Outline

1. What is Deforestation?
2. The Original Deforestation Proposal
3. Supercompilation
4. Shortcut Deforestation
5. Staged Fusion
6. The Long Way to Deforestation
1. What is Deforestation?
Basic Idea of Deforestation

*Functional programming* languages tend to *allocate* lots of short-lived objects. This is due to a focus on high-level programming and the use of composable abstractions.
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**Example:** list combinators, such as `map`, `filter`, `concatMap`, etc.

Even when implemented *lazily* (as in Haskell), these require allocating intermediate values often *used only once* and then immediately discarded.
Basic Idea of Deforestation

Functional programming languages tend to allocate lots of short-lived objects. This is due to a focus on high-level programming and the use of composable abstractions.

**Example:** list combinators, such as `map`, `filter`, `concatMap`, etc.

Even when implemented lazily (as in Haskell), these require allocating intermediate values often used only once and then immediately discarded.

**Deforestation:** the act of removing the unnecessary creation of trees from functional programs without changing their semantics.
1.1. Manual Deforestation
Manual Deforestation Example

Consider the following Haskell program:

```haskell
map f xs = case xs of { [] → []; x : xs → f x : map f xs }
incr x = x + 1
double x = x * 2

main ls = map incr (map double ls)
```

Problem?
Manual Deforestation Example

Consider the following Haskell program:

\[
\begin{align*}
\text{map } f \ x s &= \text{case } x s \ of \ { [ ] } \rightarrow [ ] ; \\
&\phantom{= } x : x s \rightarrow f \ x : \text{map } f \ x s \\
\text{incr } x &= x + 1 \\
\text{double } x &= x * 2
\end{align*}
\]

\[\text{main } l s = \text{map incr } (\text{map double } l s)\]

Problem?

The intermediate list \text{map double } l s is immediately consumed by \text{map incr}!
Manual Deforestation Example

map f xs = case xs of { [] → []; x : xs → f x : map f xs }

main ls = map incr (map double ls)

The intermediate list \texttt{map double ls} is immediately consumed by \texttt{map incr}!
Manual Deforestation Example

```haskell
map f xs = case xs of { [] → []; x : xs → f x : map f xs }
main ls = map incr (map double ls)
```

The intermediate list `map double ls` is immediately consumed by `map incr`!

The following code is typically 40% more efficient:

```haskell
map2 f g xs = case xs of { [] → []; x : xs → f (g x) : map2 f g xs }
main ls = map2 incr double ls
```
Manual Deforestation Example

```haskell
map f xs = case xs of { [] → []; x : xs → f x : map f xs }
main ls = map incr (map double ls)
```

The intermediate list `map double ls` is immediately consumed by `map incr`!

The following code is typically 40% more efficient:

```haskell
map2 f g xs = case xs of { [] → []; x : xs → f (g x) : map2 f g xs }
main ls = map2 incr double ls
```

No more intermediate list created!
Manual Deforestation

It is possible to rewrite your programs to avoid the creation of intermediate data structures manually.
Manual Deforestation

It is possible to rewrite your programs to avoid the creation of intermediate data structures manually.

This requires heavy refactoring, duplication, and breaks modular abstractions, as it requires exposing and rewriting implementations.
Manual Deforestation

It is possible to rewrite your programs to avoid the creation of intermediate data structures manually. This requires heavy refactoring, duplication, and breaks modular abstractions, as it requires exposing and rewriting implementations.

⟹ BAD!
1.2. The Pie in The Sky: Automatic Deforestation
Automatic Deforestation

This lecture is a *high-level introduction* on various *approaches to deforestation* that have been proposed over the years.
2. The Original Deforestation Proposal
The Original Deforestation Idea

Not the first, but the one that coined the name, and one of the simplest
Proposed by Philip Wadler in 1990, then at University of Glasgow (Wadler 1990).
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Proposed by Philip Wadler in 1990, then at University of Glasgow (Wadler 1990).

High-level ideas:
• restrict the input language to simplify the problem
• unroll recursive definitions and tie the knot to avoid infinite loops
The Original Deforestation Idea

The Language: $t$ stands for term; $p$ stands for pattern

$$
t ::= v \quad \text{variable} \\
| \ c \ t_1 \ldots \ t_k \quad \text{constructor application} \\
| \ f \ t_1 \ldots \ t_k \quad \text{function application} \\
| \ \text{case} \ t_0 \ \text{of} \ p_1 : t_1 \ | \ \cdots \ | \ p_n : t_n \quad \text{case term} \\
$$

$$
p ::= c \ v_1 \ldots \ v_k \quad \text{pattern} \\
$$

- $c$ the name of the constructor, can be arbitrary
- patterns $p$ are not nested for simplicity
- $\overline{a_i}$ denotes $a_1 \ldots a_n$
- $t\{v \rightarrow t'\}$ denotes replacing all occurrences of variable $v$ inside $t$ with $t'$
The Original Deforestation Idea

**Key Idea:** simulating the evaluation of the program to bring together the production of data structures to their corresponding consumption sites (case terms), then the elimination of intermediate data structures is trivial.

```kotlin
case (Cons 1 Nil) of Nil: branch1 | Cons h t: branch2
```

can be easily transformed into branch2 with h replaced by 1 and t replaced by Nil
The Original Deforestation Idea

The core transformation algorithm $T$ simulates the evaluation of programs

1. $T[v] = v$

2. $T[c \ t_i] = c \ T[t_i]$

3. $T[f \ t_i] = T[t \ (v_i \rightarrow t_i)]$, where $f$ is defined as $f \overline{v_i} = t$

4. $T[\text{case } v \text{ of } p_i : t_i] = \text{case } v \text{ of } p_i : T[t_i]$

5. $T[\text{case } c_n \ t_j \text{ of } p_i : t_i] = T[t_n \ (v_j \rightarrow t_j)]$ if $p_n \equiv c_n \overline{v_j}$

6. $T[\text{case } f \ t_j \text{ of } p_i : t_i] = T[\text{case } (t \ (v_j \rightarrow t_j)) \text{ of } p_i : t_i]$, where $f$ is defined as $f \overline{v_j} = t$
The Original Deforestation Idea

7. \[ T\left[ \text{case } (\text{case } t_0 \text{ of } p_i : t_i) \text{ of } p'_j : t'_j \right] = \]

\[ T\left[ \text{case } t_0 \text{ of } p_i : \text{case } t_i \text{ of } p'_j : t'_j \right] \]
The Original Deforestation Idea

7. $T[\text{case } (\text{case } t_0 \text{ of } p_i : t_i) \text{ of } p'_j : t'_j] =$

$T[\text{case } t_0 \text{ of } p_i : \text{case } t_i \text{ of } p'_j : t'_j]$

Example:

`case (case v of None : Just 1 | Just a : None) of None : 0 | Just a : a`

is transformed in one step to:

`case v of None : case Just 1 of None : 0 | Just a : a`

Just a : case None of None : 0 | Just a : a
The Original Deforestation Idea

7. \( T[\text{case } (\text{case } t_0 \text{ of } p_i : t_i) \text{ of } p'_j : t'_j] = \)

\( T[\text{case } t_0 \text{ of } p_i : \text{case } t_i \text{ of } p'_j : t'_j] \)

Example:

case (case v of None : Just 1 | Just a : None) of None : 0 | Just a : a

is transformed in one step to:

case v of None : case Just 1 of None : 0 | Just a : a

Just a : case None of None : 0 | Just a : a

into case v of None : 1 | Just a : 0
The Original Deforestation Idea

The transformation algorithm is designed to proceed as much as possible, in spite of missing the actual run-time information, to bring together data constructor applications and `case` terms.

- 3 and 6: unfold function definitions
- 5: eliminate intermediate data structure
- 7: case-of-case commuting
Example

An more meaningful example:

\[
\text{flip (flip } t) \\
\text{where flip tr = case tr of} \\
\quad \text{Leaf } z: \text{ Leaf } z \\
\quad \text{Branch } l \ r: \text{ Branch } (\text{flip } r) (\text{flip } l)
\]
Example

- \( \text{flip (flip } t \text{)} \)
- \( \text{case (flip } t \text{) of ...} \) by 3
- \( \text{case (case } t \text{ of ... ) ...} \) by 6

\[
\begin{align*}
\text{case } t \text{ of} \\
\text{Branch } l \ r : \ & \text{case (Branch (flip } r \text{)(flip } l \text{)) of Branch } l \ r : \ ...
\end{align*}
\]

by 7

- \( \text{case } t \text{ of}

  \begin{align*}
  \text{Branch } l \ r : \ & \text{Branch (flip (flip } l \text{))(flip (flip } r \text{)) } \\
  \text{Leaf } z : \ & \text{Leaf } z
  \end{align*}

by 4, 5, 5

We encounter \( \text{flip (flip } r \text{) and flip (flip } l \text{)} \) again!

\( T \) Loops forever on unfolding \( \text{flip} \)?
Tying the Recursive Knot

Keep track of function call terms we have already processed, and later when similar terms are encountered again, stop unfolding and tie the knot by introducing new recursive function definitions.

- **Similar** terms: up to renaming of variables

  Examples:
  
  \[
  f \; x \quad g \; x \quad f \; (g \; x) \quad f \; x \\
  f \; x \quad g \; y \quad f \; (g \; y) \quad f \; (g \; x)
  \]
Tying the Recursive Knot

\[ \text{flip (flip r) and flip (flip l) are both renamings of the initial term flip (flip t), so the unfolding stops by introducing a new definition } h, \text{ whose body is the current term we get from running } T \text{ with flip (flip r) and flip (flip l) replaced by } h \ r \text{ and } h \ l: } \]

\[ h \ t = \text{case } t \text{ of} \]
\[ \text{Branch } l \ r: \text{ Branch } (h \ l) \ (h \ r) \]
\[ \text{Leaf } z: \text{ Leaf } z \]
Our goal is to eliminate the allocation of intermediate data structures. There is a syntactic property of programs that approximately describes our goal: Treeless Form.

- Every argument of a function call or selector of a case term must be a variable — no possible intermediate data structure allocation
- Every variable must be used only once — no possible duplication of work after unfolding
Treeless Form

\[
\text{app} \; \text{xs} \; \text{ys} = \\
\text{case} \; \text{xs of} \\
\quad \text{Nil: ys} \\
\quad \text{Cons} \; h \; t: \\
\quad \quad \text{Cons} \; h \; (\text{app} \; t \; \text{ys})
\]

treeless: \text{ys} appears in two different branches

\[
\text{double} \; \text{xs} = \\
\quad \text{app} \; \text{xs} \; \text{xs}
\]

not treeless: \text{xs} used twice

\[
\text{appapp} \; \text{xs} \; \text{ys} \; \text{zs} = \\
\quad \text{app} \; \text{xs} \; (\text{app} \; \text{ys} \; \text{zs})
\]

not treeless: \text{app} \; \text{ys} \; \text{zs} passed as an argument; but can be transformed to a treeless definition
Termination

Treeless form ensures that the algorithm $T$ can always terminate with a program no less efficient than the original one.
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Treeless form ensures that the algorithm $T$ can always terminate with a program no less efficient than the original one.

**Deforestation Theorem.** Every composition of functions with treeless definitions can be effectively transformed to a single function with a treeless definition, without loss of efficiency.

Pretty strong result!
Limitations

Though the original deforestation algorithm is simple and elegant, its applicability is limited, explaining why it has not been used by practical compilers.
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- The treeless form is very restrictive
  - first-order language (!)
  - linear uses of variables
  - no internal data structures

  (later approaches lifted some restrictions but the reasoning was still limited)
Limitations

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- The treeless form is very restrictive
  - first-order language (!)
  - linear uses of variables
  - no internal data structures

  (later approaches lifted some restrictions but the reasoning was still limited)

- Tying the knot on the fly is expensive
3. Supercompilation
Supercompilation

A powerful program transformation technique originally due to Turchin (1986), which shares many similarities with Deforestation that it statically simulates the evaluation of a program to expose its internal logic and find optimization opportunities.

Uses:

• Prove theorems about programs
• Specialize function definitions
• Deforestation
Supercompilation

What does a supercompiler do?

- **Driving**: Simulate the evaluation of programs, but with free variables
- **Folding**: Introduce new recursive function definitions
Supercompilation

- **Driving** propagates more information about free variables instead of simply ignoring them like the original deforestation algorithm
Supercompilation

- **Driving** propagates more information about free variables instead of simply ignoring them like the original deforestation algorithm
- **Folding** together with **Generalization** ensure the termination of Supercompilation on *general programs* (not limited to treeless form)
3.1. Positive Supercompilation
Positive Supercompilation: Driving Rules

Positive supercompilation is a simplified form of full supercompilation: it only propagates positive information (to be explained in the next slide).

The driving rules are similar to the algorithm $T[[t]]$ presented in the original deforestation, with one core difference:

$$
\mathcal{D}[[\text{case } v \ \text{of } p_i \rightarrow t_i]] = \text{case } v \ \text{of } p_i : \mathcal{D}[[t_i \{v \rightarrow p_i\}]]
$$
Positive Supercompilation: Driving Rules

\[ \mathcal{D}[\text{case } v \text{ of } p_i \rightarrow t_i] = \text{case } v \text{ of } p_i : \mathcal{D}[t_i \{v \rightarrow p_i\}] \]

- The original deforestation simply does \text{case } v \text{ of } p_i : \mathcal{D}[t_1], but a positive supercompiler will propagate the information of the exact shape of \( v \) in each branch.

- “Positive” means the supercompiler will only propagate equality information (i.e. \( v \equiv p_i \)), but not inequality information.

- A more powerful supercompiler may propagate both kinds of information at a higher cost, and using different approaches other than substitution.
Positive Supercompilation: Driving Rules

```haskell
firstPlusLast ls
where
  firstPlusLast xs = case xs of
    Nil: Nothing
    Cons h t: Some (h + fromJust (last xs))
last xs = case xs of Nil: Nothing | Cons h t: Some (last' h t)
last' a xs = case xs of Nil: a | Cons h t: last' h t
fromJust m = case m of Just a: a
```

Having the positive information propagated helps to reduce the allocation of Some (last' h t) and the call to fromJust, by changing last xs to last (Cons h t). More examples in paper (Sørensen, Glück, Jones 1996).
Positive Supercompilation: Folding Strategies

For non-treeless programs, the driving processes of the following programs never terminates without a more powerful folding strategy,

\[
\text{nrev } \text{xs} \\
\text{where} \\
\text{nrev } [[]] = [[]] \\
\text{nrev } (h:t) = \text{app} (\text{nrev } t) (h:[]) \\
\text{app } [[]] \text{ ys} = \text{ys} \\
\text{app} (x:xs) \text{ ys} = x:(\text{app } xs \text{ ys})
\]

\[
\text{arev } \text{xs } [[]] \\
\text{where} \\
\text{arev } [[]] \text{ a} = [[]] \\
\text{arev} (x:xs) \text{ a} = \text{arev } xs (x:a)
\]

\[
\text{nrev } \text{xs} \\
\text{case} (\text{nrev } \text{xs}) \text{ of } ... \\
\text{case} (\text{case} (\text{nrev } \text{xs}) \text{ of } ... ) \text{ of } ...
\]

\[
\text{arev } \text{xs } [[]] \\
\text{arev } \text{xs } (x:[]) \\
\text{arev } \text{xs } (x:x':[]) \\
... 
\]
Positive Supercompilation: Folding Strategies

The following two techniques are used so that recursive knots can be tied during the folding process to ensure termination.

- **Homeomorphic embedding**: detect similar terms
- **Generalization**: handle similar terms and ensure termination
Positive Supercompilation: Folding Strategies

Homeomorphic embedding

- \( t_1 \triangleleft t_2 \) if \( t_1 \triangleleft_d t_2 \) (diving) or \( t_1 \triangleleft_c t_2 \) (coupling)
- **Diving**: \( t_1 \triangleleft_d t_2 \) if there is a subterm \( t_{2_i} \) of \( t_2 \) such that \( t_1 \triangleleft t_{2_i} \)
- **Coupling**: \( t_1 \triangleleft_c t_2 \) if \( t_1 \) and \( t_2 \) share the same top-level term constructor and all their corresponding subterms \( t_{1_i} \) and \( t_{2_i} \) satisfy \( t_{1_i} \triangleleft t_{2_i} \)
- the homeomorphic embedding relation \( t_1 \leq t_2 \), if there is a renaming \( t_r \) of \( t_1 \) such that \( t_r \triangleleft_c t_2 \)
Positive Supercompilation: Folding Strategies

Homeomorphic embedding

- Some examples

\[ \lambda x.x \leq \lambda y.y \quad \quad f (g x) \leq f (g y) \quad \quad f (h x) \leq f (g (h y)) \]

\[ \lambda x.x \not\leq \lambda y.x \quad \quad f z z \not\leq f x y \quad \quad f (g x) \not\leq g (f y) \]
Positive Supercompilation: Folding Strategies

**Generalization** of two similar terms $t_1, t_2$:

A triple $(t, \theta_1, \theta_2)$, where $t$ is a term, and $\theta_1, \theta_2$ are substitutions from variables to terms, such that $t\theta_1 = t_1$ and $t\theta_2 = t_2$
Positive Supercompilation: Folding Strategies

**Generalization** of two similar terms $t_1, t_2$:

A triple $(t, \theta_1, \theta_2)$, where $t$ is a term, and $\theta_1, \theta_2$ are substitutions from variables to terms, such that $t\theta_1 = t_1$ and $t\theta_2 = t_2$

**Some examples**

- for $f \ x \ y$ and $f \ z \ z$ we have $(f \ v_1 \ v_2, \{v_1 \rightarrow x, v_2 \rightarrow y\}, \{v_1 \rightarrow z, v_2 \rightarrow z\})$
- for $f \ (g \ x)$ and $f \ (g \ y)$ we have $(f \ (g \ v), \{v \rightarrow x\}, \{v \rightarrow y\})$
- for $f \ x \ x$ and $f \ (g \ y) \ (h \ y)$ we have
  $$(f \ v_1 \ v_2, \{v_1 \rightarrow x, v_2 \rightarrow x\}, \{v_1 \rightarrow g \ y, v_2 \rightarrow h \ y\})$$
Positive Supercompilation: Folding Strategies

These folding and generalization strategies ensure termination: the homeomorphic embedding generalizes the idea of similarity between terms up to renaming, such that all non-terminating possibilities can be detected during the driving process.
Positive Supercompilation: Folding Strategies

These **folding and generalization strategies** ensure termination: the homeomorphic embedding generalizes the idea of similarity between terms up to renaming, such that all non-terminating possibilities can be detected during the driving process.

**Downside**: very *complicated* to implement and *expensive* to execute; to the best of our knowledge, no practical compiler actually does this
3.2. Distillation
Distillation

Process Trees: the trace of the driving process
Distillation

The homeomorphic embedding and generalization processes are then extended to *process trees* (intuitively, “unrollings” themselves), giving a more powerful and expensive transformation algorithm.
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**Downside:** *even more* complicated and expensive!

We are not aware of any implementation that’s not patently broken even on basic examples.
4. Shortcut Deforestation
Shortcut Deforestation

“Shortcut Deforestation”, also known as “Shortcut Fusion” or just “Fusion” was introduced by Gill, Launchbury, Peyton Jones (1993) as a practical way of achieving deforestation without the complexities of supercompilation.
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Key ideas:
• leave recursive definitions alone
• only focus on rewriting the use of combinators
Shortcut Deforestation

“Shortcut Deforestation”, also known as “Shortcut Fusion” or just “Fusion” was introduced by Gill, Launchbury, Peyton Jones (1993) as a practical way of achieving deforestation without the complexities of supercompilation.

Key ideas:
- leave recursive definitions alone
- only focus on rewriting the use of combinators

Example: rewrite \( \text{map } f (\text{map } g \ x s) \) to \( \text{map } (f \ . \ g) \ x s \)

Problem: huge set of rules to account for all possible pairs of functions...?
4.1. List Fusion
Essence of Functional Lists

Functional lists can be boiled down to two fundamental operations:

- **building** a new list based on some `cons` and `nil` constructors
- **folding** a list by replacing all these `cons` and `nil` operations by function calls
Essence of Functional Lists

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Example of **building** a list:

```haskell
-- syntax sugar for List.cons(1, List.cons(2, List.cons(3, List.nil)))
1 : 2 : 3 : []
```
Essence of Functional Lists

Functional lists can be boiled down to two fundamental operations:

- **building** a new list based on some `cons` and `nil` constructors
- **folding** a list by replacing all these `cons` and `nil` operations by function calls

Example of building a list:

```plaintext
-- syntax sugar for List.cons(1, List.cons(2, List.cons(3, List.nil)))
1 : 2 : 3 : []
= build (\ c n -> c 1 (c 2 (c 3 n)))
```

Definition of `build`:

```plaintext
build g = g (:)([])
```
**Essence of Functional Lists**

**Definition of `fold`:**

\[
\text{foldr } k \ z \; [] = z \\
\text{foldr } k \ z \; (x : \xs) = k \; x \; (\text{foldr } k \ z \; \xs)
\]

**Rephrasing classical list functions in terms of `foldr`:**

\[
\begin{align*}
\text{sum } \xs &= \text{foldr } (+) \; 0 \; \xs \\
\text{elem } x \; \xs &= \text{foldr } (\lambda a \; b \rightarrow a \; \mathbin{||} \; b) \; \text{False} \; \xs \\
\text{map } f \; \xs &= \text{foldr } (\lambda a \; b \rightarrow f \; a \; : \; b) \; [] \; \xs \\
\text{filter } f \; \xs &= \text{foldr } (\lambda a \; b \rightarrow \text{if } f \; a \text{ then } a \; : \; b \text{ else } b) \; [] \; \xs \\
\text{xs }@\text{ }++\text{ }\ys &= \text{foldr } (::) \; \ys \; \xs \\
\text{concat } \xs &= \text{foldr } (++\) \; [] \; \xs \\
\text{foldl } f \; z \; \xs &= \text{foldr } (\lambda b \; g \; a \rightarrow g \; (f \; a \; b)) \; \text{id} \; \xs \; z
\end{align*}
\]
Digression: Typing build and foldr

What type should foldr have?
Digression: Typing `build` and `foldr`

What type should `foldr` have?

\[ \text{foldr} :: (b \to a \to a) \to a \to [b] \to a \]\n
How about `build`?

\[ \text{build } g = g (:) [] \]
Digression: Typing build and foldr

What type should foldr have?

\[
\text{foldr} :: (b \rightarrow a \rightarrow a) \rightarrow a \rightarrow [b] \rightarrow a
\]

How about build?

\[
\text{build } g = g (: ) []
\]

Tentative:

\[
\text{build} :: ((a \rightarrow [a] \rightarrow [a])) \rightarrow [a] \rightarrow [a]) \rightarrow [a]
\]
Digression: Typing `build` and `foldr`  

What type should `foldr` have?  

```
foldr :: (b -> a -> a) -> a -> [b] -> a
```

How about `build`?  

```
build g = g (:) []
```

Tentative:  

```
build :: ((a -> [a] -> [a]) -> [a] -> [a]) -> [a]
```

Too specific... More general type?
Digression: Typingbuild andfoldr

What type shouldfoldrhavе?

\[ \text{foldr} :: (b \rightarrow a \rightarrow a) \rightarrow a \rightarrow [b] \rightarrow a \]

How aboutbuild?

\[ \text{build } g = g (:) [] \]

Tentative:

\[ \text{build} :: ((a \rightarrow [a] \rightarrow [a]) \rightarrow [a] \rightarrow [a]) \rightarrow [a] \]

Too specific... More general type?

\[ \text{build} :: (\forall a. (b \rightarrow a \rightarrow a) \rightarrow a \rightarrow a) \rightarrow [b] \]
Crucial Equation of \texttt{build} and \texttt{foldr}

The following equation is crucial to list fusion:

\[
\text{foldr} \ c \ n \ (\text{build} \ g) = g \ c \ n
\]
Crucial Equation of `build` and `foldr`

The following equation is crucial to list fusion:

\[ \text{foldr} \ c \ n \ (\text{build} \ g) = g \ c \ n \]

Notice that `g`, which was originally used to build a list in `build`, is now used to compute a result which may be something else, such as an `Int`!

This works thanks to the higher-ranked polymorphic type of `build`, meaning that `g` is itself required to be polymorphic.
Back to the Motivating Example

Recall:

\[
\text{main \textit{ls}} = \text{map incr (map double \textit{ls})}
\]
Back to the Motivating Example

Recall:

\[
main \, ls = map \, incr \,(map \, double \, ls)
\]

Desugared into combinators:

\[
main \, ls = build \,(\,\ c1 \, n1 \rightarrow \\
\quad foldr \,(\,\ a1 \, b1 \rightarrow c1 \,(incr \, a1) \, b1) \, n1 \,(map \, double \, ls))
\]

Desugared further:

\[
main \, ls = build \,(\,\ c1 \, n1 \rightarrow \\
\quad foldr \,(\,\ a1 \, b1 \rightarrow c1 \,(incr \, a1) \, b1) \, n1 \,(build \,(\,\ c2 \, n2 \rightarrow \\
\quad foldr \,(\,\ a2 \, b2 \rightarrow c2 \,(double \, a2) \, b2) \, n2 \, ls)))
\]
List Fusion in Practice

The *Glasgow Haskell Compiler* (**GHC**) allows registering user-defined *rewrite rules*, which can be used to implement automatic list fusion (Peyton Jones, Tolmach, Hoare 2001)
List Fusion in Practice

The *Glasgow Haskell Compiler* (**GHC**) allows registering user-defined *rewrite rules*, which can be used to implement automatic list fusion (Peyton Jones, Tolmach, Hoare 2001)

Note: several fundamental and practical limitations to this approach (see later)
4.2. Other Shortcut Fusion Approaches
Other Shortcut Fusion Approaches

Many related approaches following the same technique were proposed. They have different tradeoffs: some programs fuse better than others.
Other Shortcut Fusion Approaches

Many related approaches following the same technique were proposed. They have different tradeoffs: some programs fuse better than others.

For instance, *Stream Fusion* (Coutts, Leshchinskiy, Stewart 2007) supports fusing \texttt{zip}, left folds, and nested lists.

Streams are like lists but have an additional \texttt{Skip} constructor.
4.3. Limitations of Shortcut Fusion
Limitations of Staged Fusion

Fundamental limitations:

• Cannot rewrite user-defined functions
  
  The entire program must be rewritten in terms of combinators
  ○ not always practical
  ○ can have performance implications (may make things slower)

• There isn’t always a best set of combinators to use
Limitations of Staged Fusion

Fundamental limitations:

• Cannot rewrite user-defined functions
  The entire program must be rewritten in terms of combinators
  ◦ not always practical
  ◦ can have performance implications (may make things slower)

• There isn’t always a best set of combinators to use

Practical limitations:

• Quite unreliable; extremely dependent on heuristic inlining

• User-defined rewrite rules not checked for correctness
5. Staged Fusion
High-level Idea

Staged Fusion uses *multi-stage programming*, a metaprogramming technique, to *guarantee* that all constructed programs are *completely fused*. 
High-level Idea

**Staged Fusion** uses *multi-stage programming*, a metaprogramming technique, to *guarantee* that all constructed programs are *completely fused*.

Users typically have to rewrite their programs (the library’s interface becomes *staged*)
Complete Stream Fusion

Table: Library Interface

<table>
<thead>
<tr>
<th>Function</th>
<th>Type signature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Of array</td>
<td>$\alpha$ array code $\rightarrow$ $\alpha$ stream</td>
</tr>
<tr>
<td>Unfold</td>
<td>$(\zeta$ code $\rightarrow$ $(\alpha \ast \zeta$) option code) $\rightarrow$ $\zeta$ code $\rightarrow$ $\alpha$ stream</td>
</tr>
<tr>
<td>Fold</td>
<td>$(\zeta$ code $\rightarrow$ $\alpha$ code $\rightarrow$ $\zeta$ code) $\rightarrow$ $\zeta$ code $\rightarrow$ $\alpha$ stream $\rightarrow$ $\zeta$ code</td>
</tr>
<tr>
<td>Map</td>
<td>$(\alpha$ code $\rightarrow$ $\beta$ code) $\rightarrow$ $\alpha$ stream $\rightarrow$ $\alpha$ stream $\rightarrow$ $\beta$ stream</td>
</tr>
<tr>
<td>Filter</td>
<td>$(\alpha$ code $\rightarrow$ $\text{bool}$ code) $\rightarrow$ $\alpha$ stream $\rightarrow$ $\alpha$ stream</td>
</tr>
<tr>
<td>Take</td>
<td>int code $\rightarrow$ $\alpha$ stream $\rightarrow$ $\alpha$ stream</td>
</tr>
<tr>
<td>Flat map</td>
<td>$(\alpha$ code $\rightarrow$ $\beta$ stream) $\rightarrow$ $\alpha$ stream $\rightarrow$ $\beta$ stream</td>
</tr>
<tr>
<td>Zip With</td>
<td>$(\alpha$ code $\rightarrow$ $\beta$ code $\rightarrow$ $\gamma$ code) $\rightarrow$ $(\alpha$ stream $\rightarrow$ $\beta$ stream $\rightarrow$ $\gamma$ stream)</td>
</tr>
</tbody>
</table>

**Figure 1:** The library interface
Limitations of Staged Fusion

Only *partially* automated: experts need to define the staged libraries

*Intrusive*: users need to rewrite their programs

*Inflexible*: staging boundaries are fixed and can’t easily be changed
Hybrid Approaches

Example: “Quoted staged rewriting” by Parreaux, Shaikhha, Koch (2017)

```scala
@bottomUp @fixedPoint val Flow = rewrite {
  // Floating out pullable info
  case code pull(sas) map $f $g
    => code pull(sas map $f) $g
  case code pull(sas) filter $pred $f
    => code pull(sas filter $pred) $f
  case code pull(sas) take $n
    => code pull(sas take $n) $f
  case code pull(sas) flatMap $f => code sas flatMap $f
    // Folding
    // flatMap is not 'pullable'
  case code pull(sas) doWhile $f => code sas doWhile $f
    // case code $sas map $f doWhile $g
    => code $sas doWhile ($f andThen $g)
  case code $sas filter $pred doWhile $f
    => code $sas doWhile (a => !$pred(a) || $f(a))
  case code $sas take $n doWhile $f
    => code var tk = 0
      $sas doWhile (a => tk <= 1; tk <= $n && $f(a))
  case code $sas flatMap $f doWhile $g
    => code var c = false
      $f(a) doWhile (b => c = $g(b); c)

  // Zipping
  case code $sas zip pull($bs) doWhile $f => code
    $sas.doZip($bs.zipIterator)((a, b) => $f((a, b))
  case code pull($sas) zip $bs doWhile $f => code
    $bs.doZip($sas.zipIterator)((b, a) => $f((a, b))
}
```

Figure 7. Algebraic rewrite rules for stream fusion.
Hybrid Approaches

Example: “Quoted staged rewriting” by Parreaux, Shaikhha, Koch (2017)

```scala
@bottomUp @fixedPoint val Flow = rewrite {
  // Floating out pullable info
  case code"pull($as) map $f" => code"pull($as map $f)"
  case code"pull($as) filter $pred" => code"pull($as filter $pred)"
  case code"pull($as) take $n" => code"pull($as take $n)"
  case code"pull($as) flatMap $f" => code"pull($as flatMap $f)"
  // Folding
  case code"doWhile $f" => code"doWhile $f"
  case code"doWhile $f" => code"doWhile $f"
  case code"map $f doWhile $g" => code"map $f doWhile $g"
  case code"filter $pred doWhile $f" => code"filter $pred doWhile $f"
  case code"$as doWhile ($f andThen $g)" => code"$as doWhile ($f andThen $g)"
  case code"$as take $n doWhile $f" => code"$as take $n doWhile $f"
  case code"$as flatMap $f doWhile $g" => code"$as flatMap $f doWhile $g"
  case code"$as doWhile ( a => ( $pred(a) || $f(a) )" => code"$as doWhile ( a => ( $pred(a) || $f(a) )"
  case code"$as doWhile ( a => false $f(a) doWhile (b => c = $g(b); c); c )"""
  // Zipping
  case code"$as zip pull($bs) doWhile $f" => code"
    $as.zip($bs producer())( (a,b) => $f((a,b)) )"""
  case code"pull($as) zip $bs doWhile $f" => code"
    $as.zip($bs producer())( (a,b) => $f((a,b)) )"""
}
```

Figure 7. Algebraic rewrite rules for stream fusion.
6. The Long Way to Deforestation
6.1. Type Inference
Type Inference

- **Type Inference**: assign a type to each term \((t : \tau)\) in the program such that the types describe the behavior of the value represented by terms (how the value is produced / consumed)

- **Type Check**: make sure that values are *consumed* as intended when they are *produced* so that we will not end up in weird errors, such as using lists as booleans \((\text{List} \neq \text{Bool})\)
6.2. Subtype Inference
Subtype Inference

In languages like Haskell, **type inference** propagates *equalities* between types: \( \tau_1 = \tau_2 \), which discards the direction of the flow of data.

Subtype information is more flexible because it can encode data flow information of programs: \( \tau_1 <: \tau_2 \) ("\( \tau_1 \) is a subtype of \( \tau_2 \)"") means that the data of the term with type \( \tau_1 \) flows into another term with type \( \tau_2 \).

```haskell
fromMaybe p 0
where p = Just 1
  fromMaybe x d = case x of Just a: a | Nothing: d
```

The *data flow* from the data structure allocation \texttt{Just 1} to its consuming case expression: \texttt{Just 1 \rightarrow p \rightarrow x \rightarrow case x of Just ... | Nothing ...}
Subtype Inference

```haskell
fromMaybe p 0
where p = Just 1
  fromMaybe x d = case x of Just a: a | Nothing: d
```

- *subtyping information* collected during subtype inference:
  
  \[ \text{Just } 1 <: \tau_p \quad \tau_p <: \tau_x \quad \tau_x <: \{\text{Just } \tau_a | \text{Nothing}\} \]
  
  after resolving the above inequalities (chaining them together), we get
  
  \[ \text{Just } 1 <: \{\text{Just } \tau_a | \text{Nothing}\} \]
  
  which naturally brings together the producer and consumer of the data structure \text{Just } 1, indicating an opportunity to eliminate it.
6.3. Elaboration
Elaboration

By efficiently *resolving* the subtyping inequalities collected when doing subtyping inference using Simple-sub (Parreaux 2020) and *keeping track of* types with their corresponding terms, deforestation can be done in a novel way.

\[
\text{fromMaybe } p \ 0 \\
\text{where } p = \text{Just } 1 \\
\quad \text{fromMaybe } x \ d = \text{case } x \text{ of } \text{Just } a : a \mid \text{Nothing: } d
\]

can be eventually transformed to

\[
\text{fromMaybe'} p \ 0 \\
\text{where } p' = \text{let } a = 1 \text{ in } a \\
\quad \text{fromMaybe'} x \ d = x
\]
Elaboration

The transformation is done through *elaboration*, which rewrites a term according to the type information attached to it. Fusible *producers* will have types of data constructors, with the information that they are subtypes of types of their consumers; similarly for fusible *consumers*.

Rewriting is done by *importing* the body of consumer into the site where the data constructor is called, binding arguments using *let*, and leaving the body of new “consumer” empty.

- \[ p = \text{Just} \ 1 \rightarrow \ p' = \text{let} \ a = 1 \ \text{in} \ a \]
- \[ \text{fromMaybe} \ x \ d = \text{case} \ x \ \text{of} \ \text{Just} \ a : \ a \ \mid \ \text{Nothing} : \ d \rightarrow \ \text{fromMaybe}' \ x \ d = x \]
6.4. A Recursive Example
A Recursive Example

sum (enumerate x)
where
  enumerate n = if n ≥ 0 then n : enumerate (n - 1) else []
  sum xs = case xs of { [] → 0; x : xs → x + sum xs }

• n:enumerate (n - 1) → (enumerate x) → xs(parameter of sum) →
  case xs of { ... } , so this constructor call is transformed into
  let x = n; xs = numerate' (n - 1) in x + sum' xs

• [] → (enumerate x) → xs(parameter of sum) → case xs of { ... } , so this
  constructor call is transformed into 0
A Recursive Example

sum (enumerate x)
where
  enumerate n = if n ≥ 0 then n : enumerate (n - 1) else []
  sum xs = case xs of { [] → 0; x : xs → x + sum xs }

sum' (enumerate' x)
where
  enumerate' n = if n ≥ 0
  then let x = n; xs = enumerate' (n - 1) in x + sum' xs
  else 0
  sum' xs = xs
Benchmark Results (51 tests in the *nofib* benchmark suite)

- average speedup: 14%
- leftmost: original program; rightmost: after all the steps of our transformation
Benchmark Results (51 tests in the nofib benchmark suite)

- average code size increases by 1.8x
Bibliography


