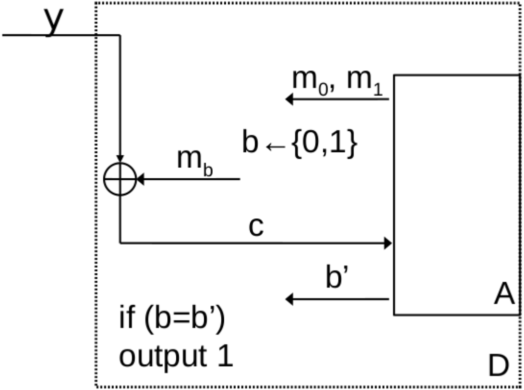
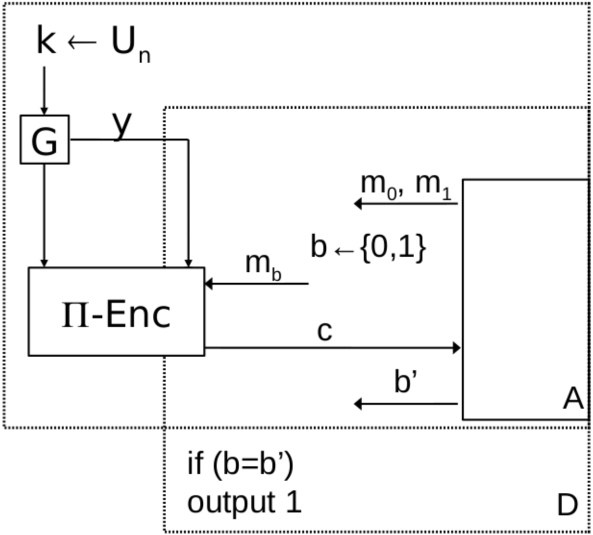
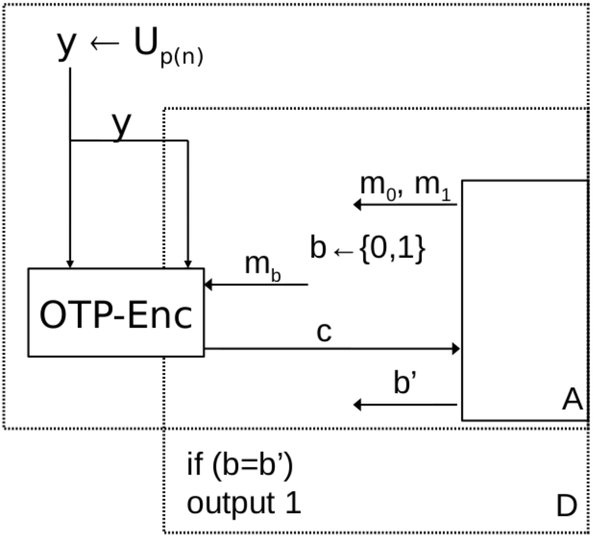


The Reduction



The Reduction



The Proof

Proof by Reduction

- Implement $\boldsymbol{D}$ by using $\boldsymbol{A}$ as a subroutine
  - If $\boldsymbol{A}$ runs in polynomial time, then so does $\boldsymbol{D}$
- Relate the success $\mathbf{Pr}$ of $\boldsymbol{D}$ and $\boldsymbol{A}$
- Prove that if $\boldsymbol{A}$ succeeds in breaking POTP then $\boldsymbol{D}$ succeeds in breaking $\boldsymbol{G}$
- i.e. **reduce the security of the POTP to the security of the underlying $\boldsymbol{G}$**

The Attacker A

A attacks POTP via $\mathsf{PrivK}_{A,\Pi}(n)$

- $A(1^n)$ outputs m_0, m_1
- $k \leftarrow \mathsf{Gen}(1^n)$, $b \leftarrow \{0, 1\}$, $c \leftarrow \mathsf{Enc}_k(m_b)$
- $b' \leftarrow A(c)$
- If $b = b'$ return 1 (success)

If POTP is computationally ind. (EAV-secure) then

$$\mathbf{Pr}[\mathsf{PrivK}_{A,\Pi}(n) = 1] \leq \frac{1}{2} + \epsilon(n)$$

$\implies$ sufficient to prove the above inequality in order to prove the security of the POTP

The Attacker A

A attacks OTP via $\mathsf{PrivK}_{A,\mathsf{OTP}}$

1. A outputs m_0, m_1
2. $k \leftarrow \mathsf{Gen}$, $b \leftarrow \{0, 1\}$, $c \leftarrow \mathsf{Enc}_k(m_b)$
3. $b' \leftarrow A(c)$
4. If $b = b'$ return 1 (success)

Since OTP is perfectly secret:

$$\mathbf{Pr}[\mathsf{PrivK}_{A,\mathsf{OTP}} = 1] = \frac{1}{2}$$

The Distinguisher $\boldsymbol{D}$

$\boldsymbol{D}$ attacks $\boldsymbol{G}$

Since $\boldsymbol{G}$ is a PRG (by assumption) $\implies \exists \boldsymbol{\epsilon(n) = \mathbf{negl}}$ s.t.

$$\boldsymbol{|\mathbf{Pr}_{x \leftarrow U_n}[D(G(x)) = 1] - \mathbf{Pr}_{y \leftarrow U_{p(n)}}[D(y) = 1]| \leq \epsilon(n)}$$

World $\mathbf{0}$: D with a Truly Random Input

$D(y)$ for uniform y

D simulates the $\mathsf{PrivK}_{A,\mathsf{OTP}}$ experiment for A for a truly random input y:

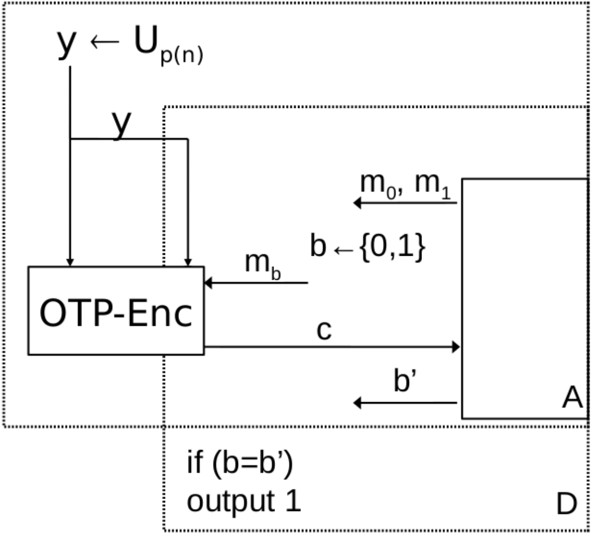
- $A(1^n)$ outputs m_0, m_1
- Simulation:
  1. D generates $b \leftarrow \{0, 1\}$
  2. D computes $c = m_b \oplus y$
  3. D sends c to A
- $b' \leftarrow A(c)$
- If $b = b'$ then $D(y) = 1$

World $\mathbf{0}$: $\boldsymbol{D}$ with a Truly Random Input

Since $\boldsymbol{y}$ is truly random, from the viewpoint of $\boldsymbol{A}$ it is as if $\boldsymbol{A}$ is interacting with the OTP in World $\mathbf{0}$. Therefore:

$$\mathbf{Pr}_{\boldsymbol{y} \leftarrow \boldsymbol{U}_{\boldsymbol{p(n)}}}[\boldsymbol{D(y)} = \mathbf{1}] = \mathbf{Pr}[\mathsf{PrivK}_{\boldsymbol{A},\mathsf{\mathbf{OTP}}} = \mathbf{1}] = \frac{\mathbf{1}}{\mathbf{2}}$$

World **0**: ***A*** interacting with OTP



World $\mathbf{1}$: D with a Pseudorandom Input

$D(G(x))$ for pseudorandom $G(x)$

D simulates the $\mathsf{PrivK}_{A,\Pi}(n)$ experiment for A for a pseudorandom input $G(x)$:

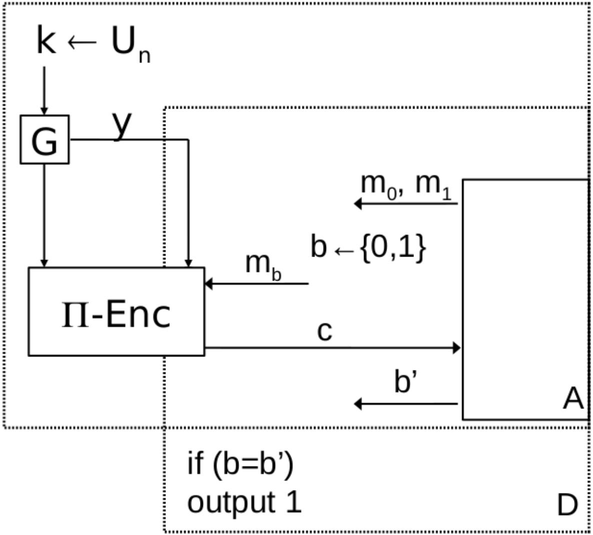
- $A(1^n)$ outputs m_0, m_1
- Simulation:
  1. D generates $b \leftarrow \{0,1\}$
  2. D computes $c = m_b \oplus G(x)$
  3. D sends c to A
- $b' \leftarrow A(c)$
- If $b = b'$ then $D(G(x)) = 1$

World $\mathbf{1}$: $\boldsymbol{D}$ with a Pseudorandom Input

Since $\boldsymbol{G(x)}$ is pseudorandom, from the viewpoint of $\boldsymbol{A}$ it is as if $\boldsymbol{A}$ is interacting with the POTP in World $\mathbf{1}$. Therefore:

$$\mathbf{Pr}_{x \leftarrow U_n}[D(G(x)) = 1] = \mathbf{Pr}[\mathsf{PrivK}_{A,\Pi}(n) = 1]$$

World **1**: ***A*** interacting with POTP



Proof.

1) By the assumption that G is a PRG $\exists \epsilon(n) = \mathbf{negl}$:

$$\left|\mathbf{Pr}_{x \leftarrow U_n}[D(G(x)) = 1] - \mathbf{Pr}_{y \leftarrow U_{p(n)}}[D(y) = 1]\right| \leq \epsilon(n)$$

2) By the simulation of $\mathsf{PrivK}_{A,\Pi}$ by $D(y)$:

$$\mathbf{Pr}_{y \leftarrow U_{p(n)}}[D(y) = 1] = \mathbf{Pr}[\mathsf{PrivK}_{A,\mathsf{OTP}} = 1] = \frac{1}{2}$$

3) By the simulation of $\mathsf{PrivK}_{A,\Pi}(n)$ by $D(G(x))$:

$$\mathbf{Pr}_{x \leftarrow U_n}[D(G(x)) = 1] = \mathbf{Pr}[\mathsf{PrivK}_{A,\Pi}(n) = 1]$$

Therefore

$$\mathbf{Pr}[\mathsf{PrivK}_{A,\Pi}(n) = 1] \leq \frac{1}{2} + \epsilon(n)$$

$\Longrightarrow$ $\mathbf{\Pi}$ (i.e. POTP) is EAV-secure. $\square$

Summary

- Proof that the pseudo OTP is secure...
- We have a provably secure scheme, rather than just a heuristic construction!

Summary

- Proof that the pseudo OTP is secure...
- ...with some caveats
  - Assuming $\boldsymbol{G}$ is a pseudorandom generator
  - Relative to our definition
- The only ways the scheme can be broken are:
  - If a weakness is found in $\boldsymbol{G}$
  - **If the definition isn't sufficiently strong** (next lecture!)

Have we gained anything?

- Yes! The **POTP has a key shorter than the message**
  - $\boldsymbol{n}$ bits vs. $\boldsymbol{p(n)}$ bits
- $\implies$ **Solved one of the limitations of the OTP**

- The fact that the parties internally generate a $\boldsymbol{p(n)}$-bit temporary string to encrypt/decrypt is irrelevant
- The key is what the parties share in advance
- Parties do not store the $\boldsymbol{p(n)}$-bit temporary value
- What about the other limitation? (next lectures)

End

Reference: Section 3.3.2