Introduction to Algorithms and Data Structures

Greedy Algorithms: Dijkstra's algorithm for shortest paths
What is the fastest way to go from the School of Informatics to the EICC?
Shortest Paths in Graphs
Shortest Paths in Graphs

- **Input:** A directed graph $G = (V, E)$, and a designated node $s$ in $V$. We also assume that every node $u$ in $V$ is reachable from $s$. We are also given a length $\ell_e$ for every edge $e$ in $E$. 
Shortest Paths in Graphs

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• **Output:** For every node $u$ in $V$, a shortest path $s\sim u$ from $s$ to $u$. 
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Shortest Paths in Graphs

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- **Output:** For every node \( u \) in \( V \), a shortest path \( s \sim u \) from \( s \) to \( u \).

- To be more precise, a list of paths:

  \[
  \begin{array}{c|c|c|c|c|c}
  P(s) & P(u_1) & P(u_2) & P(u_3) & \ldots \\
  \end{array}
  \]
What about undirected graphs?

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Running Example
(KT Figure 5.7)
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Not a shortest path!
not a shortest path!
Running Example
(KT Figure 5.7)

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a shortest path!
Dijkstra’s Algorithm
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- Proposed by Edsger Dijkstra in 1959.
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  - We may refer to $S$ as the *explored* part of the graph.
  - Initially, $S = \{s\}$ and $d(s) = 0$. 
Running Example (KT Figure 5.7)

[Diagram of a graph with nodes labeled s, u, v, x, y, z and edges labeled with distances.]

Shortest path distances from s:

- s to u: 1
- s to v: 2
- s to x: 4
- s to y: 3
- s to z: 3
Running Example
(KT Figure 5.7)

shortest path distances from $s$
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  • We may refer to \( S \) as the *explored* part of the graph.

  • Initially, \( S = \{s\} \) and \( d(s) = 0 \).

• For every node \( v \in V - S \), we determine the shortest path that can be constructed by traveling along a path \( s \sim u \) for \( u \in S \), followed by \( (u, v) \).
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• In other words, we choose node \( v \in V - S \) such that

\[
  d'(v) = \min_{e=(u,v):u\in S} d(u) + \ell_e
\]
Dijkstra’s Algorithm

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- Add \( v \) to \( S \) and define \( d(v) = d'(v) \).
Running Example
(KT Figure 5.7)

shortest path distances from $s$
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Running Example
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Iteration 1
Running Example (KT Figure 5.7)

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Running Example
(KT Figure 5.7)

$d'(u) = d(s) + \ell_{(s,u)} = 1$

$d'(v) = d(s) + \ell_{(s,v)} = 2$

Iteration 1

In other words, we choose node \( v \in V - S \) such that

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Iteration 1

\[
d'(u) = d(s) + \ell_{(s,u)} = 1
\]

\[
d'(v) = d(s) + \ell_{(s,v)} = 2
\]

\[
d'(x) = d(s) + \ell_{(s,x)} = 4
\]

shortest path distances from \( s \)
In other words, we choose node $v \in V - S$ such that

$$d'(v) = \min_{e = (u,v): u \in S} d(u) + \ell_e$$

Add $v$ to $S$ and define $d(v) = d'(v)$. 

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shortest path distances from $s$
Running Example
(KT Figure 5.7)

$d'(u) = d(s) + \ell_{(s,u)} = 1$

$d'(v) = d(s) + \ell_{(s,v)} = 2$

Iteration 1

Iteration 2

$d'(x) = d(s) + \ell_{(s,x)} = 4$

shortest path distances from $s$

In other words, we choose node $v \in V - S$ such that

$d'(v) = \min_{e=(u,v):u \in S} d(u) + \ell_e$

Add $v$ to $S$ and define $d(v) = d'(v)$. 
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In other words, we choose node \( v \) such that
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d'(v) = \min_{e=(u,v): u \in S} d(u) + \ell_e
\]
Add \( v \) to \( S \) and define \( d(v) = d'(v) \).

Iteration 1

\[d'(u) = d(s) + \ell_{(s,u)} = 1\]
\[d'(v) = d(s) + \ell_{(s,v)} = 2\]
\[d'(v) = d(s) + \ell_{(s,v)} = 2\]

Iteration 2

\[d'(y) = d(u) + \ell_{(u,y)} = 4\]
\[d'(x) = d(s) + \ell_{(s,x)} = 4\]
In other words, we choose node $v$ such that $v \in V - S$ such that

$$d'(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e$$

Add $v$ to $S$ and define $d(v) = d'(v)$.
running example
(KT Figure 5.7)

\[d'(y) = d(u) + \ell_{(u,y)} = 4\]

\[d'(x) = d(u) + \ell_{(u,x)} = 2\]

\[d'(v) = \min_{e=(u,v):u \in S} d(u) + \ell_e\]

Add \(v\) to \(S\) and define \(d(v) = d'(v)\).
Running Example
(KT Figure 5.7)

In other words, we choose node \( v \) \( \in V - S \) such that

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d'(v) = \min_{e=(u,v): u \in S} d(u) + \ell_e
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Add \( v \) to \( S \) and define \( d(v) = d'(v) \).
Dijkstra’s Algorithm (Pseudocode)

Let $S$ be the set of explored nodes, $A$ be a list of distances.

Initially $S = \{s\}$ and $d(s) = 0$, $A[s] = d(s) = 0$

While $S \neq V$

Select a node $v \in V - S$ connected via an edge with at least one node in $S$ for which

$$d'(v) = \min_{e = (u,v): u \in S} d(u) + \ell_e$$

Add $v$ to $S$ and define $d(v) = d'(v)$

Let $A[v] = d(v)$
Running Example (KT Figure 5.7)

Let \( S \) be the set of explored nodes, \( A \) be a list of distances.

Initially \( S = \{s\} \) and \( d(s) = 0 \), \( A[s] = d(s) = 0 \)

While \( S \neq V \)

Select a node \( v \in V - S \) connected via an edge with at least one node in \( S \) for which

\[
d'(v) = \min_{e=(u,v),u\in S} d(u) + e
\]

Add \( v \) to \( S \) and define \( d(v) = d'(v) \)

Let \( A[v] = d(v) \)
Running Example
(KT Figure 5.7)

Let $S$ be the set of explored nodes, $A$ be a list of distances.

Initially $S = \{s\}$ and $d(s) = 0$, $A[s] = d(s) = 0$

While $S \neq V$

Select a node $v \in V - S$ connected via an edge with at least one node in $S$

for which

$$d'(v) = \min_{e=(u,v): u \in S} d(u) + \ell_e$$

Add $v$ to $S$ and define $d(v) = d'(v)$

Let $A[v] = d(v)$
Running Example
(KT Figure 5.7)

\[ d'(x) = d(u) + \ell_{(u,x)} = 2 \]

**Dijsktra \((G, \ell)\)**

Let \( S \) be the set of explored nodes, \( A \) be a list of distances.

Initially \( S = \{s\} \) and \( d(s) = 0, A[s] = d(s) = 0 \)

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Select a node \( v \in V - S \) connected via an edge with at least one node in \( S \) for which

\[ d'(v) = \min_{e=(u,v):u\in S} d(u) + \ell_e \]

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Running Example (KT Figure 5.7)

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While $S \neq V$

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$$d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e$$

Add $v$ to $S$ and define $d(v) = d'(v)$

Let $A[v] = d(v)$
Running Example (KT Figure 5.7)

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Add $v$ to $S$ and define $d(v) = d'(v)$

Let $A[v] = d(v)$
Running Example (KT Figure 5.7)

$S$

$\begin{align*}
d'(x) &= d(u) + \ell_{(u,x)} = 2 \\
d'(y) &= d(u) + \ell_{(u,y)} = 4 \\
d'(z) &= d(v) + \ell_{(v,z)} = 5
\end{align*}$

Let $S$ be the set of explored nodes, $A$ be a list of distances.

Initially $S = \{s\}$ and $d(s) = 0$, $A[s] = d(s) = 0$

While $S \neq V$

Select a node $v \in V - S$ connected via an edge with at least one node in $S$

for which

$$d'(v) = \min_{e=(u,v) : u \in S} d(u) + \ell_e$$

Add $v$ to $S$ and define $d(v) = d'(v)$

Let $A[v] = d(v)$

shortest path distances from $s$
Running Example
(KT Figure 5.7)

\[ d'(x) = d(u) + \ell_{(u,x)} = 2 \]
\[ d'(y) = d(u) + \ell_{(u,y)} = 4 \]
\[ d'(z) = d(v) + \ell_{(v,z)} = 5 \]

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Let \( S \) be the set of explored nodes, \( A \) be a list of distances.

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While \( S \neq V \)

Select a node \( v \in V - S \) connected via an edge with at least one node in \( S \) for which

\[ d'(v) = \min_{e=(u,v):u\in S} d(u) + \ell_e \]

Add \( v \) to \( S \) and define \( d(v) = d'(v) \)

Let \( A[v] = d(v) \)
Running Example (KT Figure 5.7)

\[ d'(x) = d(u) + \ell_{(u,x)} = 2 \]
\[ d'(y) = d(u) + \ell_{(u,y)} = 4 \]
\[ d'(z) = d(v) + \ell_{(v,z)} = 5 \]

Let \( S \) be the set of explored nodes, \( A \) be a list of distances.

Initially \( S = \{s\} \) and \( d(s) = 0, A[s] = d(s) = 0 \)

While \( S \neq V \) 

Select a node \( v \in V - S \) connected via an edge with at least one node in \( S \) for which

\[ d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e \]

Add \( v \) to \( S \) and define \( d(v) = d'(v) \)

Let \( A[v] = d(v) \)

\[ A[x] = 2 \]
Running Example
(KT Figure 5.7)

\[ d'(x) = d(u) + \ell_{u,x} = 2 \]
\[ d'(y) = d(u) + \ell_{u,y} = 4 \]
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**Dijsktra \((G, \ell)\)**

Let \( S \) be the set of explored nodes, \( A \) be a list of distances.

Initially \( S = \{s\} \) and \( d(s) = 0, A[s] = d(s) = 0 \)

While \( S \neq V \)

Select a node \( v \in V - S \) connected via an edge with at least one node in \( S \) for which

\[ d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e \]

Add \( v \) to \( S \) and define \( d(v) = d'(v) \)

Let \( A[v] = d(v) \quad A[x] = 2 \)
Running Example
(KT Figure 5.7)

Dijsktra \((G, \mathcal{E})\)

Let \(S\) be the set of explored nodes, \(A\) be a list of distances.

Initially \(S = \{s\}\) and \(d(s) = 0, A[s] = d(s) = 0\)

While \(S \neq V\)

Select a node \(v \in V - S\) connected via an edge with at least one node in \(S\) for which

\[
d'(v) = \min_{e=(u,v), u \in S} d(u) + \mathcal{E}_e
\]

Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

Let \(A[v] = d(v)\)
Running Example (KT Figure 5.7)

Let $S$ be the set of explored nodes, $A$ be a list of distances.

Initially $S = \{s\}$ and $d(s) = 0, A[s] = d(s) = 0$

While $S \neq V$

Select a node $v \in V \setminus S$ connected via an edge with at least one node in $S$ for which

$$d'(v) = \min_{e=(u,v),u \in S} d(u) + \ell_e$$

Add $v$ to $S$ and define $d(v) = d'(v)$

Let $A[v] = d(v)$
Running Example (KT Figure 5.7)

\[ d'(y) = d(x) + \ell_{(x,y)} = 3 \]

Dijsktra \((G, \ell)\)

Let \(S\) be the set of explored nodes, \(A\) be a list of distances.

Initially \(S = \{s\} \) and \(d(s) = 0, A[s] = d(s) = 0\)

While \(S \neq V\)

Select a node \(v \in V - S\) connected via an edge with at least one node in \(S\) for which

\[ d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e \]

Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

Let \(A[v] = d(v)\)
Running Example
(KT Figure 5.7)

Dijsktra \((G, ℓ)\)

Let \(S\) be the set of explored nodes, \(A\) be a list of distances.

Initially \(S = \{s\}\) and \(d(s) = 0, A[s] = d(s) = 0\)

While \(S \neq V\):

Select a node \(v \in V - S\) connected via an edge with at least one node in \(S\) for which

\[d'(v) = \min_{e=(u,v), u \in S} d(u) + ℓ_e\]

Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

Let \(A[v] = d(v)\)

\[d'(y) = d(x) + ℓ_{(x,y)} = 3\]

\[d'(z) = d(x) + ℓ_{(x,z)} = 4\]
Running Example (KT Figure 5.7)

\[ d'(y) = d(x) + \ell_{(x,y)} = 3 \]
\[ d'(z) = d(x) + \ell_{(x,z)} = 4 \]
Running Example
(KT Figure 5.7)

Dijsktra \((G, \ell)\)

Let \(S\) be the set of explored nodes, \(A\) be a list of distances.

Initially \(S = \{s\}\) and \(d(s) = 0\), \(A[s] = d(s) = 0\)

While \(S \neq V\)

Select a node \(v \in V - S\) connected via an edge with at least one node in \(S\) for which

\[
d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e
\]

Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

Let \(A[v] = d(v)\)

\[
d'(y) = d(x) + \ell_{(x,y)} = 3
\]

\[
d'(z) = d(x) + \ell_{(x,z)} = 4
\]
Running Example
(KT Figure 5.7)

Dijsktra \((G, \ell)\)

Let \(S\) be the set of explored nodes, \(A\) be a list of distances.

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While \(S \neq V\)

Select a node \(v \in V - S\) connected via an edge with at least one node in \(S\) for which

\[
d'(v) = \min_{e = (u, v), u \in S} d(u) + \ell_e
\]

Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

Let \(A[v] = d(v)\)

\[A[y] = 3\]

\(d'(y) = d(x) + \ell_{(x, y)} = 3\)

\(d'(z) = d(x) + \ell_{(x, z)} = 4\)
Running Example
(KT Figure 5.7)

Dijsktra \((G, \ell)\)

Let \(S\) be the set of explored nodes, \(A\) be a list of distances.

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Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

Let \(A[v] = d(v)\)

\(A[y] = 3\)

\[d'(y) = d(x) + \ell_{(x,y)} = 3\]
\[d'(z) = d(x) + \ell_{(x,z)} = 4\]
Running Example (KT Figure 5.7)

Dijsktra \((G, \ell)\)

Let \( S \) be the set of explored nodes, \( A \) be a list of distances.

Initially \( S = \{s\} \) and \( d(s) = 0 \), \( A[s] = d(s) = 0 \)

While \( S \neq V \)

Select a node \( v \in V - S \) connected via an edge with at least one node in \( S \) for which

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d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e
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Add \( v \) to \( S \) and define \( d(v) = d'(v) \)

Let \( A[v] = d(v) \)
Running Example (KT Figure 5.7)

**Dijsktra \((G, \ell)\)**

Let \(S\) be the set of explored nodes, \(A\) be a list of distances.

Initially \(S = \{s\} \text{ and } d(s) = 0, A[s] = d(s) = 0\)

While \(S \neq V\)

Select a node \(v \in V - S\) connected via an edge with at least one node in \(S\) for which

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d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e
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Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

Let \(A[v] = d(v)\)
Running Example (KT Figure 5.7)

Let $S$ be the set of explored nodes, $A$ be a list of distances.

Initially $S = \{s\}$ and $d(s) = 0, A[s] = d(s) = 0$

While $S \neq V$

Select a node $v \in V - S$ connected via an edge with at least one node in $S$ for which

$$d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e$$

Add $v$ to $S$ and define $d(v) = d'(v)$

Let $A[v] = d(v)$

$\begin{array}{cccccccc}
  s & u & v & x & y & z \\
  0 & 1 & 2 & 2 & 3 & \\
\end{array}$

$\text{shortest path distances from } s$

$d'(z) = d(x) + \ell_{(x,z)} = 4$
Running Example
(KT Figure 5.7)

Let $S$ be the set of explored nodes, $A$ be a list of distances.

Initially $S = \{s\}$ and $d(s) = 0$, $A[s] = d(s) = 0$.

While $S \neq V$

Select a node $v \in V - S$ connected via an edge with at least one node in $S$ for which

$$d'(v) = \min_{e=(u,v):u\in S} d(u) + \ell_e$$

Add $v$ to $S$ and define $d(v) = d'(v)$

Let $A[v] = d(v)$

$d'(z) = d(x) + \ell_{(x,z)} = 4$
Running Example
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Let $S$ be the set of explored nodes, $A$ be a list of distances.

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Add $v$ to $S$ and define $d(v) = d'(v)$

Let $A[v] = d(v)$

\[
d'(z) = d(x) + \ell_{(x,z)} = 4
\]
Running Example
(KT Figure 5.7)

\[ d'(z) = d(x) + \ell_{(x,z)} = 4 \]

**Dijkstra** \( (G, \ell) \)

Let \( S \) be the set of explored nodes, \( A \) be a list of distances.

Initially \( S = \{ s \} \) and \( d(s) = 0, A[s] = d(s) = 0 \)

While \( S \neq V \)

Select a node \( v \in V - S \) connected via an edge with at least one node in \( S \) for which

\[ d'(v) = \min_{e=(u,v), u\in S} d(u) + \ell_e \]

Add \( v \) to \( S \) and define \( d(v) = d'(v) \)

Let \( A[v] = d(v) \)

\[ A[z] = 4 \]
Running Example (KT Figure 5.7)

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\[
\begin{align*}
    d'(v) &= \min_{e=(u,v), u \in S} d(u) + \ell_e \\
    A[v] &= d'(v) \\
    d(v) &= d'(v)
\end{align*}
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Running Example (KT Figure 5.7)

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Running Example (KT Figure 5.7)

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While $S \neq V$

Select a node $v \in V - S$ connected via an edge with at least one node in $S$ for which

$$d'(v) = \min_{e=(u,v), u \in S} d(u) + \ell_e$$

Add $v$ to $S$ and define $d(v) = d'(v)$

Let $A[v] = d(v)$
Are we done?
Are we done?

- **Output:** For every node $u$ in $V$, a shortest path $s \sim u$ from $s$ to $u$. 
Are we done?

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• To be more precise, a list of paths:
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- To be more precise, a list of paths:

| \( P(s) \) | \( P(u_1) \) | \( P(u_2) \) | \( P(u_3) \) | \( \ldots \) |
Are we done?

- **Output:** For every node \( u \) in \( V \), a shortest path \( s \sim u \) from \( s \) to \( u \).

- To be more precise, a list of paths:

\[
\begin{array}{c|c|c|c|c|c}
P(s) & P(u_1) & P(u_2) & P(u_3) & \cdots \\
\hline
s & u & v & x & y & z \\
0 & 1 & 2 & 2 & 3 & 4 \\
\end{array}
\]
• Output: For every node $u$ in $V$, a shortest path $s\sim u$ from $s$ to $u$.

• To be more precise, a list of paths:

<table>
<thead>
<tr>
<th>$P(s)$</th>
<th>$P(u_1)$</th>
<th>$P(u_2)$</th>
<th>$P(u_3)$</th>
<th>...</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>$s$</th>
<th>$u$</th>
<th>$v$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

This is only a list of shortest path lengths, not the paths themselves!
From lengths to paths
From lengths to paths

• When we add a node $v$ to $S$, we record the edge $(u,v)$ that led us to explore $v$. 
From lengths to paths

• When we add a node $v$ to $S$, we record the edge $(u,v)$ that led us to explore $v$.

• This is enough to recursively recover the path $P_v$. $P_v$ is just $P_u + (u,v)$. In turn, $P_u$ is $P_w + (w,u)$, where $w$ is the node from which we explored $u$, and so on.
Correctness

**Theorem:** Consider the set $S$ at any point in the execution of the algorithm. For each $u \in S$, the path $P_u$ is a shortest $s-u$ path.
Correctness

**Theorem:** Consider the set $S$ at any point in the execution of the algorithm. For each $u \in S$, the path $P_u$ is a shortest $s-u$ path.

This is enough to prove correctness. *Why?*
Correctness

**Theorem:** Consider the set $S$ at any point in the execution of the algorithm. For each $u \in S$, the path $P_u$ is a shortest $s$-$u$ path.
Correctness

**Theorem:** Consider the set $S$ at any point in the execution of the algorithm. For each $u \in S$, the path $P_u$ is a shortest $s-u$ path.

Proof by *induction* on the size of $S$. 
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Proof by *induction* on the size of $S$.

**Base Case:** $|S| = 1$
Running Example
(KT Figure 5.7)

shortest path distances from $s$
Correctness

**Theorem:** Consider the set \( S \) at any point in the execution of the algorithm. For each \( u \in S \), the path \( P_u \) is a shortest \( s-u \) path.

Proof by **induction** on the size of \( S \).

Base Case: \( |S| = 1, \ S=\{s\}, \ d(s) = 0 \), trivially shortest path.
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**Base Case:** $|S| = 1$, $S = \{s\}$, $d(s) = 0$, trivially shortest path.

**Induction Hypothesis:** Assume that it holds for $|S| = k$ for some $k \geq 1$. 
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**Induction Step:** We will prove that it holds for $|S| = k + 1$. 
Adding a node

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shortest path distances from $s$
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(KT Figure 5.7)

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Let $v^*$ be the node added to $S$. Let $(u^*, v^*)$ be the final edge on the path $P_{v^*}$.

Assume by contradiction that $P_{v^*}$ is not a shortest $s-v^*$ path.
Adding a node

**Induction Step:** We will prove that it holds for $|S| = k + 1$

Let $v^*$ be the node added to $S$. Let $(u^*, v^*)$ be the final edge on the path $P_{v^*}$.

Assume by contradiction that $P_{v^*}$ is *not* a shortest $s$-$v^*$ path.

That means that there exists some other path $P$ that is shorter than $P_{v^*}$.
Adding a node

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**Fact:** $P$ starts in $S$ and must leave $S$ at some point. Why?
Adding a node

Assume by contradiction that $P_{v^*}$ is not a shortest $s$-$v^*$ path.

That means that there exists some other path $P$ that is shorter than $P_{v^*}$.

**Fact:** $P$ starts in $S$ and must leave $S$ at some point. **Why?**

Let $y^*$ be the first node of $P$ that is not in $S$. 
Adding a node

Assume by contradiction that $P_{v*}$ is not a shortest $s-v^*$ path.

That means that there exists some other path $P$ that is shorter than $P_{v^*}$.

**Fact:** $P$ starts in $S$ and must leave $S$ at some point. Why?

Let $y^*$ be the first node of $P$ that is not in $S$.

Let $x^*$ be the node “just before” $y^*$, i.e., the last node of $P$ before it leaves $S$. 
Let $v^*$ be the node added to $S$. Let $(u^*, v^*)$ be the final edge on the path $P_{v^*}$.

Let $y^*$ be the first node of $P$ that is not in $S$.

Let $x^*$ be the node “just before” $y^*$, i.e., the last node of $P$ before it leaves $S$. 
Correctness
Correctness

We know that the $s-x^*$ path $P_{x^*}$ is a shortest path. Why?
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Correctness

**Theorem:** Consider the set $S$ at any point in the execution of the algorithm. For each $u \in S$, the path $P_u$ is a shortest $s-u$ path.

Proof by induction on the size of $S$.

**Base Case:** $|S| = 1$, $S = \{s\}$, $d(s) = 0$, trivially shortest path.

**Induction Hypothesis:** Assume that it holds for $|S| = k$ for some $k \geq 1$.

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We now that the $s - x^*$ path $P_{x^*}$ is a shortest path. Why? By the induction hypothesis.
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Therefore $\ell(P) \geq \ell(P') + \ell(x^*, y^*) \geq \ell(x^*, y^*) + d(x^*) \geq d'(y^*)$
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Therefore $\ell(P) \geq d'(y^*)$

At the same time, we know that $d'(y^*) \geq d'(v^*) = \ell(P_{v^*})$.

Why?
Dijkstra’s Algorithm (Pseudocode)

Dijkstra \((G, \ell)\)

Let \(S\) be the set of explored nodes, \(A\) be a list of distances.

Initially \(S = \{s\}\) and \(d(s) = 0, A[s] = d(s) = 0\)

While \(S \neq V\)

Select a node \(v \in V - S\) connected via an edge with at least one node in \(S\) such that

\[
d'(v) = \min_{e=(u,v):u \in S} d(u) + \ell_e \]

Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

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Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

Let \(A[v] = d(v)\)
Correctness

Therefore $\ell(P) \geq d'(y^*)$

At the same time, we know that $d'(y^*) \geq d'(v^*) = \ell(P_{v^*})$.

Why? Because $v^*$ was chosen by Dijkstra's Algorithm.
Putting it together

Therefore $\ell(P) \geq d'(y^*)$

At the same time, we know that $d'(y^*) \geq \ell(P_{v^*}).$

That implies that $\ell(P) \geq \ell(P_{v^*})$, a contradiction.
Therefore \( \ell(P) \geq d'(y^*) \)

At the same time, we know that \( d'(y^*) \geq \ell(P_v^*) \).

That implies that \( \ell(P) \geq \ell(P_v^*) \), a contradiction.
Running Time

Let's look at the pseudocode.
Dijkstra’s Algorithm
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Let \(S\) be the set of explored nodes, \(A\) be a list of distances.

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While \(S \neq V\)  **How many iterations here?**

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Add \(v\) to \(S\) and define \(d(v) = d'(v)\)

Let \(A[v] = d(v)\)

Here, consider every node \(v\) outside \(S\), and then consider all edges between \(S\) and \(v\).
Dijkstra’s Algorithm (Pseudocode)

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While \(S \neq V\)  

**How many iterations here?**  

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\(|E| = m\)
**Dijkstra's Algorithm (Pseudocode)**

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**Overall:** \(O(nm)\).
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Let \(A[v] = d(v)\)

Overall: \(O(nm)\).  
Not terrible, not great.
Running Time

Let's look at the pseudocode.

That was somewhat naive. Can we do better?
Priority Queues

- The values denote *priorities*.

- For Max-Priority Queues, the elements with the largest values are those with the highest priority.
Priority Queues

• **Priority queue**: A data structure that maintains
  
  • A set of elements $S$.
  
  • Each with an associated value, $\text{key}(v)$.
  
  • The values denote *priorities*.
  
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Priority Queues

- Priority queue: A data structure that maintains
  - A set of elements $S$.
  - Each with an associated value, $\text{key}(v)$.
  - The values denote *priorities*.
  
- For *Min-Priority* Queues, the elements with the smallest values are those with the highest priority.
Priority Queue Operations

- Insert($Q, v$) inserts a new item $v$ in the priority queue.

- FindMin($Q$) finds the element with the maximum priority (the smallest value) in the priority queue and returns it (but does not remove it).

- ExtractMin($Q$) finds the element with the maximum priority (smallest value) in the priority queue, returns it, and deletes it from the queue.
Priority Queue Operations

- \textbf{ExtractMin}(Q)\ finds the element with the maximum priority (smallest value) in the priority queue, returns it, and deletes it from the queue.

- \textbf{ChangeKey}(Q, v, a)\ changes the key value of element \( v \) to \( \text{key}(v) = a \).
Running Time

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That was somewhat naive. Can we do better?
Running Time

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That was somewhat naive. Can we do better?

1. Once we have computed $d'(v) = \min_{(u,v):u \in S} d(u) + \ell_e$, we store it, so we don’t have to compute it again.
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That was somewhat naive. Can we do better?

1. Once we have computed $d'(v) = \min_{(u,v): u \in S} d(u) + \ell_e$, we store it, so we don’t have to compute it again.

2. Place the nodes in a priority queue, with $d'(v)$ as the key of $v$. 
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How do we get the node \( v \) to be added to \( S \)?
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\[ \text{ExtractMin}(Q) \]
Running Time

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How do we get the node \( v \) to be added to \( S \)?

\[ \text{ExtractMin}(Q) \]

**Issue:** Once we add nodes to \( S \), we need to update the values \( d'(v) \), i.e., the keys in the priority queue.
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**Case 1:** $(v, w) \notin E$.

$$\min_{e=(u,w):u\in S} d(u) + \ell_e$$ has not changed!
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(this is because when we explore $w$ in the future, it will not be via $v$).
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Consider an iteration when $v$ is added to $S$.

Let $w \in V - S$ be a node that remains in the priority queue.
Running Time

**Issue:** Once we add nodes to $S$, we need to update the values $d'(v)$, i.e., the keys in the priority queue.

Consider an iteration when $v$ is added to $S$.

Let $w \in V - S$ be a node that remains in the priority queue.

Case 2: $(v, w) \in E.$
Running Time

**Issue:** Once we add nodes to $S$, we need to update the values $d'(v)$, i.e., the keys in the priority queue.

Consider an iteration when $v$ is added to $S$.

Let $w \in V - S$ be a node that remains in the priority queue.

**Case 2:** $(v, w) \in E$.

$$d'(w) = \min(d'(w), d(v) + \ell_{(v,w)})$$
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**ChangeKey**($Q, v, a$)  At most once per edge!
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That was somewhat naive. Can we do better?
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Overall: $O(m \log n)$
Reading

Kleinberg and Tardos Chapter 5.4. (or 4.4. in the online weird version). *Slides follow this religiously.*

Roughgarden 9.2., 9.3.

CLRS 24.3.

You can also find visualisers online and play around with them, e.g., [https://www.cs.usfca.edu/~galles/visualization/Dijkstra.html](https://www.cs.usfca.edu/~galles/visualization/Dijkstra.html) and the more general [https://visualgo.net/en/sssp?slide=1](https://visualgo.net/en/sssp?slide=1)