Introduction to Algorithms and Data Structures

Greedy Algorithms: Interval Scheduling
The Greedy approach

- The goal is to come up with a global solution.
- The solution will be built up in small consecutive steps.
- For each step, the solution will be the best possible myopically, according to some criterion.
Interval Scheduling
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• Each request has a starting time \(s(i)\) and a finishing time \(f(i)\).
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  - Two requests \( i \) and \( j \) are compatible if their respective intervals do not overlap.
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  - Alternative view: Every request is an interval \([s(i), f(i)]\).
- Two requests \( i \) and \( j \) are compatible if their respective intervals do not overlap.
- **Goal:** Output a schedule which maximises the number of compatible intervals.
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- We start by selecting an interval $[s(i), f(i)]$ for some request $i$.

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- We will continue with some compatible interval $[s(j), f(j)]$ and repeat the same process.
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- We terminate when there are no more compatible intervals to consider.
Example
Example
Example
Example
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- Let’s try to make this more concrete.
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- Let’s try to make this more concrete.
- Option 1: Choose the available interval that starts earliest.
Example
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Is this the best we can do?
Is this always optimal?
Is this always optimal?
The Greedy Approach

• We start by selecting an interval \([s(i), f(i)]\) for some request \(i\).

• Let’s try to make this more concrete.

• Option 1: Choose the available interval that starts earliest.

• Option 2: Choose the smallest available interval.
Choosing the smallest interval
Is this always optimal?
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  • Option 1: Choose the available interval that starts earliest.

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  • Option 3: Something more clever.
The Greedy Approach

• We start by selecting an interval \([s(i), f(i)]\) for some request \(i\).

• Let’s try to make this more concrete.

• Option 1: Choose the available interval that starts earliest.

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• Option 3: Something more clever.

• Find the interval that minimises the number of “conflicts”.
Minimum number of conflicts
Is this always optimal?
Is this always optimal?
Is this always optimal?
Is this always optimal?
Something even more clever
Something even more clever

• Select the interval \([s(i), f(i)]\) that finishes first (smallest \(f(i)\)).

• Intuition: The resource becomes free as soon as possible, but we still satisfy one request.
Greedy Algorithm for interval scheduling

\begin{itemize}
  \item Let \( R \) be the set of requests, let \( A \) be empty
  \item While \( R \) is not empty
    \begin{itemize}
      \item Choose a request \( i \) with the smallest \( f(i) \).
      \item Add \( i \) to \( A \)
      \item Delete all requests from \( R \) that are not compatible with request \( i \).
    \end{itemize}
\end{itemize}

Return the set \( A \) of accepted requests
Correctness
Correctness

• Does the Greedy algorithm produce an optimal schedule?
Correctness

- Does the Greedy algorithm produce an optimal schedule?

- Does the Greedy algorithm produce a feasible (or acceptable) schedule?
Correctness

• Does the Greedy algorithm produce an optimal schedule?

• Does the Greedy algorithm produce a feasible (or acceptable) schedule?

  • Yes, since it removes in each step the intervals which are not compatible with what has been chosen.
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  • Note that $|A| = k$. 
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    • Note that $|A| = k$.

  • Let $j_1, j_2, \ldots, j_m$ be the set of requests in $O$.

    • Note that $|O| = m$.

  • We will prove that $m = k$. (Why is that enough?)
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• Assume wlog that this is in order of increasing $s(j_n)$.
Arguing for optimality

- Let $j_1, j_2, \ldots, j_m$ be the set of requests in $O$.
  - Assume wlog that this is in order of increasing $s(j_h)$.
  - Since $O$ is feasible, this is also in order of increasing $f(j_h)$. 
Arguing for optimality

Let \( j_1, j_2, \ldots, j_m \) be the set of requests in \( O \).

Assume wlog that this is in order of increasing \( s(j_n) \).

Since \( O \) is feasible, this is also in order of increasing \( f(j_n) \).

Claim: \( f(i_1) \leq f(j_1) \)
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  - Assume wlog that this is in order of increasing $s(j_h)$.
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- Claim: $f(i_1) \leq f(j_1)$
  - Because $i_1$ is chosen to be the interval with the smallest $f(i_h)$. 
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- Lemma: For all indices \( r \leq k \), it holds that \( f(i_r) \leq f(j_r) \)
Arguing for optimality

- **Claim:** \( f(i) \leq f(j) \)
  - Because \( i \) is chosen to be the interval with the smallest \( f(i) \).

- **Lemma:** For all indices \( r \leq k \), it holds that \( f(i_r) \leq f(j_r) \)
  - **Proof by induction:**
Arguing for optimality

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  • Proof by induction:

    • Base Case \((r=1)\), by Claim.
Arguing for optimality

- **Claim:** $f(i_1) \leq f(j_1)$

  - Because $i_1$ is chosen to be the interval with the smallest $f(i_h)$.

- **Lemma:** For all indices $r \leq k$, it holds that $f(i_r) \leq f(j_r)$

  - **Proof by induction:**

    - **Base Case** ($r=1$), by Claim.

    - **Induction Step.** Assume it is true for $r-1$ (IH), we will prove it for $r$. 
Induction step proof
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• We know that $f(j_{r-1}) \leq s(j_r)$ (why?)
Induction step proof

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• What does that mean for the interval \( j_r = (s(j_r), f(j_r)) \)?
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• What does that mean for the interval $j_r = (s(j_r), f(j_r))$?
  • When the Greedy algorithm selected $i_r, j_r$ was in the set $R$ of available intervals.
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- What does that mean for the interval $j_r = (s(j_r), f(j_r))$?
  - When the Greedy algorithm selected $i_r$, $j_r$ was in the set $R$ of available intervals.

- This means that $f(i_r) \leq f(j_r)$, as otherwise the algorithm would have selected $j_r$ instead.
With a picture

\[ f(i_{r-1}) \leq f(j_{r-1}) \leq f(i_r) \leq s(j_r) \]
Completing the proof
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- By contradiction: To the contrary, assume that $m > k$
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- For $r=k$, the Lemma gives us that $f(i_k) \leq f(j_k)$. 
Completing the proof

• By contradiction: To the contrary, assume that $m > k$

• For $r=k$, the **Lemma** gives us that $f(i_k) \leq f(j_k)$.

• Since $m > k$, there is an extra request $j_{k+1}$ in $O$. 
Completing the proof

• **By contradiction:** To the contrary, assume that $m > k$

• For $r=k$, the Lemma gives us that $f(i_k) \leq f(j_k)$.

• Since $m > k$, there is an extra request $j_{k+1}$ in $O$.

• $s(j_{k+1}) > f(j_k) \geq f(i_k)$. 
Completing the proof

• **By contradiction:** To the contrary, assume that \( m > k \)

• For \( r=k \), the Lemma gives us that \( f(i_k) \leq f(j_k) \).

• Since \( m > k \), there is an extra request \( j_{k+1} \) in \( O \).

• \( s(j_{k+1}) > f(j_k) \geq f(i_k) \).

• The greedy algorithm would have continued with \( j_{k+1} \).
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  • If yes, we select it and continue with the same checks for this new interval.
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• For any consecutive interval $j$ in the ordering, we check if $f(i) \leq s(j)$.
  
  • If yes, we select it and continue with the same checks for this new interval.

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• The running time is $O(n \log n)$. 