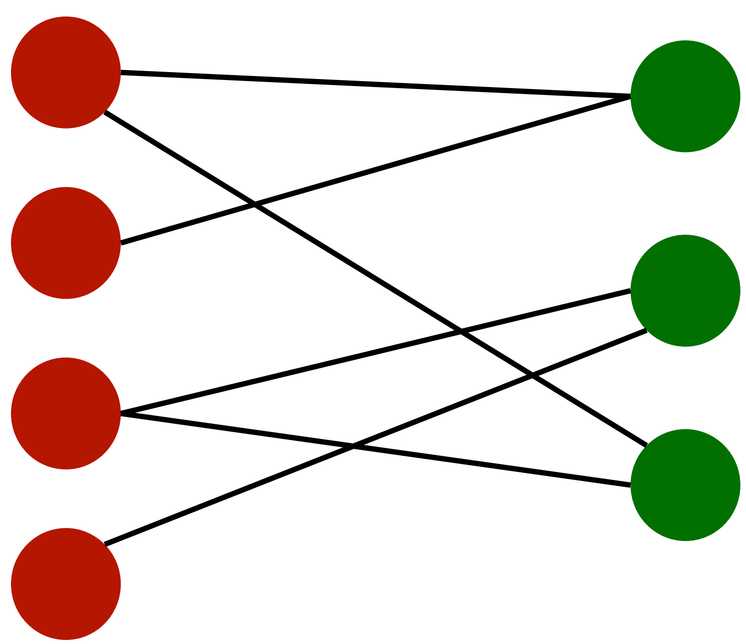
Advanced Algorithmic Techniques (COMP523)

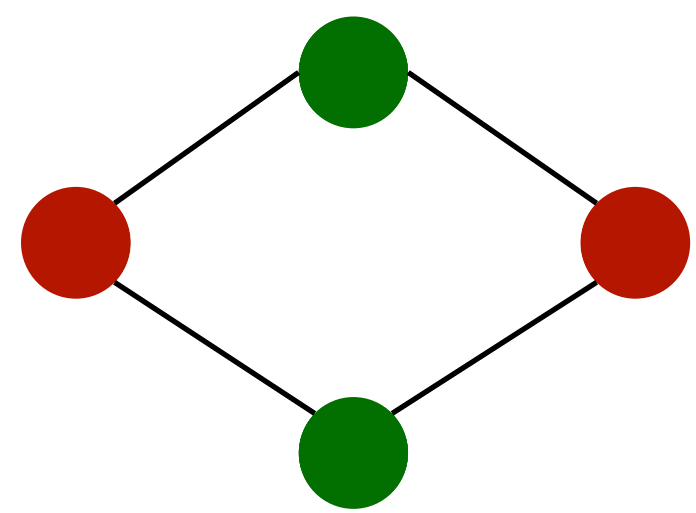
Testing for bipartiteness

Bipartite graphs

- A graph G=(V,E) is bipartite *if any only if* it can be partitioned into sets A and B such that each edge has one endpoint in A and one endpoint in B.

    - Often, we write G=(A U B,E).





Alternative definitions

- A graph G=(V,E) is bipartite *if any only if* its nodes can be coloured with 2 colours (say red and green), such that every vertex has one red endpoint and one green endpoint.
- A graph G=(V,E) is bipartite *if any only if* it does not contain any cycles of odd length.

No odd cycles

- A graph $G=(V,E)$ is bipartite *if any only if* it does not contain any cycles of odd length.
  - **=>** Assume that G is bipartite
  - Suppose that G does contain an odd cycle (proof by contradiction), $C = u_1\ u_2\ u_3\ \dots\ u_n\ u$ for some u in A *(wlog),* or alternatively, for some u that is red.
  - Because G is bipartite, u_2 must be green, and then u_3 must be red, and so on.
  - Generally, we observe that for all k in {1,2, … ,n}, u_k is red if k is odd and green if k is even.
  - By assumption, n is odd, so it must be red. But then u cannot be red, because G is bipartite.

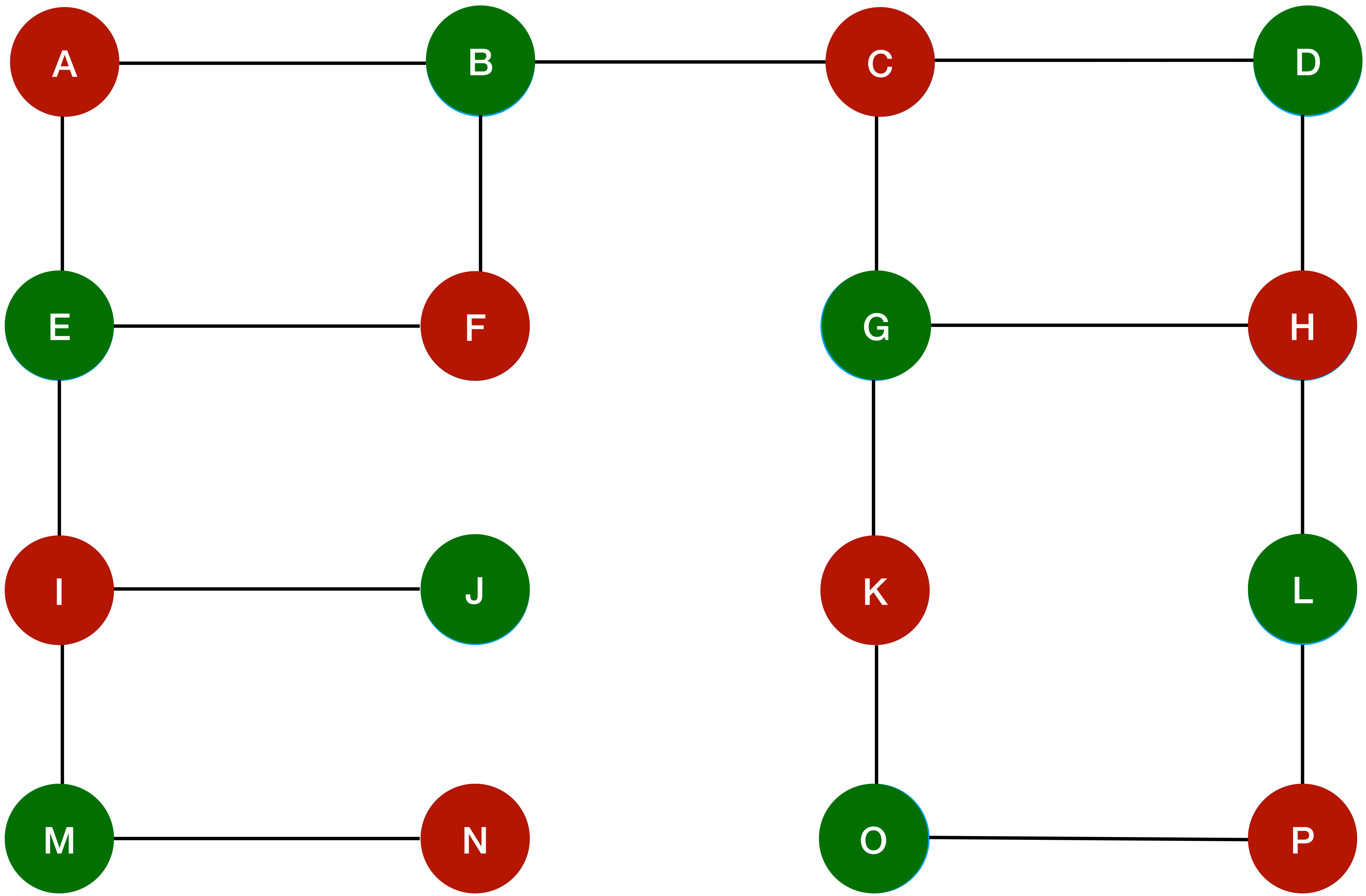
Alternative definitions

- A graph G=(V,E) is bipartite *if any only if* its nodes can be coloured with 2 colours (say red and green), such that every vertex has one red endpoint and one green endpoint.
- A graph G=(V,E) is bipartite *if any only if* it does not contain any cycles of odd length.
- Sometimes, these alternatives definitions are also called "characterisations".

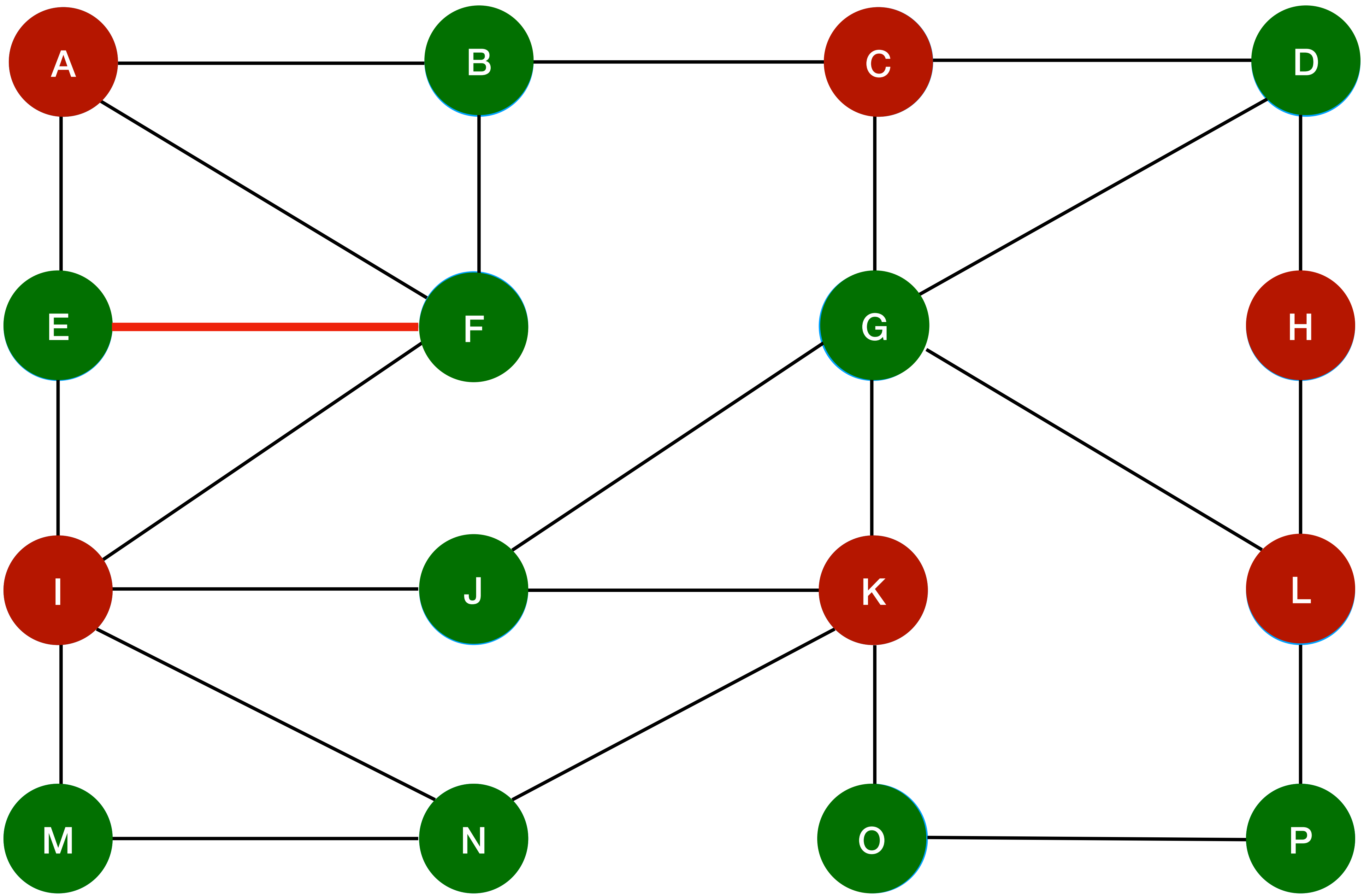
Testing bipartiteness

- Given a graph G=(V,E), decide if it is bipartite or not.
- Given a a graph G=(V,E) decide if it is 2-colourable or not.
- Given a a graph G=(V,E) decide if it is contains cycles of odd length or not.

Colouring the nodes



Colouring the nodes



Colouring the nodes

- Does this remind you of something?
  - It is essentially BFS!
  - We label the nodes of *layer 1* red, the nodes of *layer 2* green, and so on.
- Implementation:
  - Add a check for odd/even and assign a colour accordingly.
  - In the end, check all edges to see if they have endpoints of the same colour.

Breadth-First Search Pseudocode

```
Algorithm BFS(G,s)

Initialise empty list L0
Initialise colour list C
Insert s into L0
Set C[s] = red

Set i=0
While Li is not empty
    Initialise empty list Li+1
    for each node v in Li
      for all edges e incident to v
        if edge e is unexplored
          let w be the other endpoint of e
          if node w is unexplored
            label e as discovery edge
            insert w into Li+1
            If i+1 is odd, set C[w] = red, else set C[w] = green
          else
            label e as cross edge
i = i+1

For all edges e=(u,v) in G
    if C[u] = C[v] return "not bipartite"
Return "bipartite"
```

Running time

- What did we add?
  - A colour assignment for the starting node.
  - An odd/even check and a colour assignment for each node in the loop.
  - An extra loop for checking the edges of their graph for the colours of their endpoints.
- How much more do we “pay” (asymptotically)?
  - Nothing!
- Running time **O(m+n)**.

Correctness

- We started at an arbitrary node s.
- Maybe we were lucky / unlucky?

Properties of BFS

- For simplicity, assume that the graph is **connected.**
- The traversal visits all vertices of the graph.
- The *discovery edges* form a spanning tree.
- The path of the spanning tree from **s** to a node **v** at level *i* has *i* edges, and this is the shortest path.
- If **e**=(**u**,**v**) is a *cross edge*, then the **u** and **v** differ by at most one level.

Properties of BFS

- If **e**=(**u**,**v**) is a *cross edge*, then the **u** and **v** differ by at most one level.

- If **e**=(**u**,**v**) is a *discovery edge*, then the **u** and **v** differ by at most one level.

Correctness

- Suppose that G is **bipartite**. Then, all cycles must be of even length.

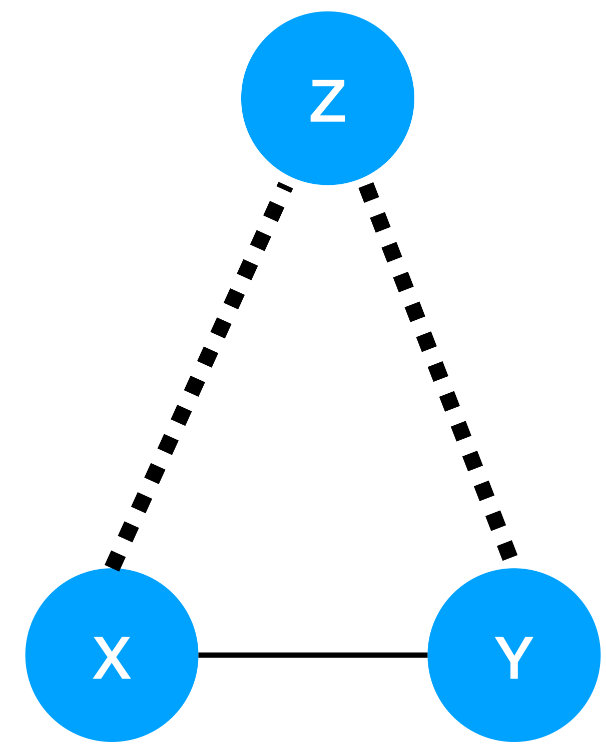
- Suppose *to the contrary* that the algorithm returns "*not bipartite*".

  - This means that it has found an edge e=(x,y) with endpoints of the same colour.

  - Since the endpoints of any edge can not differ by more than one layer and layers have alternating colours, x and y must be in the same layer.

Correctness

- Consider the lowest common ancestor z of x and y in the BFS tree.
- Let L_i be the layer of z and let L_j be the layer of x and y
- Consider the cycle (z … x), (x,y), (y … z).
- Length: (j-i) + 1 + (j-i) (odd)
- **Contradiction!**

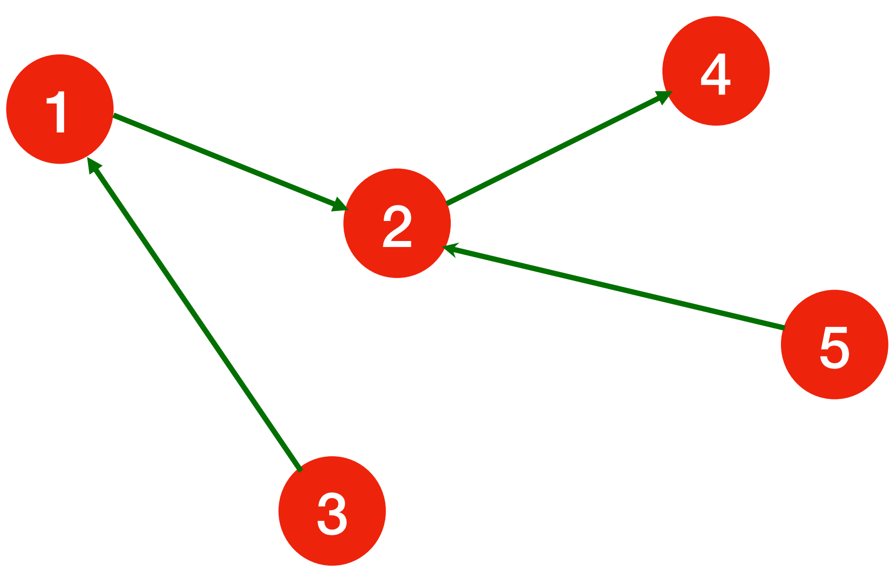


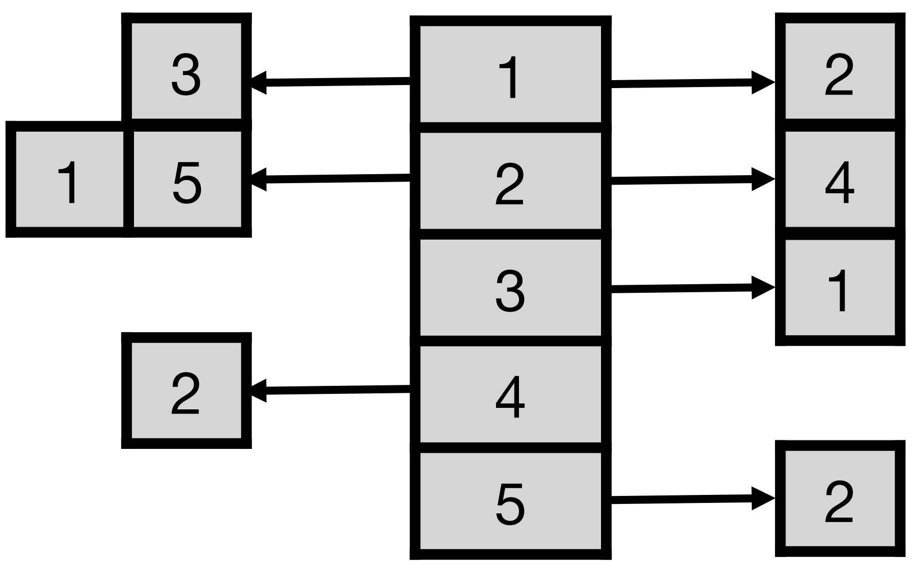
Correctness

- Suppose that G is **not bipartite**. Then, it must contain a cycle of odd length.
- Suppose *to the contrary* that the algorithm returns "*bipartite*".
  - This means that it has not found any edge e=(x,y) with endpoints of the same colour.
  - This also obviously means that there is no edge with endpoints in the same layer.
  - By the earlier discussion, all edges must have endpoints that lie in consecutive layers.
  - Take any cycle (z, … , z). Since for every edge in this cycle there is a change of layer (from j to $j+1$ or from $j+1$ to j), the cycle must have even length.
  - **Contradiction!**

Directed graphs

- Nodes are arranged as a list, each node points to the neighbours.
- For directed graphs, the node points in two directions, for in-degree and for out-degree.

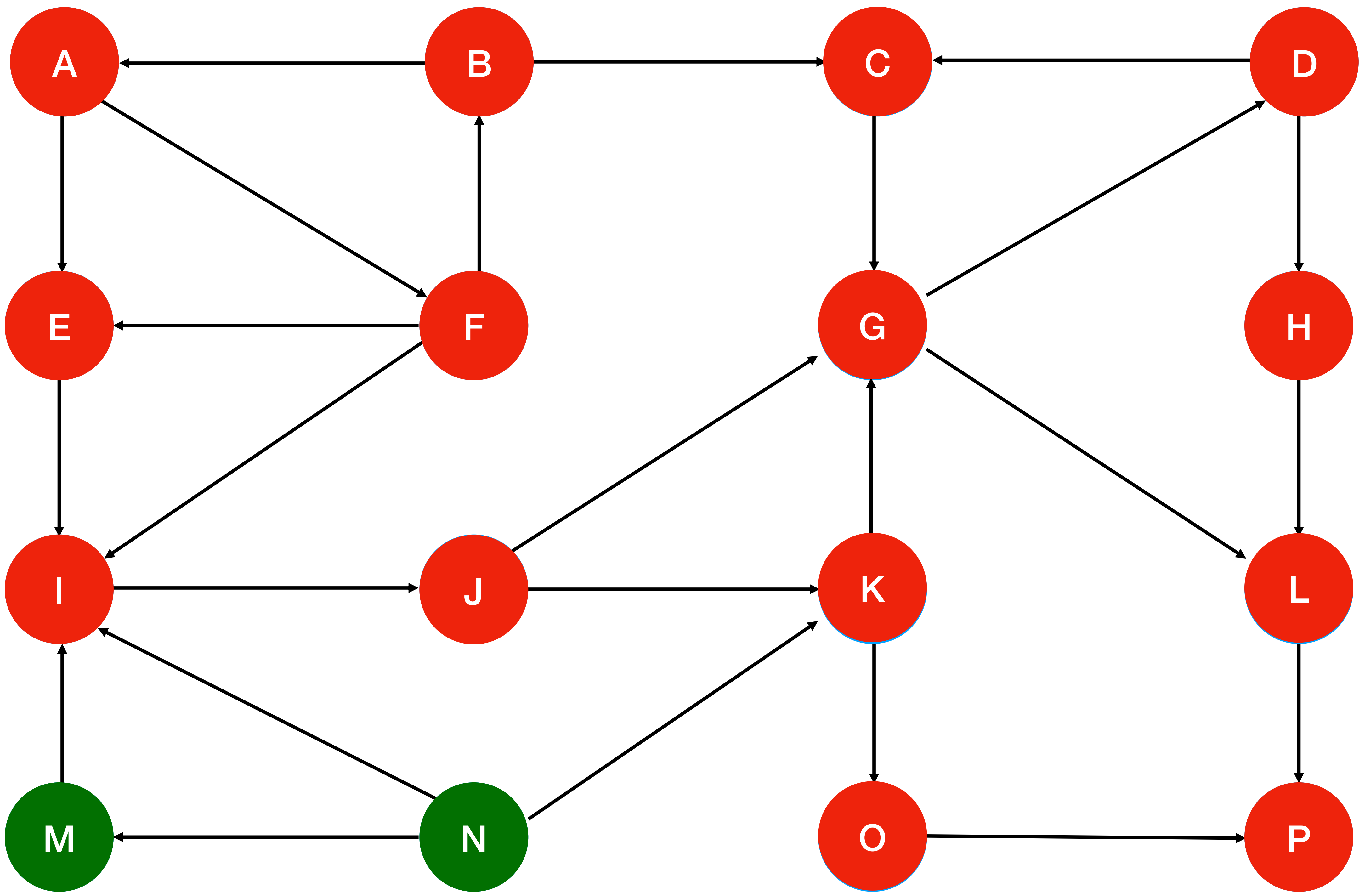




DFS and BFS on directed graphs

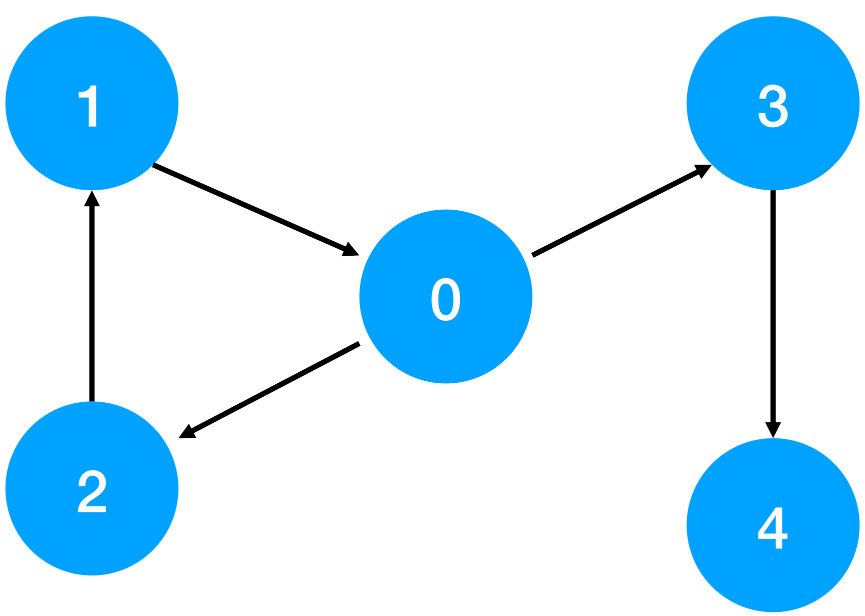
- Very similar to their version on undirected graphs.
- When we are at a node and we examine its neighbours, a neighbour is now only a node that we can reach with a directed edge.
- The running time is still **O(n+m)**.

Breadth-First Search

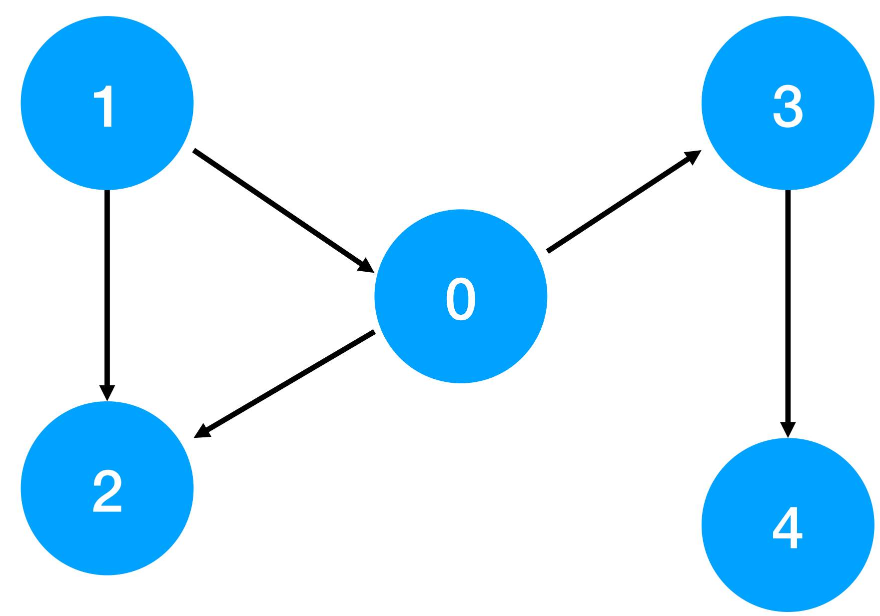


Directed Acyclic Graphs

- A directed acyclic graph (DAG) G is a graph that does not have any cycles.



not a DAG



a DAG

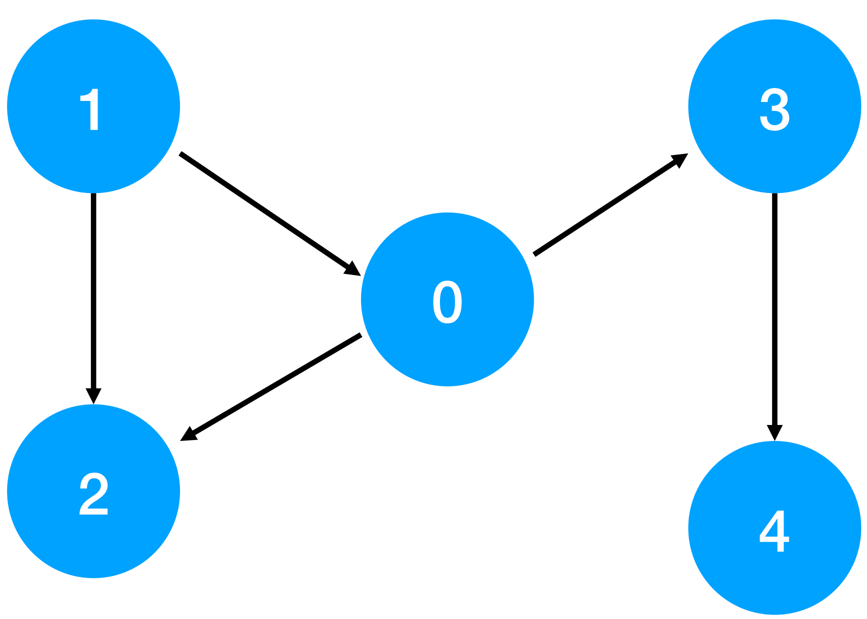
Properties of DAGs

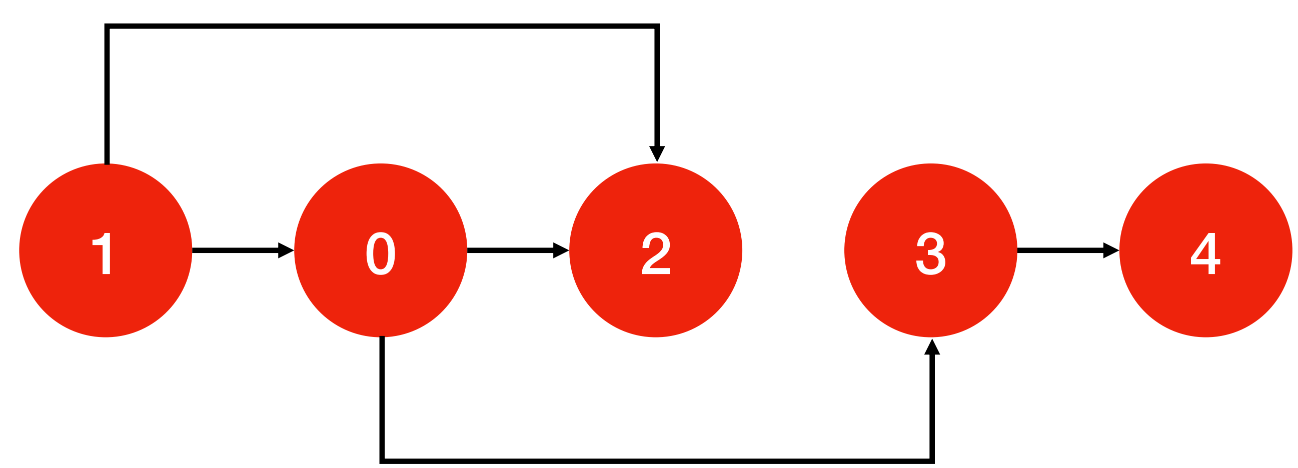
- They appear quite often in many applications.
- Example - prerequisite modules: To take module A you need to have taken module B and module C.
- If the module prerequisite relation has a cycle, then it is impossible to get a degree!

Topological Ordering

- Given a directed graph G, a topological ordering of G is an ordering of the nodes u_1, u_2, … , u_n, such that for every edge $e=(u_i, u_j)$, it holds that $i < j$.

- Intuitively, a topological ordering orders the nodes in a way such that all edges point “forward”.





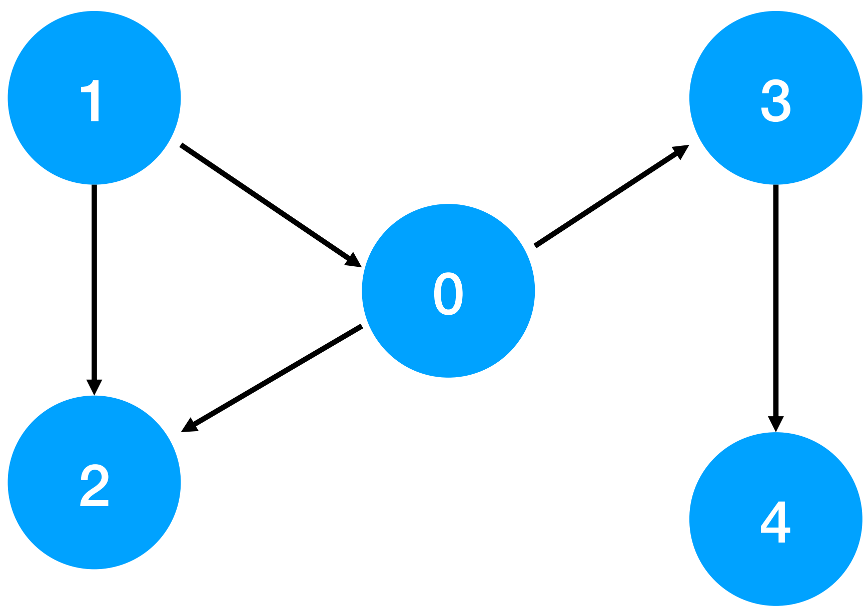
Topological Ordering implies DAG

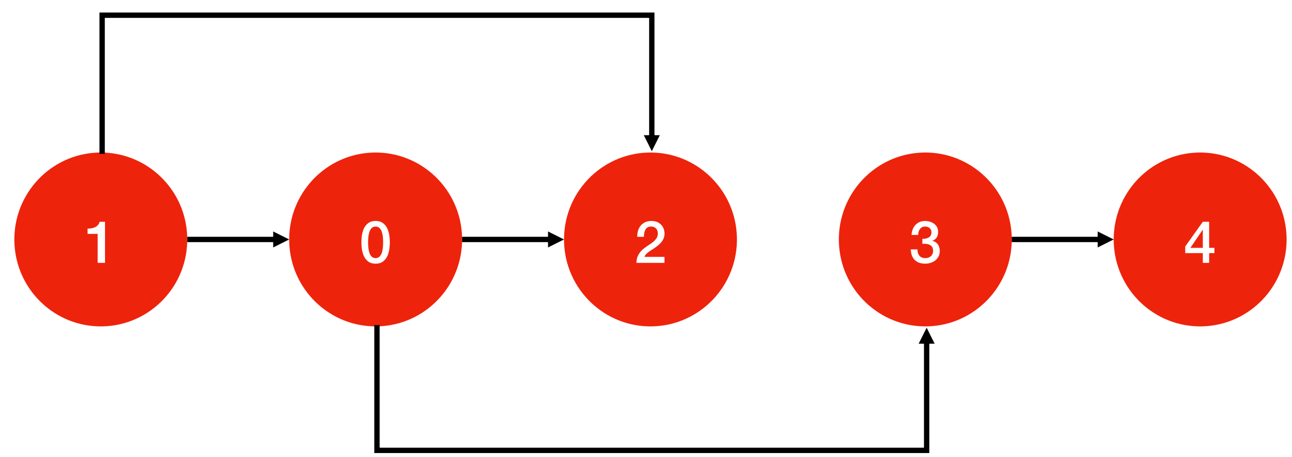
- If graph G has a topological ordering, then G is a DAG.
- Suppose by contradiction that G has a topological ordering (u_1, u_2, … , u_n) but it also has a cycle C.
- Let u_j be the smallest element of C according to the topological ordering.
- Let u_i be its *predecessor* in the cycle (i.e., there is an edge $e=(u_i, u_j)$).
- u_i must appear before u_j in the topological order, by the presence of this edge.
- This **contradicts** the fact that u_j was the smallest element of C according to the topological ordering.

Does DAG imply topological ordering?

- TO => DAG was proved via proof-by-contradiction.
- DAG => TO will be proved via “proof-by-algorithm”.
- We will design an *efficient* algorithm that, given a DAG G, finds a topological ordering of G.

How do we start?



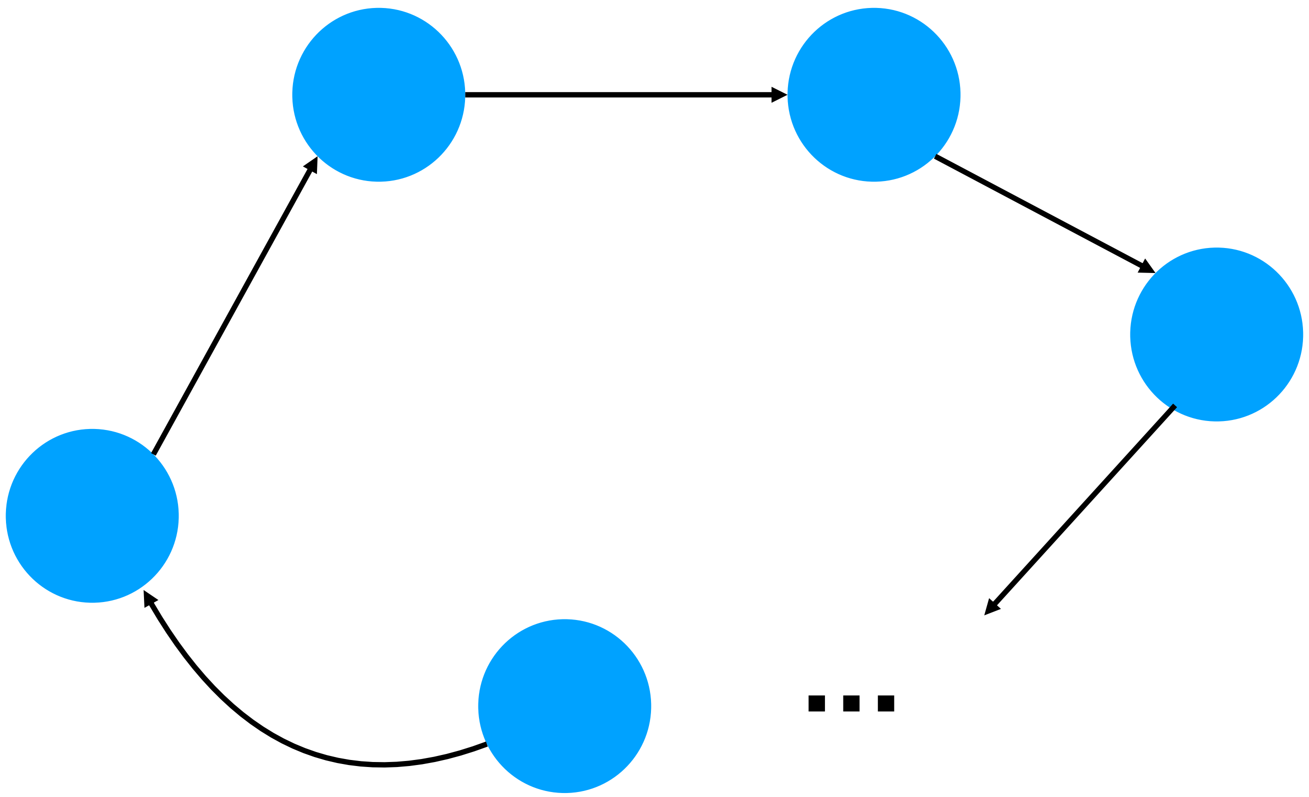


- Could we have started with anything other than node 1?
- The starting node must have no incoming edges!
- Can we always find such a node?

Source node

- A source node is a node with no incoming edges.
- **Lemma:** Every DAG has at least one source node.
- Proof by contradiction:
  - Assume that every node has at least one incoming edge.
  - Start from any node u and follow edges from u *backwards*.
    - Equivalently, we move to a neighbour of u in G^{rev}.
  - We can do that for every node, since by assumption there is no source.
  - After at least n+1 steps, we will have visited the same node twice.
  - The graph has a cycle, therefore it can't be a DAG. **Contradiction!**

Pictorially



Another simple fact

- If we remove a node u and all its incident edges from a DAG G, the resulting graph G’ is still a DAG.
  - If G’ had a cycle, the same cycle would be present in G.

DAG implies topological ordering

- Proof-by-induction:
  - Base Case: If the DAG has one or two nodes, it clearly has a topological ordering.
  - Inductive step: Assume that a DAG with up to k nodes has a topological ordering (Inductive Hypothesis). We will prove that a DAG with k+1 nodes has a topological ordering.
    - By our lemma, there is at least one source node in G, and let u be such a node.
    - Put u first in the topological ordering (safe, since u is a source).
    - Consider the graph G', obtained by G if we remove u and its incident edges.
    - G' is a DAG (by the simple fact) with k nodes.
      - It has a topological ordering by the induction hypothesis.
    - Append this ordering to u.

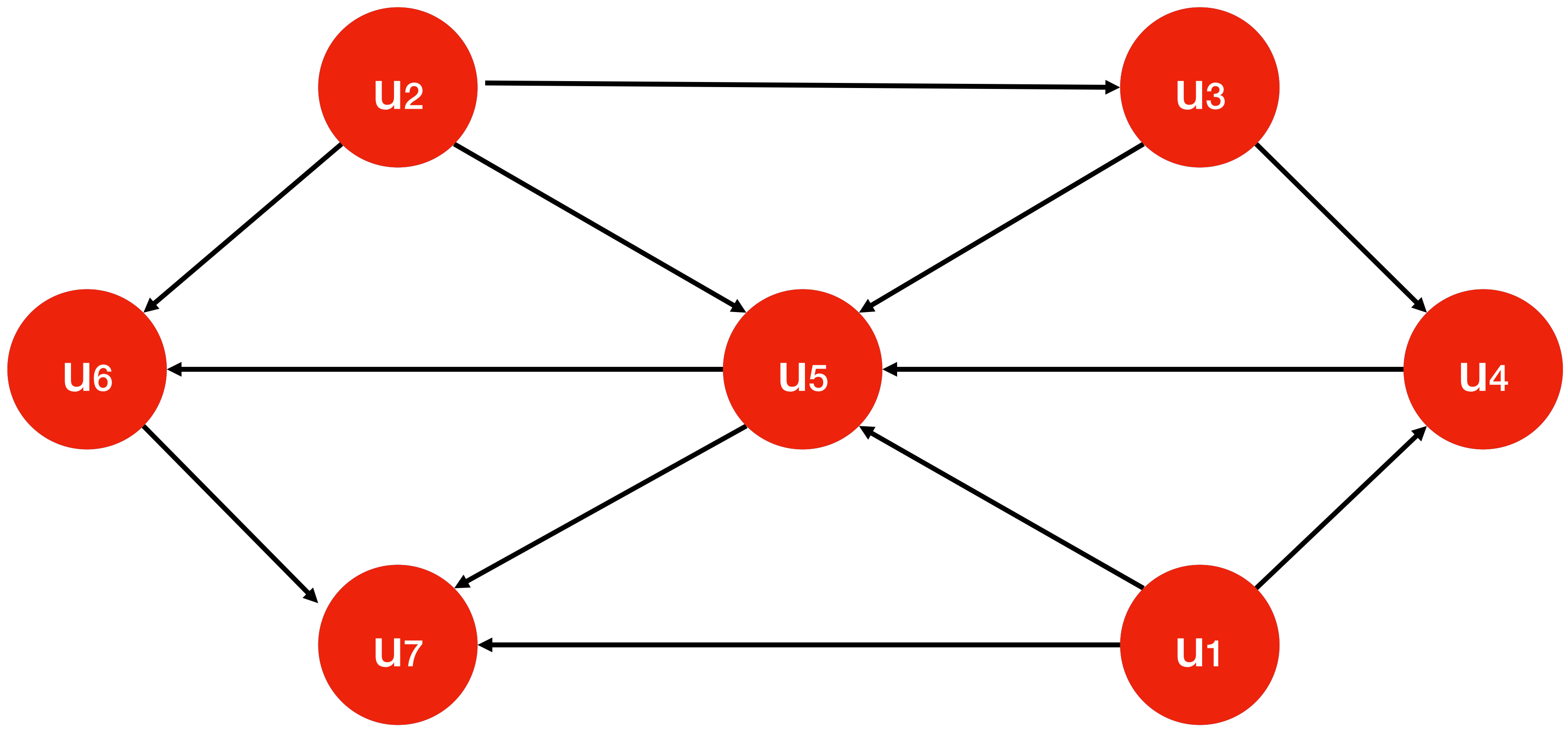
Where is the "proof-by-algorithm"?

- We can turn that induction proof into an algorithm.

Algorithm TopologicalSort(G)

    Find a source vertex u and put it first in the order.
    Let G'=G-{u}
    TopologicalSort(G')
    Append this order after u

Example



Running time

- We need to find a source u.
- We could check each node of the graph.
- We check *n* nodes in the first iteration, *n-1* nodes in the second, and so on…
- What is the running time of this?
  - **$O(n^2)$**
- Can we do better?

A faster algorithm

- We will be more efficient in the choice of sources.
- We will say that a node is active, if it has not been selected (and therefore removed) as a source by the algorithm.
- We maintain two things:
  - (a) For each node w, the number of *incoming edges* from active nodes.
  - (b) The set S of all active nodes that have *no incoming edges* from other active nodes.

A faster algorithm

- In the beginning, all nodes are active and we can initialise (a) and (b) via a pass through the graph (time **O(m+n)**)
- In each iteration:
  - We select a node u from the set S.
  - We delete u.
  - We go through all the neighbours w of u and we reduce their value in (a) (i.e., number of incoming edges from active nodes) by 1.
  - When the value of (a) for some node w goes to 0, w is added to the set S.

Reading

Kleinberg and Tardos 3.4, 3.6 (for bipartiteness and topological sort)

Roughgarden 8.5 (for topological sort)

CLRS 20.4 (for topological sort)

See you next year!