Introduction to Modern Cryptography

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(Slides courtesy of Prof. Jonathan Katz)

Lecture 5, part 2

Pseudorandomness

Terminology: random vs. uniform

Random

Sample a **random element** according to **some distribution**

Uniform

Sample an element **uniformly at random** means to sample according to the **uniform distribution**

Informally

random ≈ uniform

Pseudorandom (informally)

pseudorandom ≈ *"looks like random"*

Pseudorandomness

- Important building block for **computationally secure** encryption
- Important concept in cryptography

What does *random* mean?

Uniform

- What does **uniform** mean?
- Which of the following is a uniform string?
  - `0101010101010101`
  - `0010111011100110`
  - `0000000000000000`

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If we generate a uniform **16**-bit string, each of the above occurs with probability $\mathbf{2^{-16}}$

What does *uniform* mean?

Uniformity

- **Uniformity** is not a property of a string, but a property of a distribution
- A distribution on n-bit strings is a function $D : \{0,1\}^n \to [0,1]$ such that $\sum_x D(x) = 1$
- The **uniform distribution** on n-bit strings, denoted U_n, assigns probability 2^{-n} to every $x \in \{0,1\}^n$

What does *pseudorandom* mean?

Pseudorandom

- Cannot be distinguished from **uniform** (i.e. random)
- Which of the following is **pseudorandom**?
  - `0101010101010101`
  - `0010111011100110`
  - `0000000000000000`
- Pseudorandomness is a property of a distribution, not a string

Pseudorandomness (heuristic)

- Fix some distribution D on n-bit strings
  - $x \leftarrow D$ means *sample x according to D*
- Historically, D was considered pseudorandom if it *passed a bunch of statistical tests*
  - $\mathbf{Pr}_{x \leftarrow D}[\text{1st bit of x is 1}] \approx 1/2$
  - $\mathbf{Pr}_{x \leftarrow D}[\text{parity of of x is 1}] \approx 1/2$
  - $\mathbf{Pr}_{x \leftarrow D}[\mathbf{Test}_i(x) = 1] \approx \mathbf{Pr}_{x \leftarrow U_n}[\mathbf{Test}_i(x) = 1]$ for all $i = 1, 2, \ldots$

- This is not sufficient in an adversarial setting!
- Who knows what statistical test an attacker will use?

Pseudorandomness

Cryptographic definition

$\boldsymbol{D}$ is pseudorandom if it passes **all** efficient statistical tests

Pseudorandomness (concrete)

Definition

Let D be a distribution on p-bit strings. D is **(t, ϵ)-pseudorandom** if for all A running in time at most t it holds that:

$$|\mathbf{Pr}_{x \leftarrow D}[A(x) = 1] - \mathbf{Pr}_{x \leftarrow U_p}[A(x) = 1]| \leq \epsilon$$

Pseudorandomness (asymptotic)

- Security parameter $\boldsymbol{n}$, polynomial $\boldsymbol{p}$
- Let $\boldsymbol{D_n}$ be a distribution over $\boldsymbol{p(n)}$-bit strings
- **Pseudorandomness is a property of a sequence of distributions:**

$$\{D_n\} = \{D_1, D_2, \ldots\}$$

Pseudorandomness (asymptotic)

Definition

$\{D_n\}$ is **pseudorandom** if for all probabilistic, polynomial-time distinguishers A, there is a negligible function ϵ such that

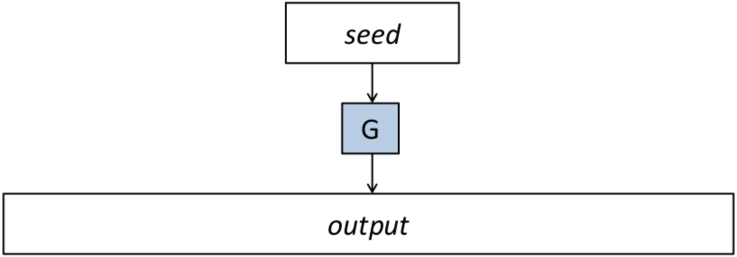
$$\left|\Pr_{x \leftarrow D_n}[A(x) = 1] - \Pr_{x \leftarrow U_{p(n)}}[A(x) = 1]\right| \leq \epsilon(n)$$

Pseudorandom Generators (PRG)

- A PRG is an efficient, deterministic algorithm that expands a **short, uniform seed** into a **longer, pseudorandom output**
- Useful whenever you have a *small* number of true random bits, and want lots of *random-looking* bits

PRGs

$\boldsymbol{G}$ is a deterministic, poly-time algorithm that is **expanding** i.e. $|\boldsymbol{G}(\boldsymbol{x})| = \boldsymbol{p}(|\boldsymbol{x}|) > |\boldsymbol{x}|$



PRGs

- G defines a sequence of distributions $\{D_n\}$
- D_n: the distribution on $p(n)$-bit strings defined by choosing $x \leftarrow U_n$ and outputting $G(x)$:

$$\begin{aligned}
\mathbf{Pr}_{D_n}[y] &= \mathbf{Pr}_{U_n}[G(x) = y] \\
&= \sum_{x:G(x)=y} \mathbf{Pr}_{U_n}[x] \\
&= \sum_{x:G(x)=y} 2^{-n} \\
&= \frac{|\{x : G(x) = y\}|}{2^n}
\end{aligned}$$

- Note that most y occur with probability **0**
  - i.e. D_n **is far from uniform**

PRGs

- G is a PRG $\iff$ $\{D_n\}$ is pseudorandom
- i.e. for all efficient distinguishers A, there is a negligible function ϵ such that

$$\left|\Pr_{x \leftarrow U_n}[A(G(x)) = 1] - \Pr_{y \leftarrow U_{p(n)}}[A(y) = 1]\right| \leq \epsilon(n)$$

- i.e. no efficient A can distinguish whether it is given $G(x)$ (for uniform x) or a uniform string y

Is the Following PRG Secure?

PRG
$\boldsymbol{G(x) = 0 \ldots 0}$

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Distinguisher

$\boldsymbol{A} =$ [all bits equal to $\mathbf{0}$]

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Analysis

$$\Pr_{x \leftarrow U_n}[A(G(x)) = 1] = 1$$

$$\Pr_{y \leftarrow U_{p(n)}}[A(y) = 1] = \frac{1}{2^{p(n)}}$$

$$1 - \frac{1}{2^{p(n)}} \approx 1 \not\leq \text{negl}$$

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$\boldsymbol{G(x) = x \mid \texttt{OR}(\textbf{bits of } x)}$

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PRG

$G(x) = x \mid \texttt{OR}(\text{bits of } x)$

Distinguisher

$A = [\text{ least-significant bit} \neq 0]$

Analysis

$$\Pr_{x \leftarrow U_n}[A(G(x)) = 1] = 1 - \frac{1}{2^n} \approx 1$$

$$\Pr_{y \leftarrow U_{p(n)}}[A(y) = 1] = \frac{1}{2}$$

$$1 - \frac{1}{2} \approx \frac{1}{2} \not\lesssim \text{negl}$$

Do PRGs Exist?

- We don't know...
- Most of cryptography requires the unproven assumption that $\mathcal{P} \neq \mathcal{NP}$
- We will assume certain algorithms are PRGs
  - This is what is done in practice
- Can construct PRGs from weaker assumptions

So far

- We saw that there are some inherent limitations if we want **perfect secrecy**
- In particular, **key must be as long as the message**
- We defined **computational secrecy**, a relaxed notion of security
- We defined **PRG**
- Can we use **computational secrecy** + **PRG** to overcome prior limitations?

End

References: from Page 60 until the last paragraph of Page 64