Introduction to Algorithms and Data Structures

Greedy Algorithms: Dijkstra's algorithm for shortest paths

Going to the EICC

What is the fastest way to go from the School of Informatics to the EICC?

Shortest Paths in Graphs

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- **Input:** A directed graph $G = (V, E)$, and a designated node s in V. We also assume that every node u in V is reachable from s. We are also given a length ℓ_e for every edge e in E.

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$P(s)$	$P(u_1)$	$P(u_2)$	$P(u_3)$	...

What about undirected graphs?

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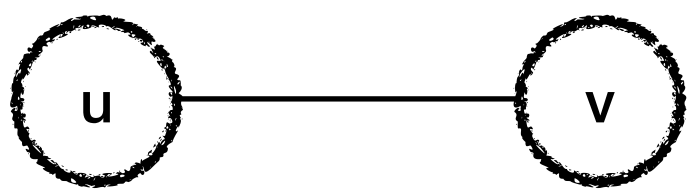

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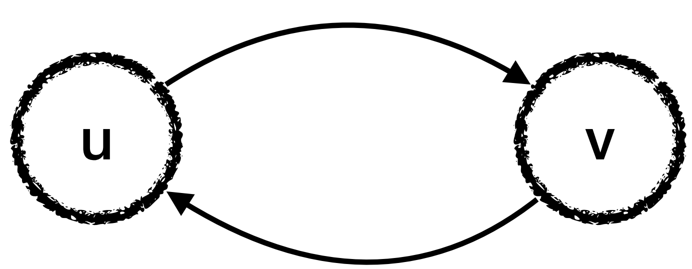
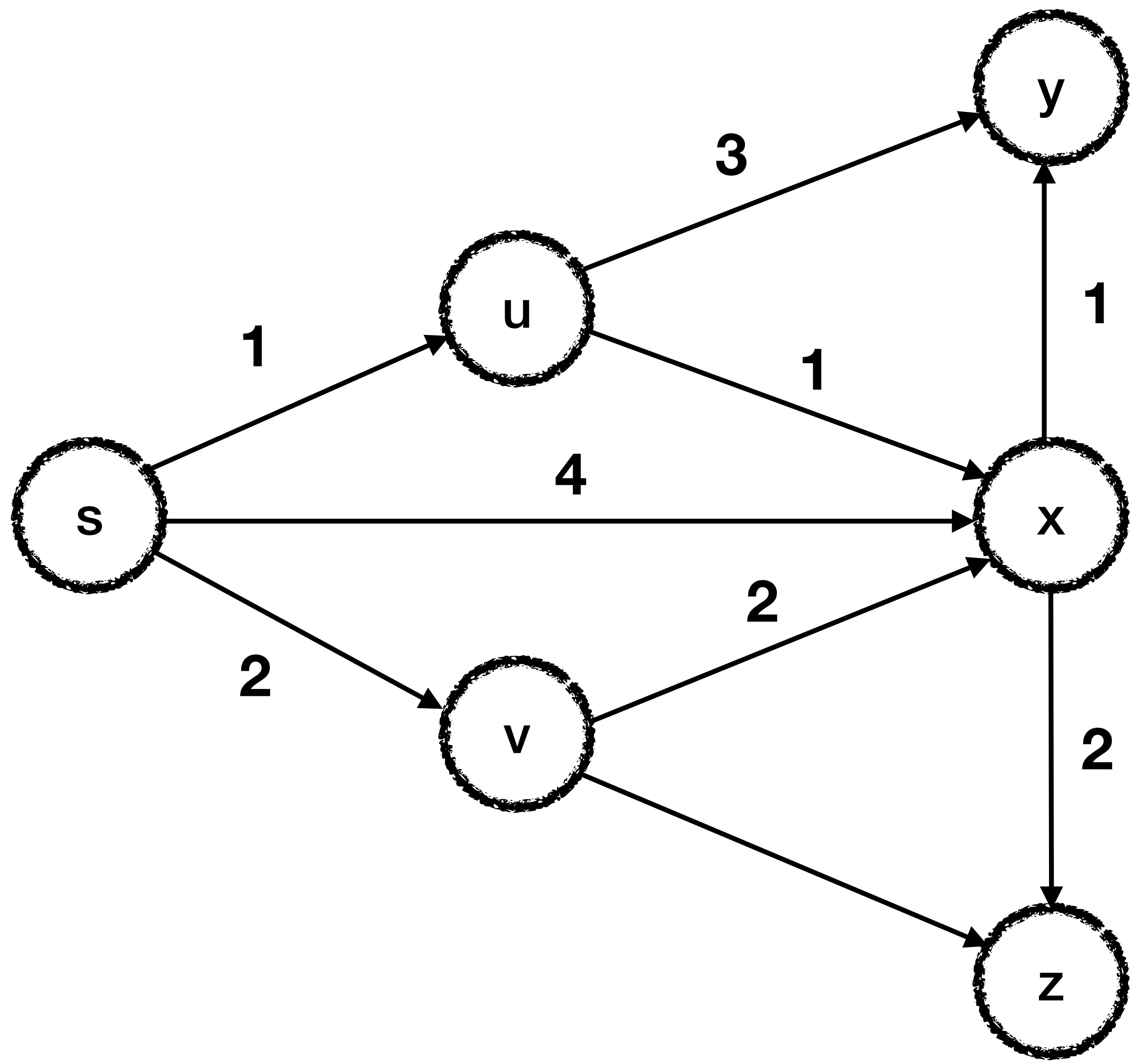


Running Example (KT Figure 5.7)



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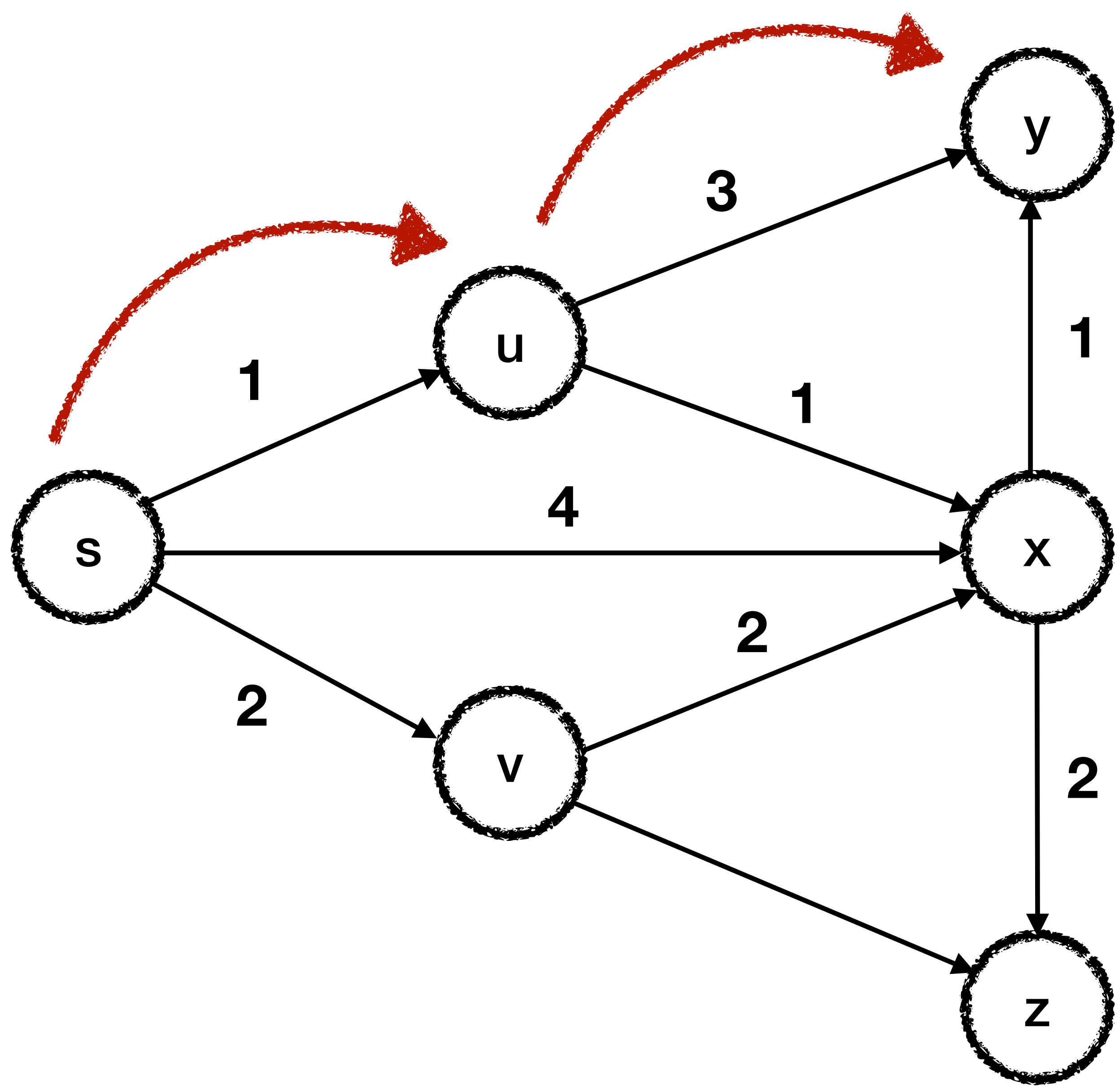

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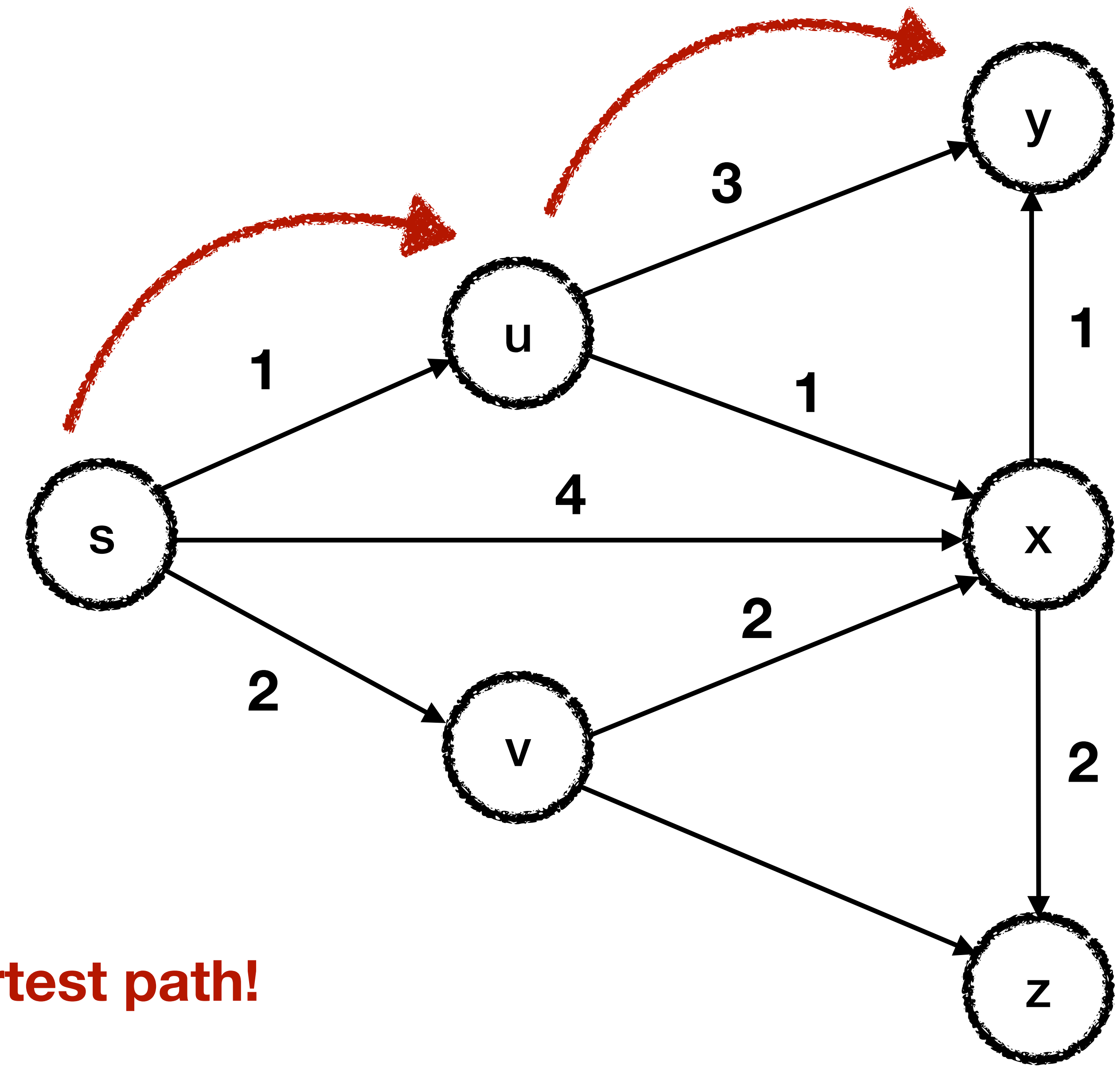

not a shortest path!

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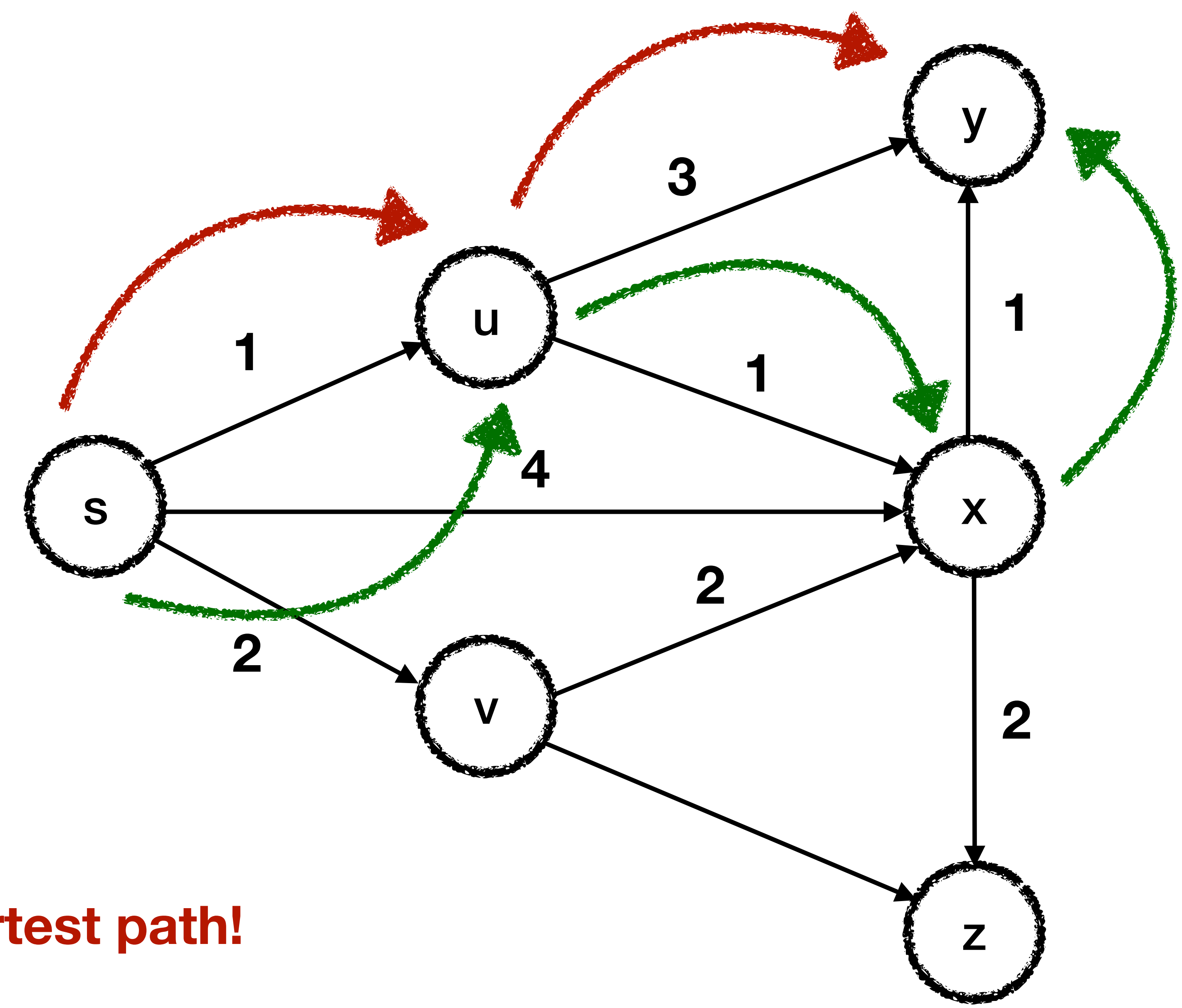

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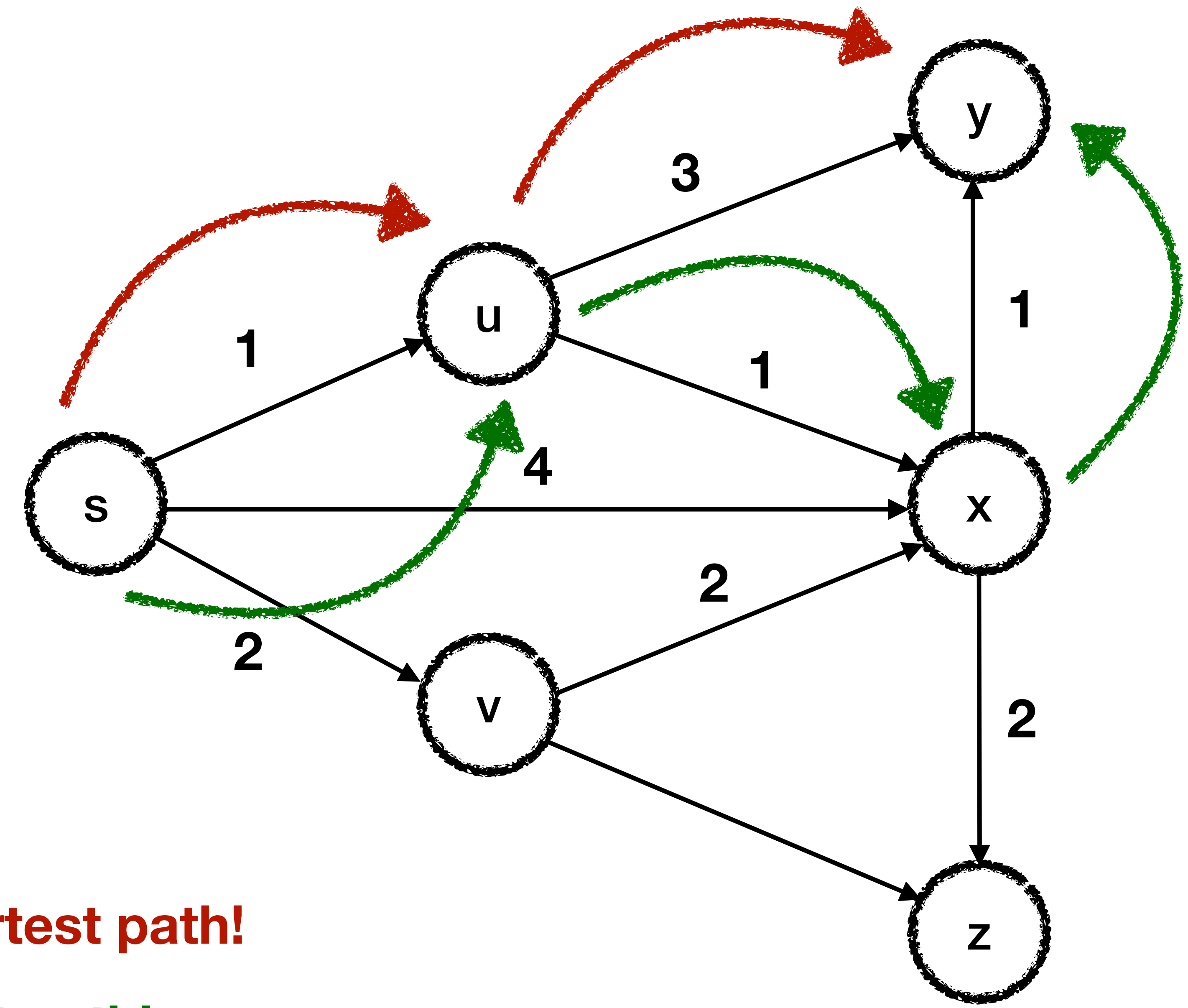

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Dijkstra’s Algorithm

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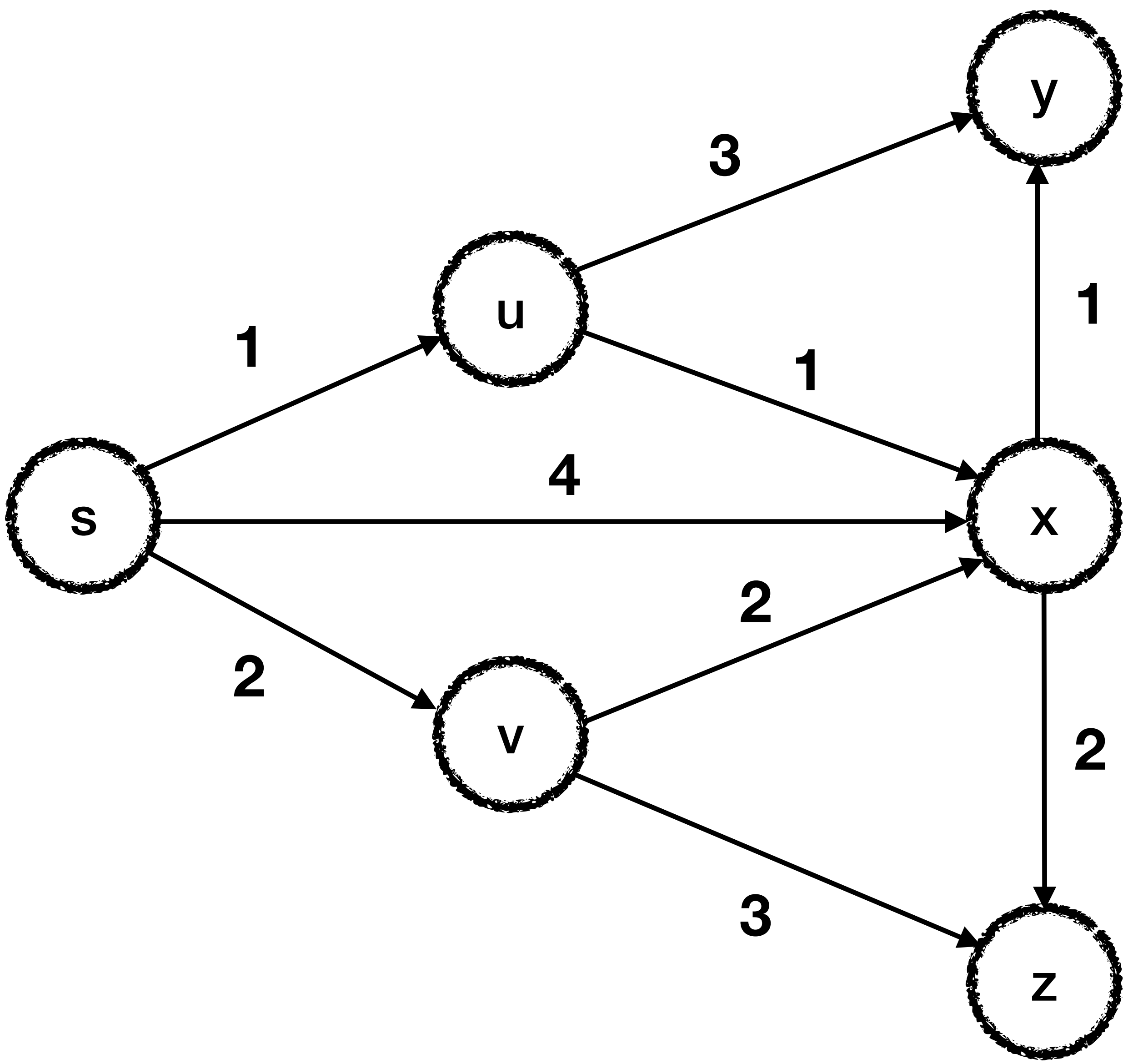
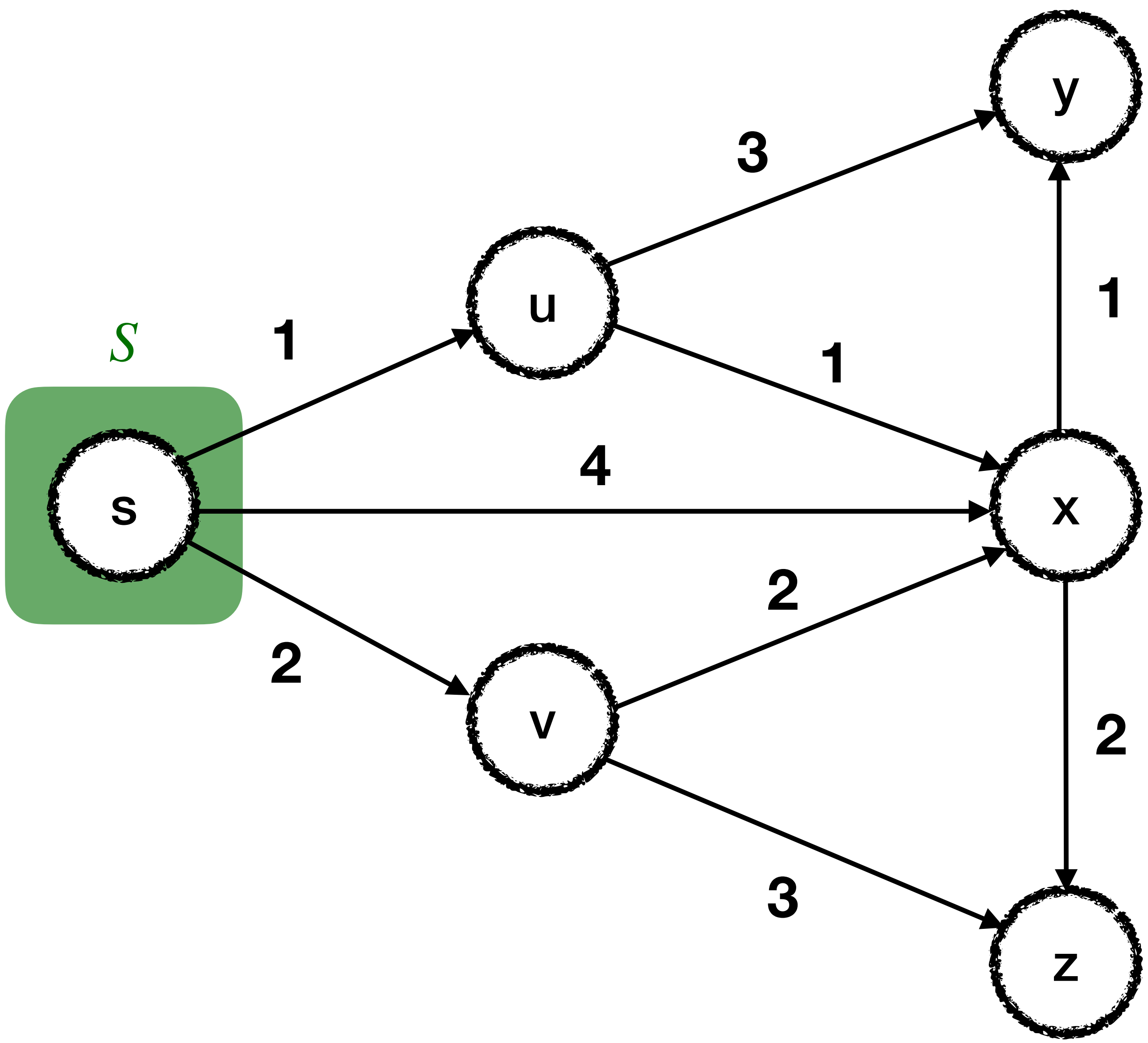
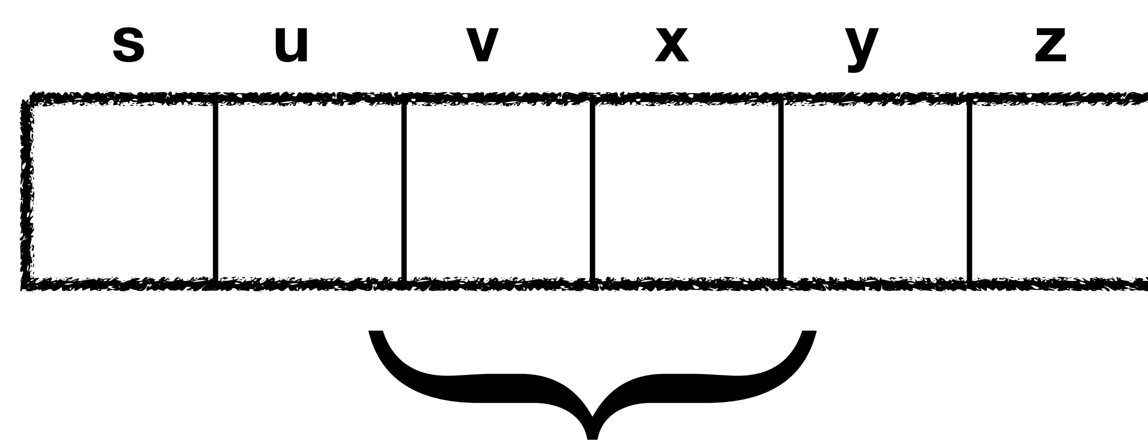


s	u	v	x	y	z

shortest path distances from s

Running Example (KT Figure 5.7)





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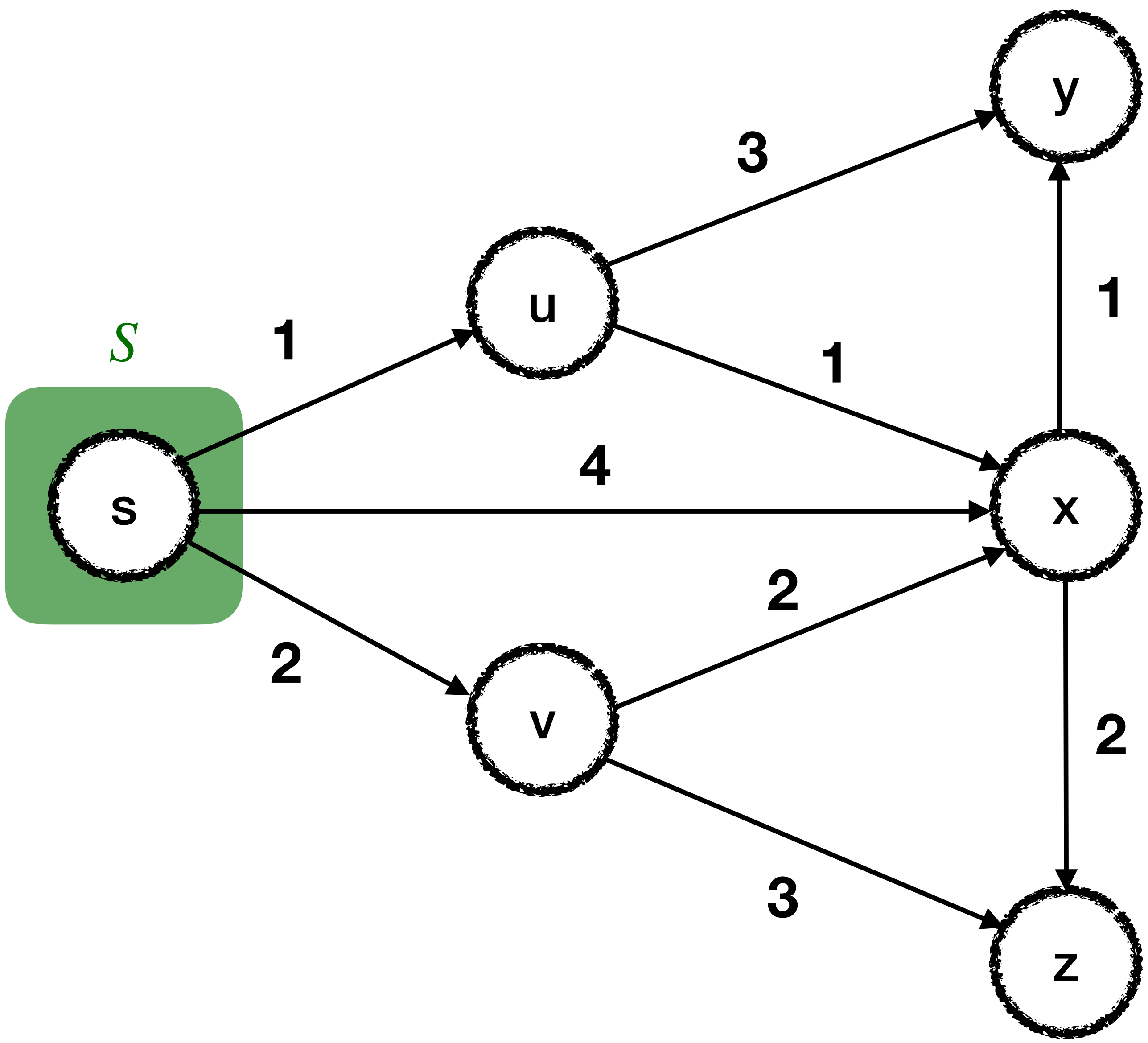

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0					

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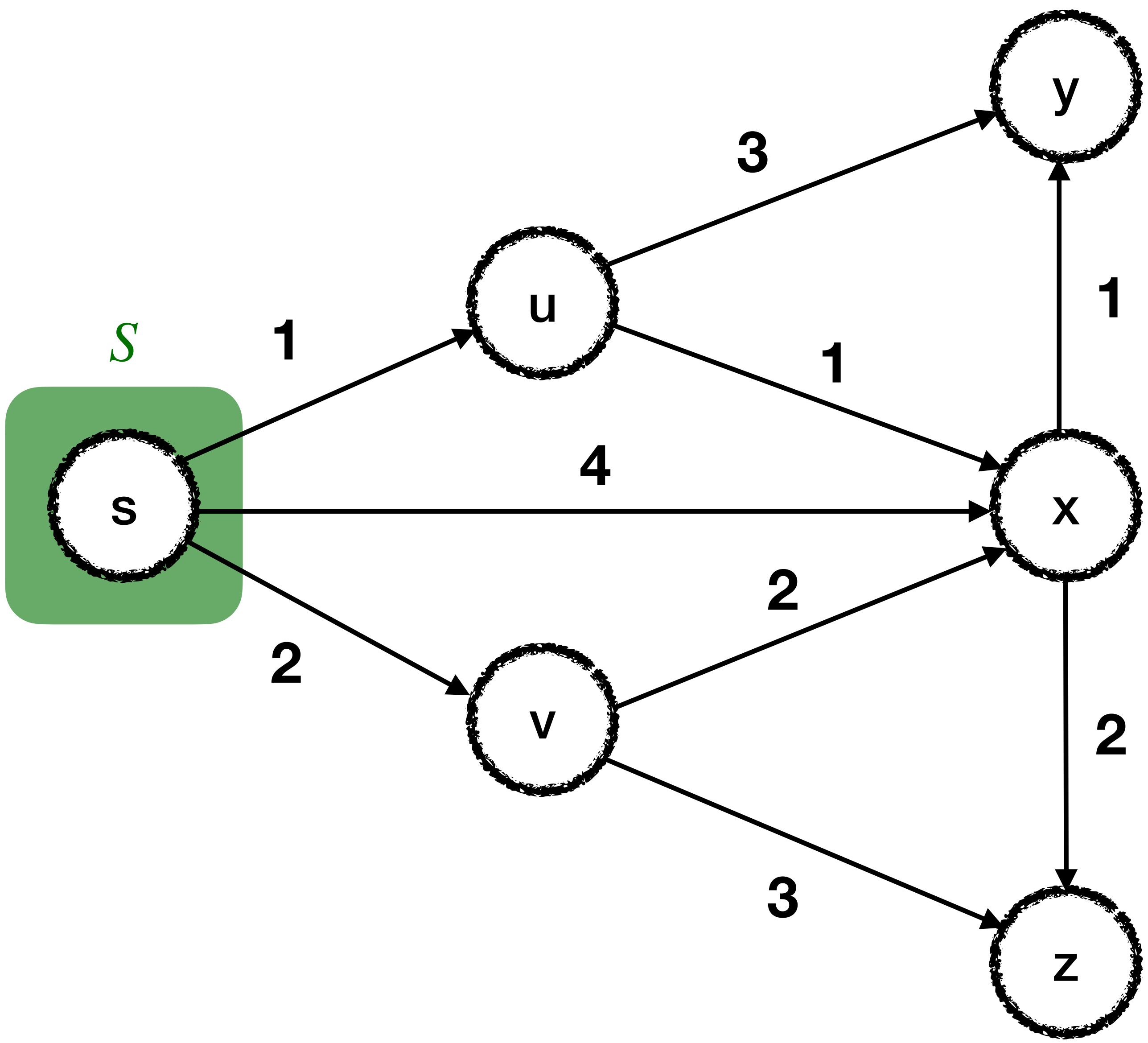

s	u	v	x	y	z
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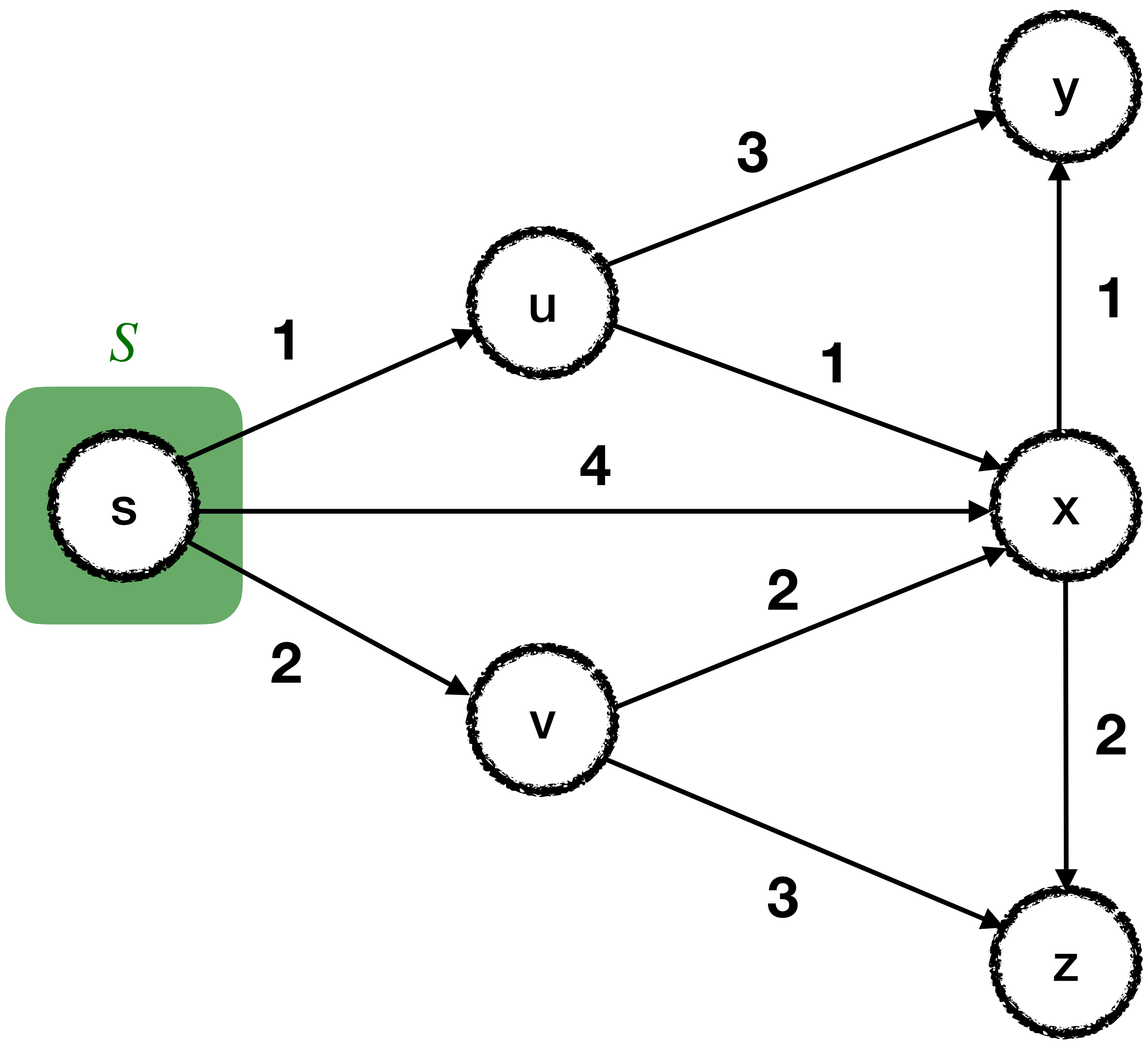

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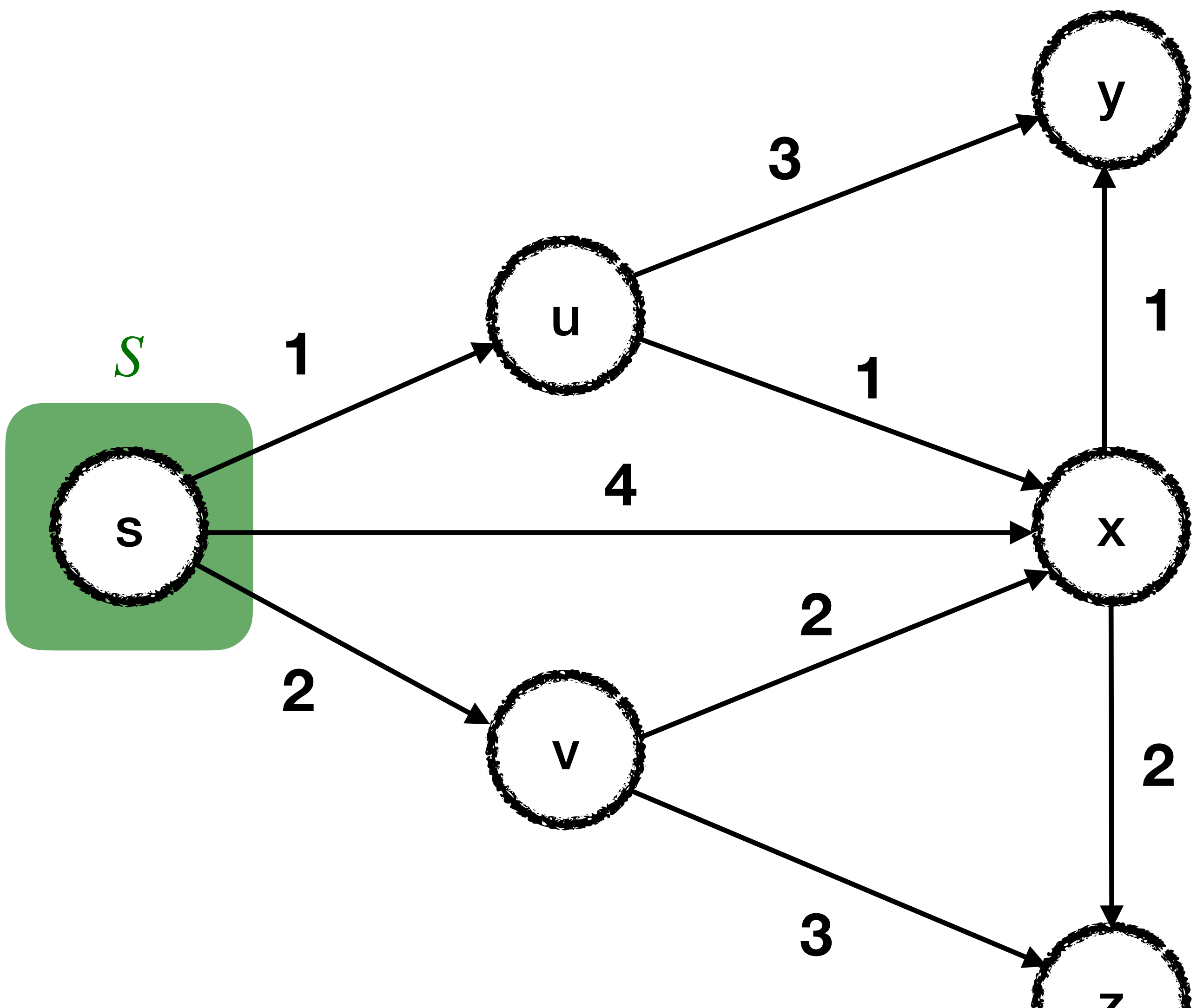

Iteration 1

s	u	v	x	y	z
0					

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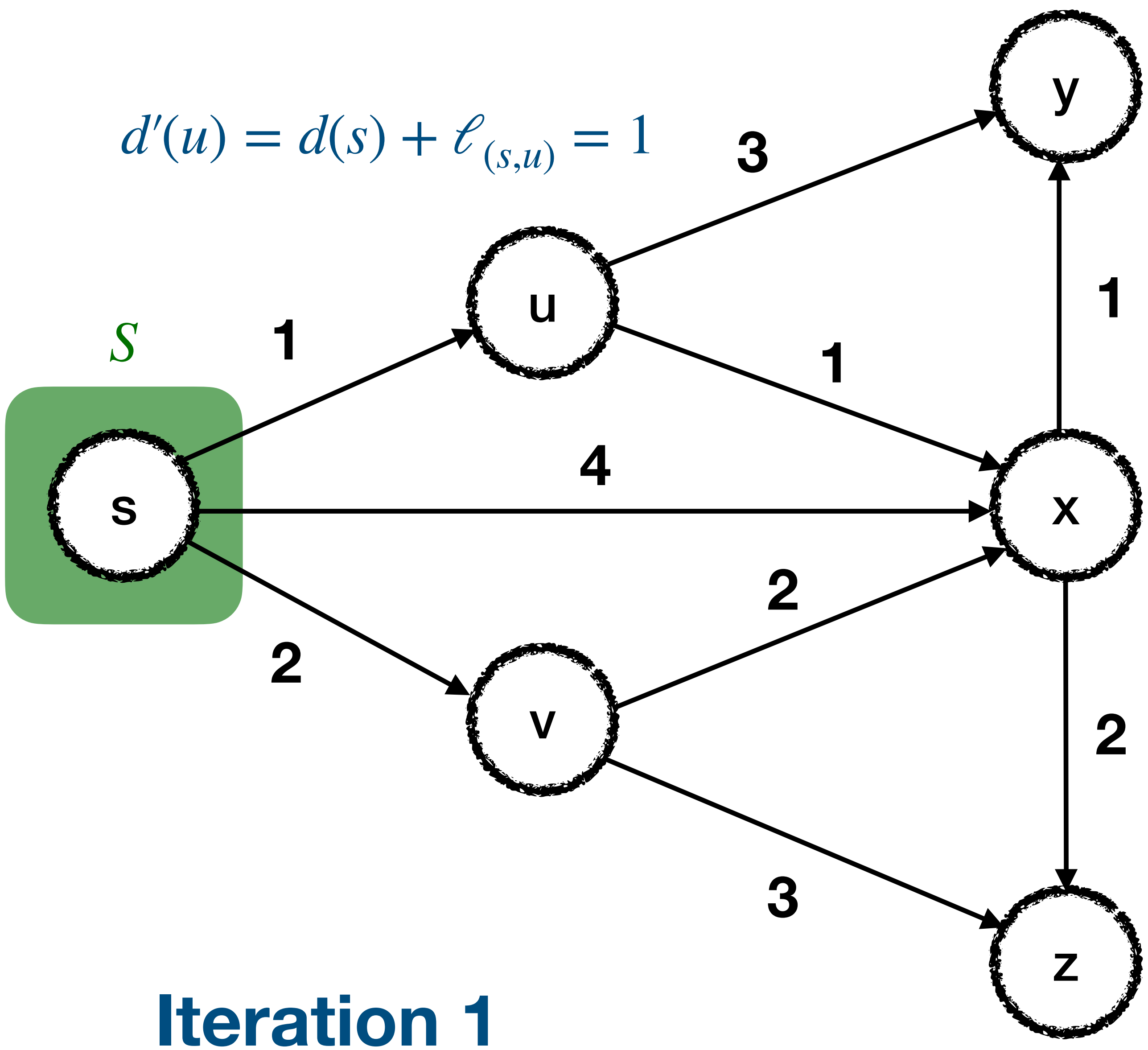
Running Example (KT Figure 5.7)

$d'(u) = d(s) + \ell_{(s,u)} = 1$



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0					

$\underbrace{\hspace{6em}}$

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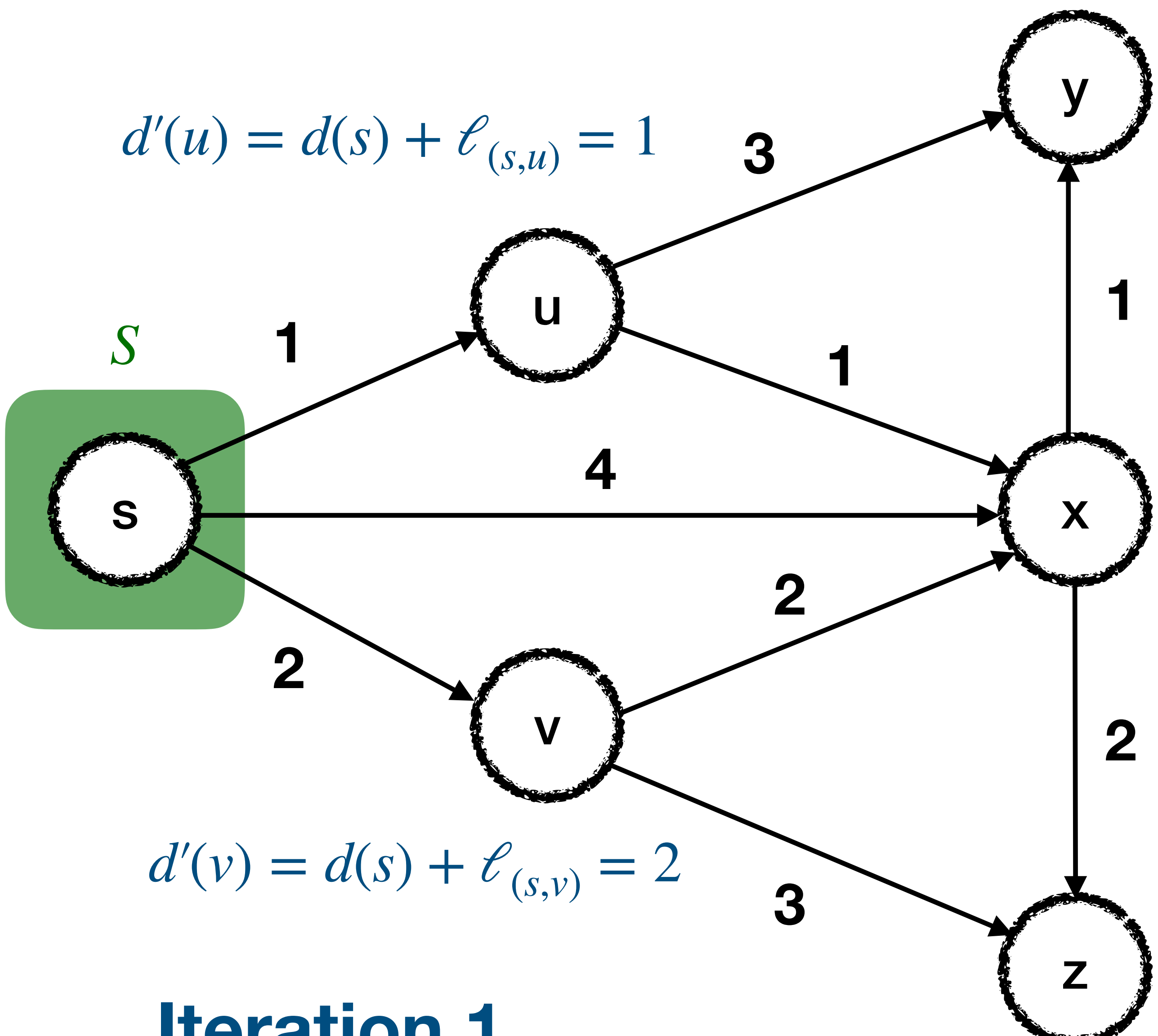

$d'(v) = d(s) + \ell_{(s,v)} = 2$

Iteration 1

s	u	v	x	y	z
0					

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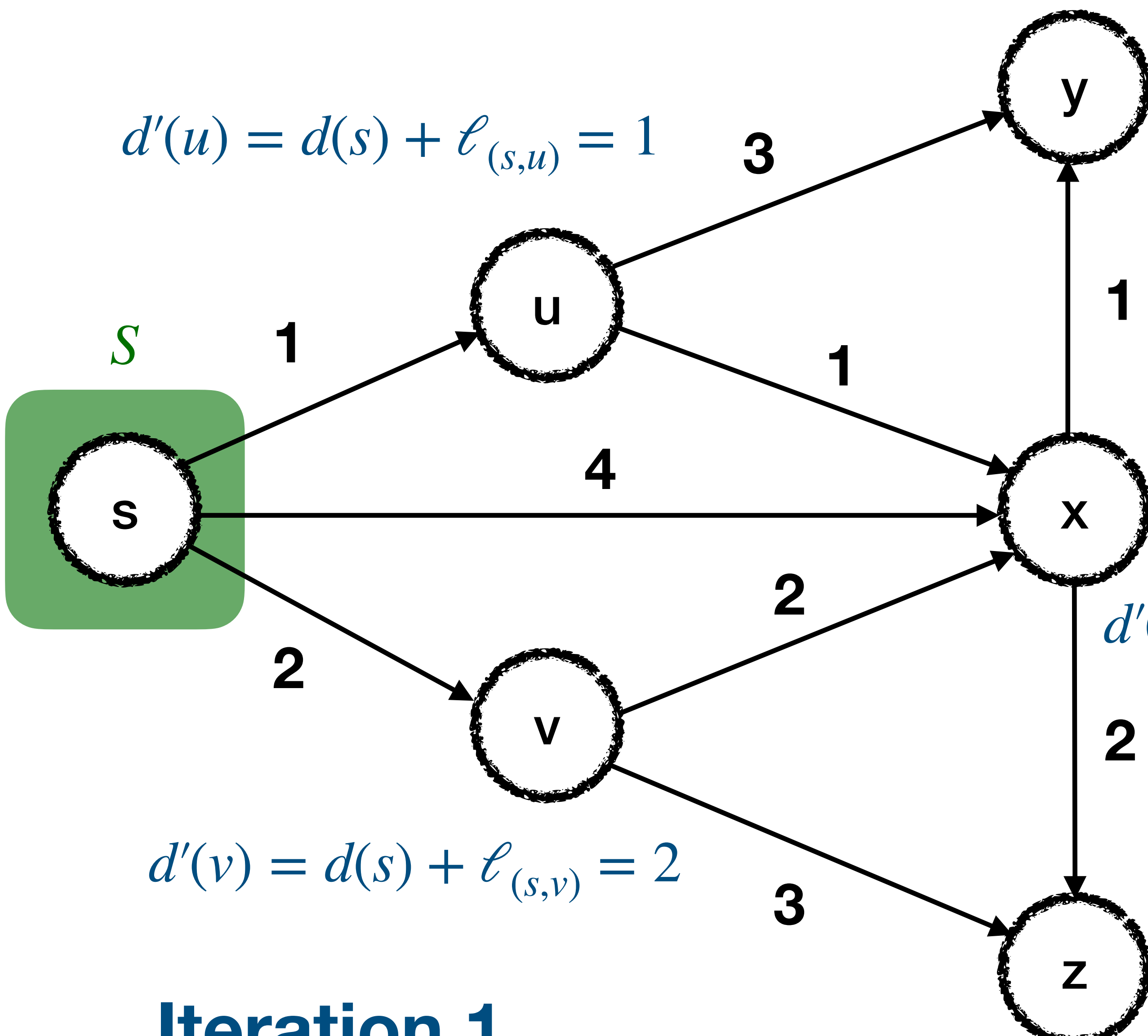

$d'(u) = d(s) + \ell_{(s,u)} = 1$

$d'(x) = d(s) + \ell_{(s,x)} = 4$

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Iteration 1

s	u	v	x	y	z
0					

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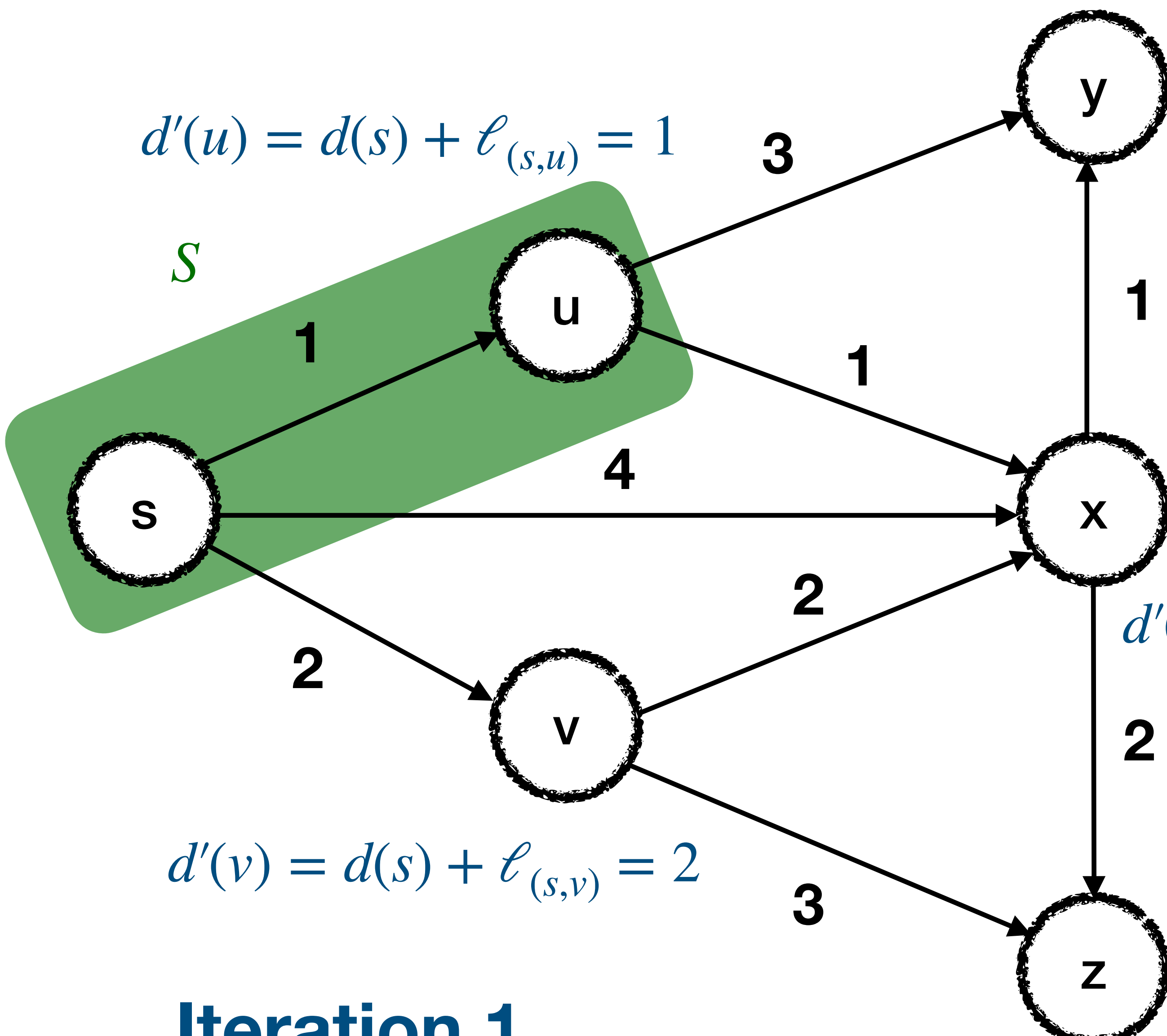

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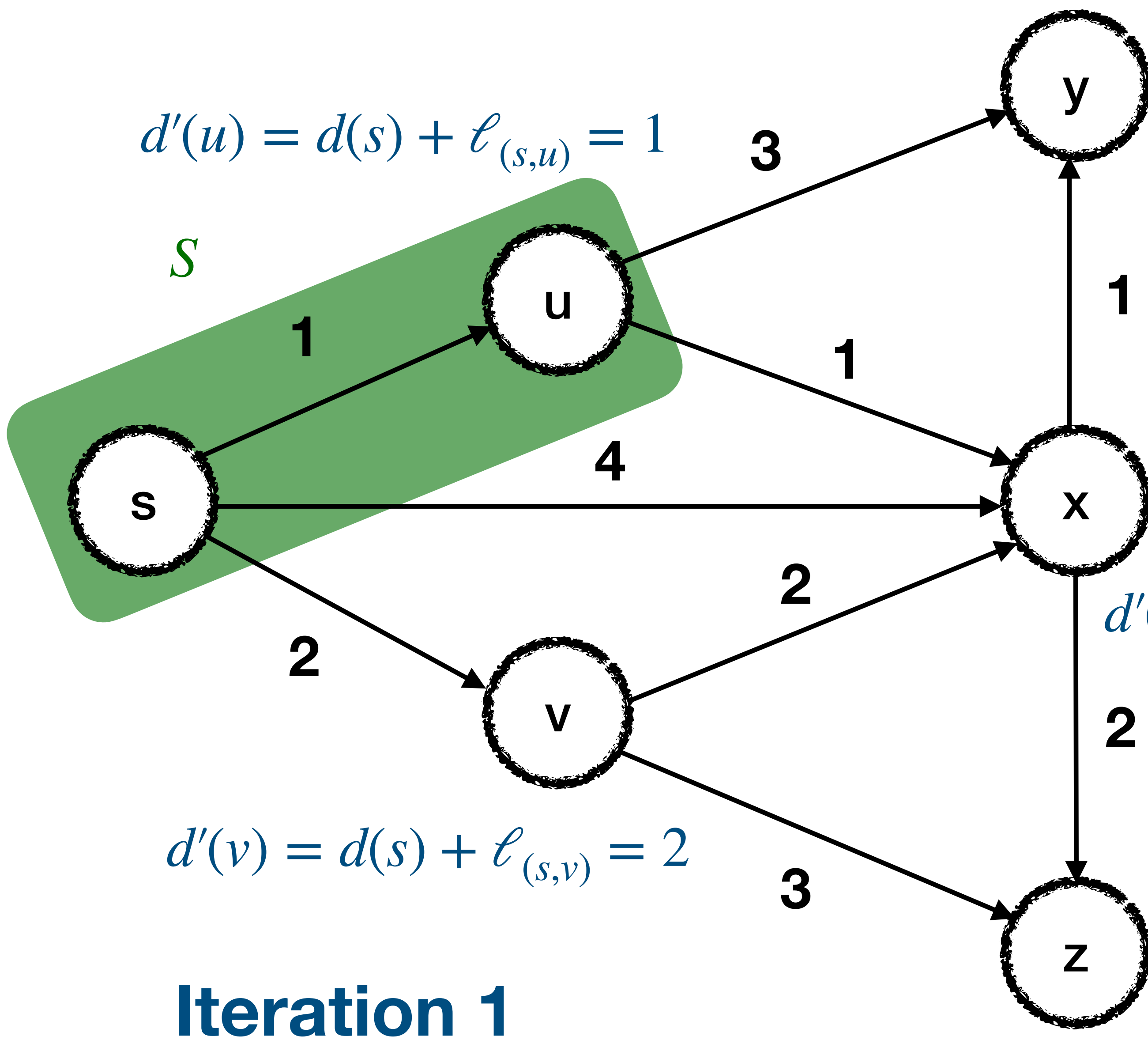

Iteration 1

s	u	v	x	y	z
0	1				

shortest path distances from s

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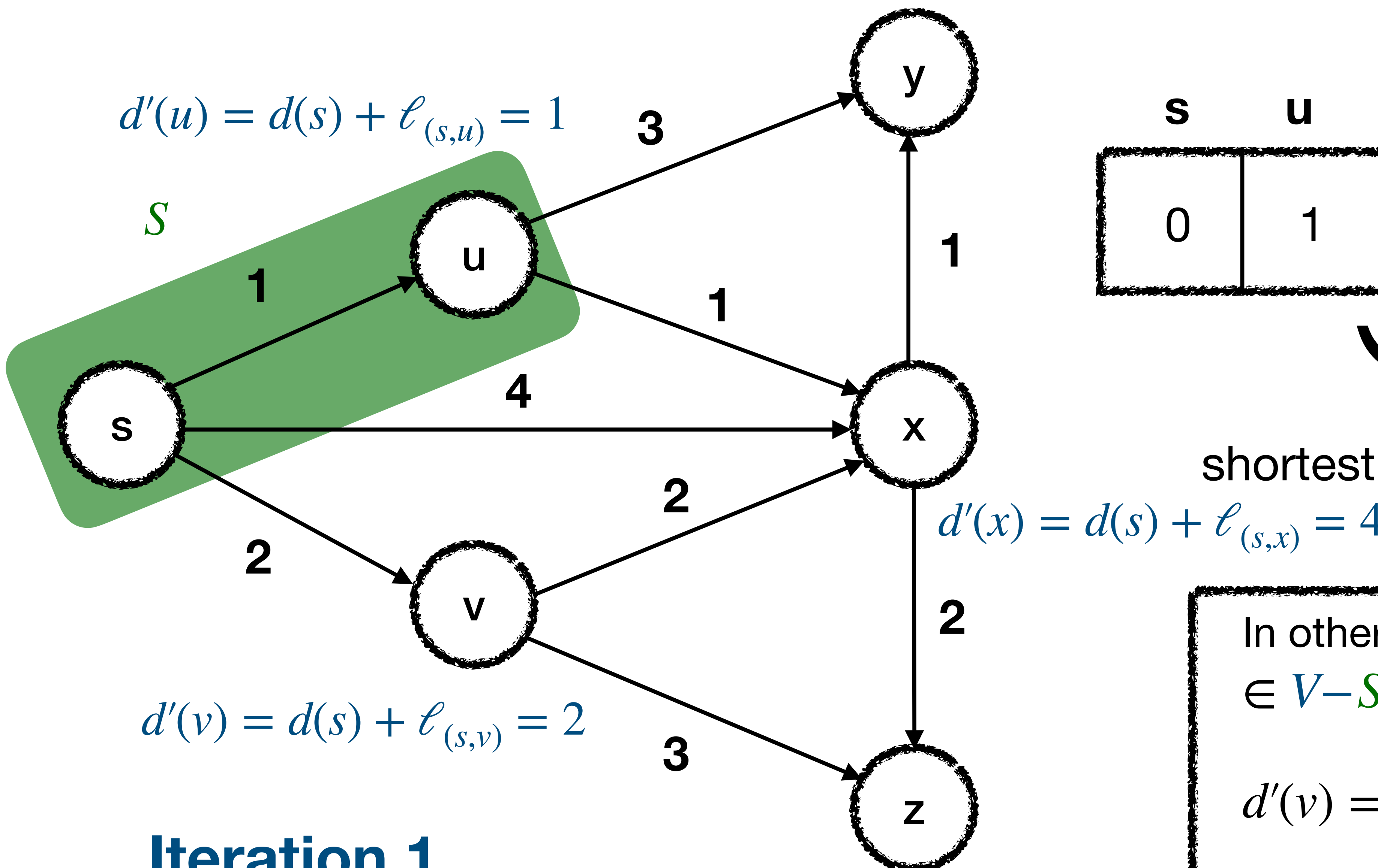

s	u	v	x	y	z
0	1				

shortest path distances from s

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Iteration 1

Iteration 2

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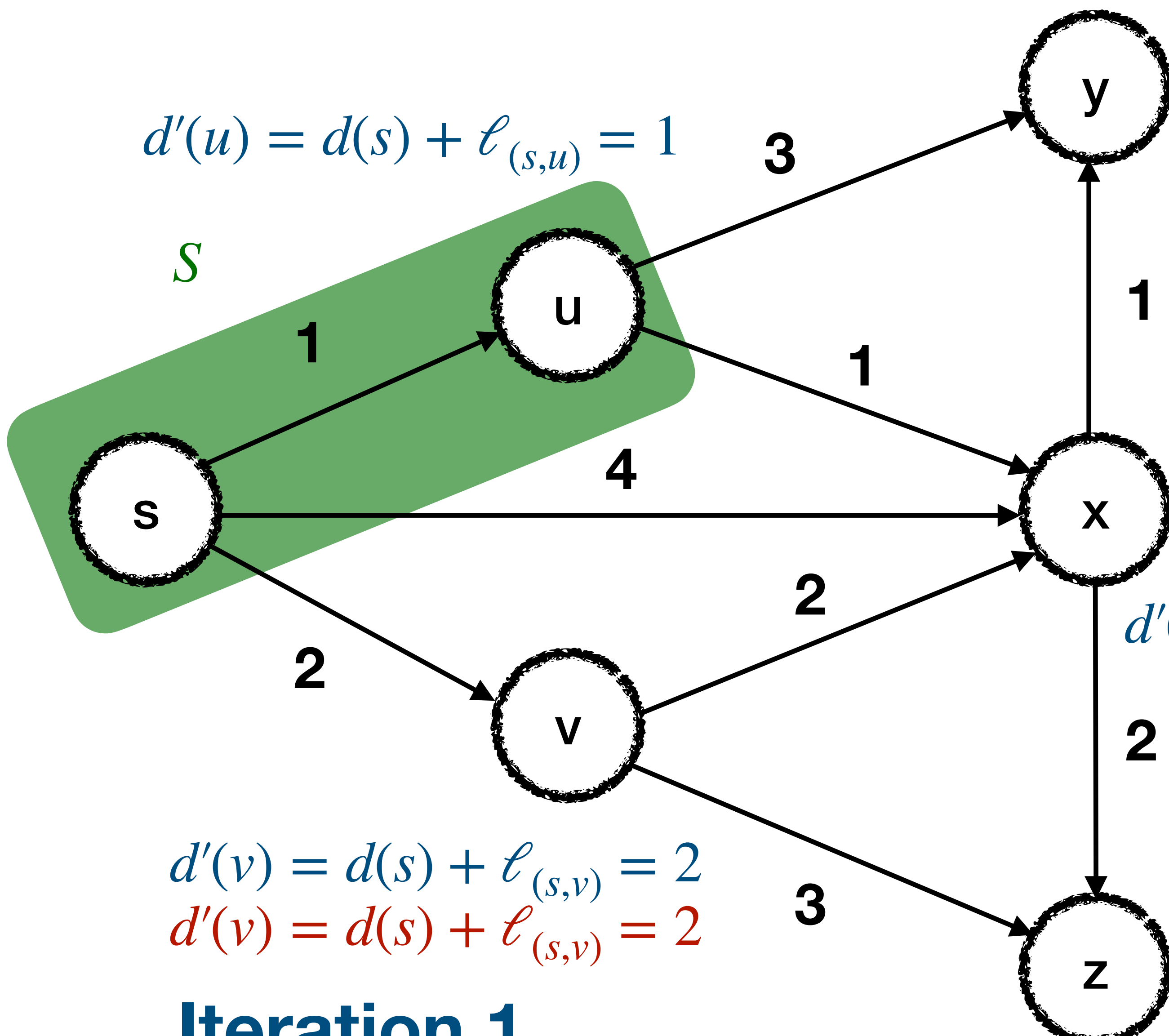

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$d'(v) = d(s) + \ell_{(s,v)} = 2$

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Iteration 1

Iteration 2

s	u	v	x	y	z
0	1				

$\underbrace{\qquad\qquad}$ shortest path distances from s

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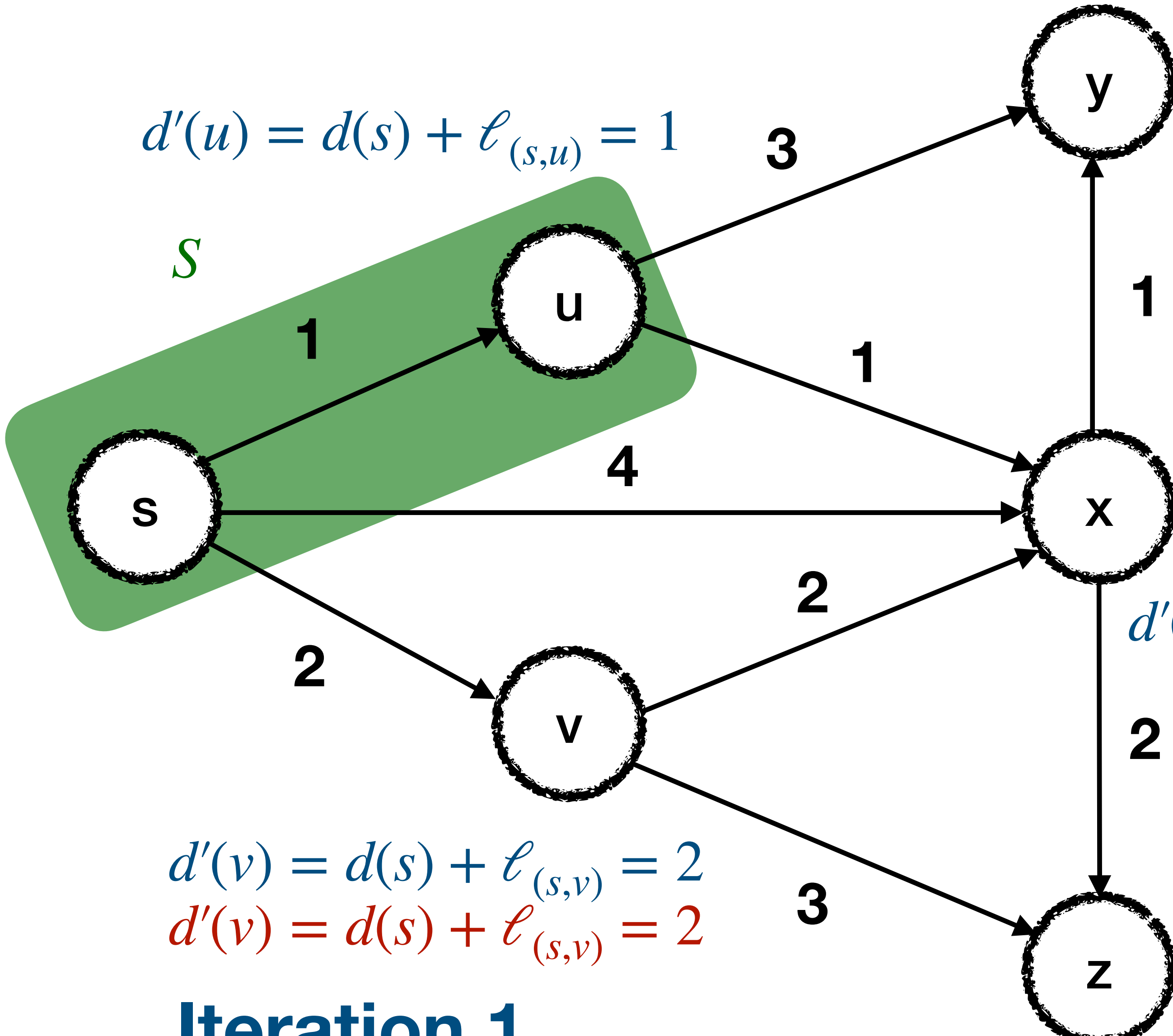
Running Example (KT Figure 5.7)

$d'(y) = d(u) + \ell_{(u,y)} = 4$

$d'(u) = d(s) + \ell_{(s,u)} = 1$

S



$d'(x) = d(s) + \ell_{(s,x)} = 4$

$d'(v) = d(s) + \ell_{(s,v)} = 2$

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Iteration 1

Iteration 2

s	u	v	x	y	z
0	1				

shortest path distances from s

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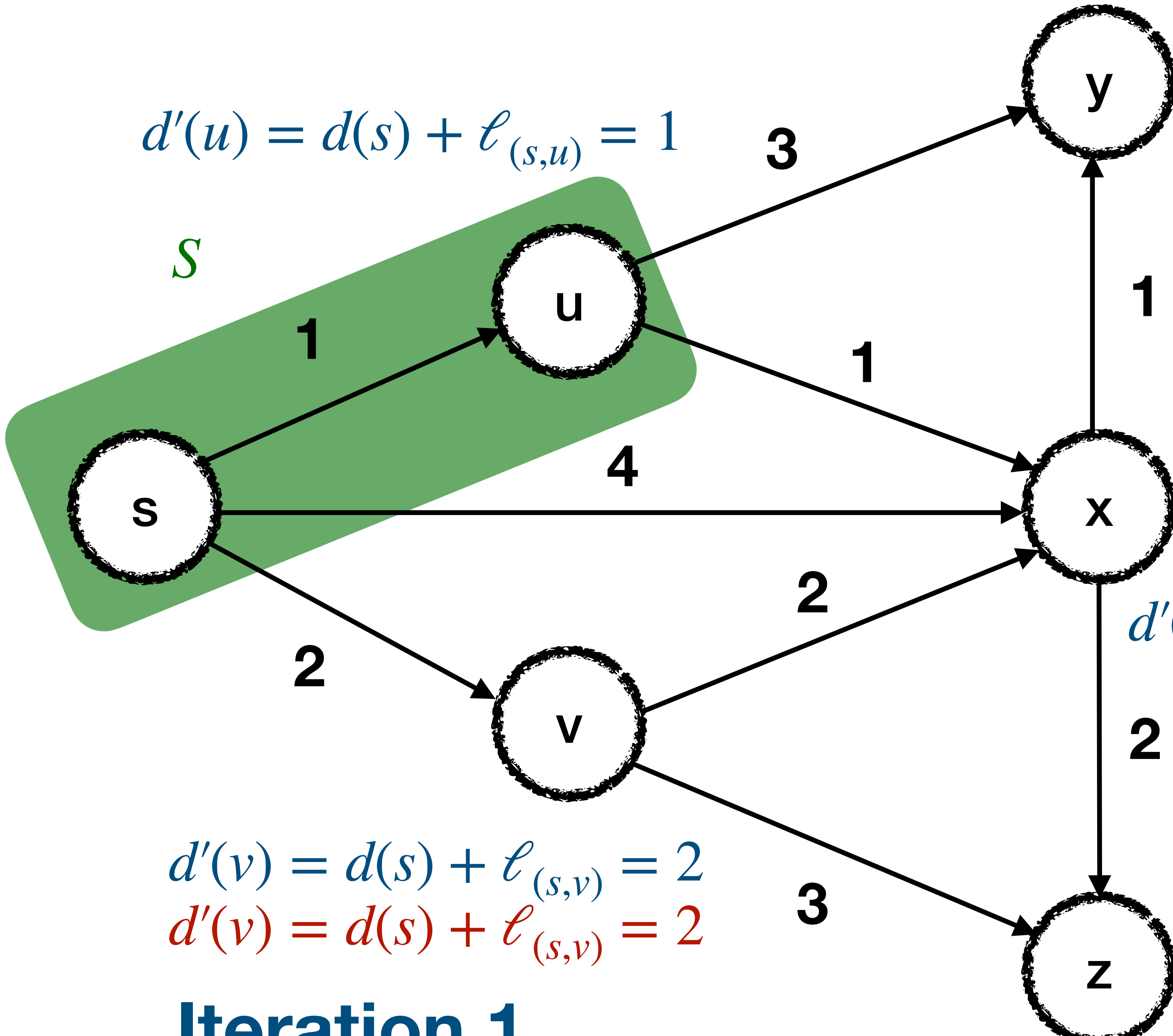

$d'(v) = d(s) + \ell_{(s,v)} = 2$

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Iteration 1

Iteration 2

s	u	v	x	y	z
0	1				

shortest path distances from s

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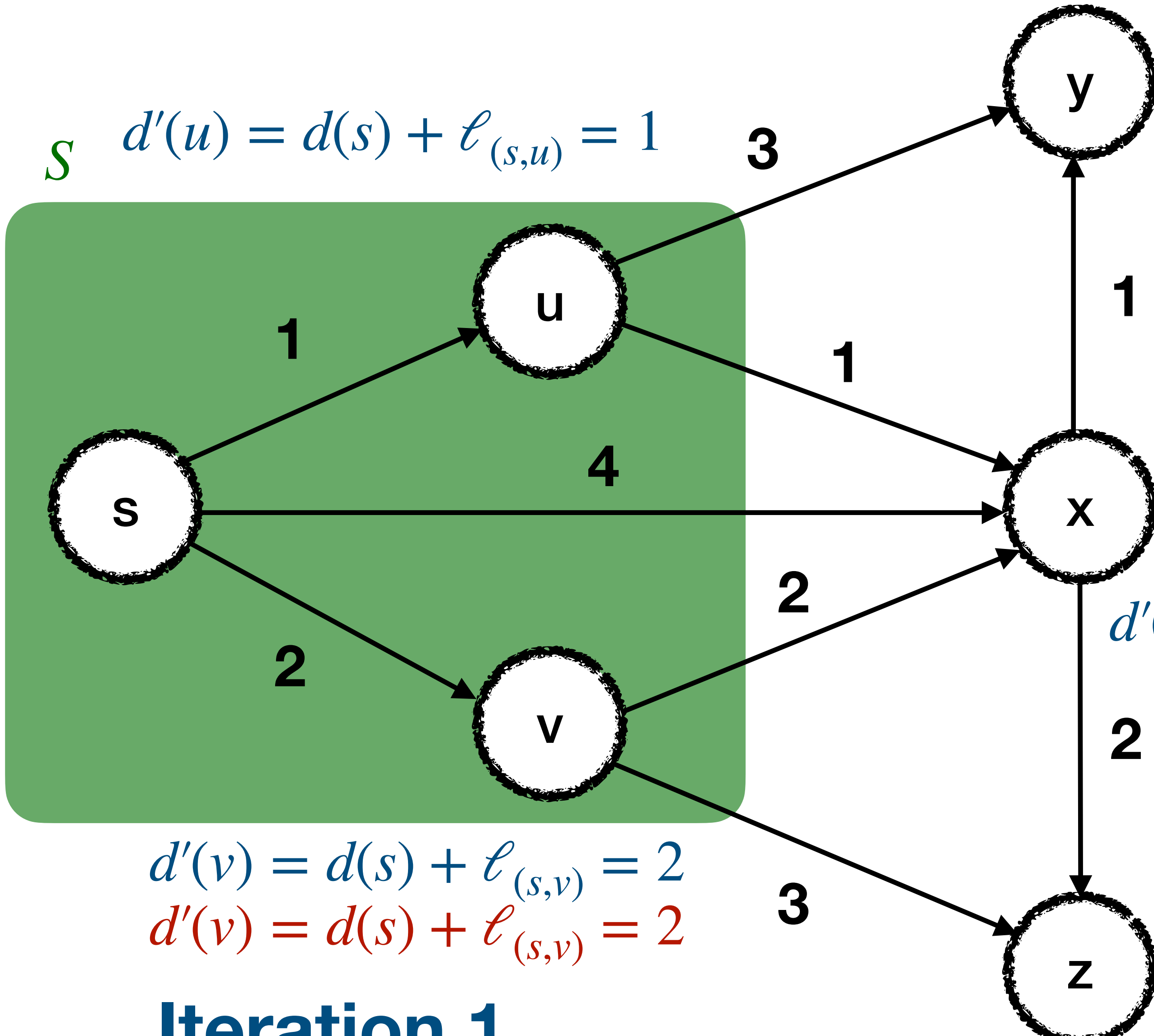
$d'(u) = d(s) + \ell_{(s,u)} = 1$



s	u	v	x	y	z
0	1				

shortest path distances from s

$d'(x) = d(s) + \ell_{(s,x)} = 4$ $\quad d'(x) = d(u) + \ell_{(u,x)} = 2$

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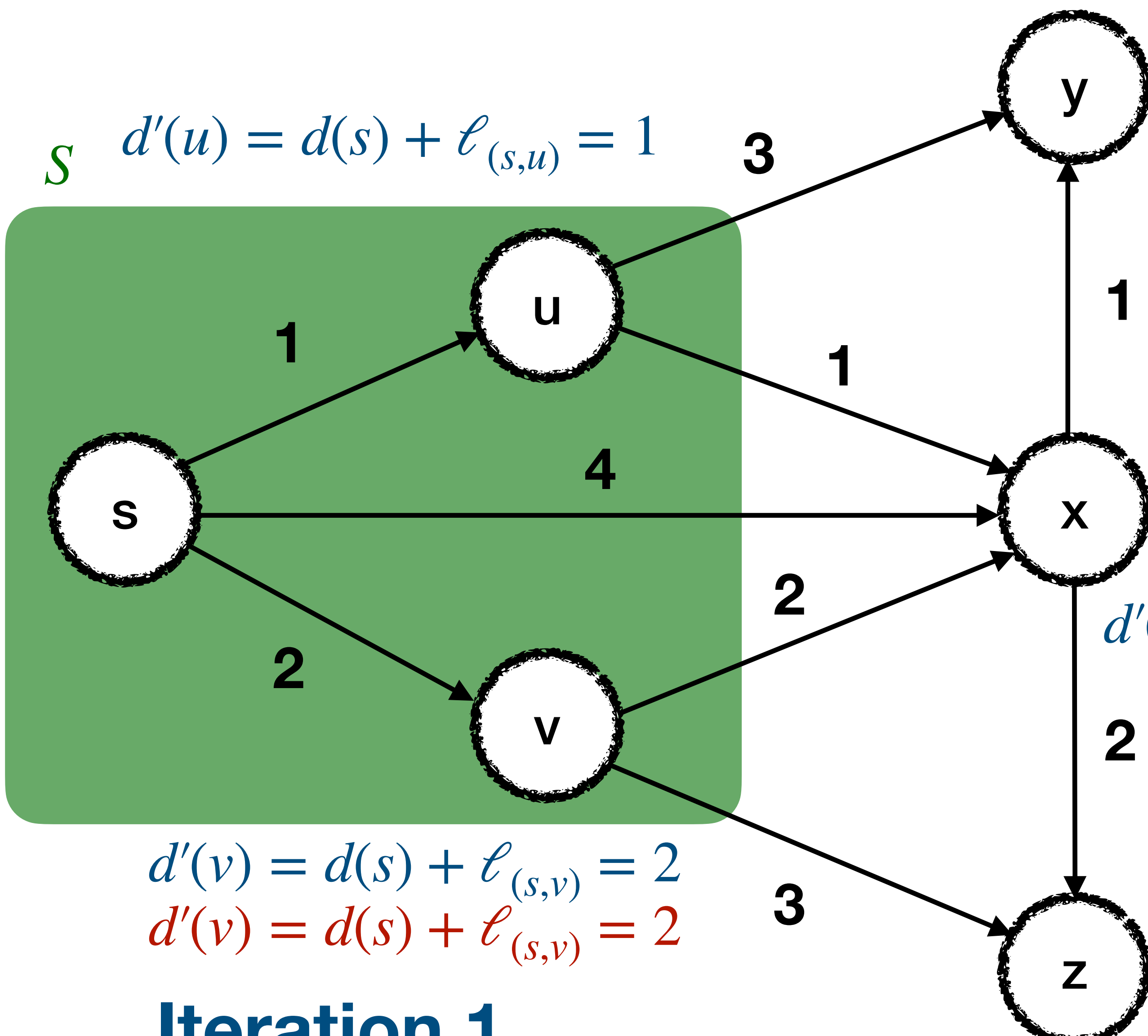
S $d'(u) = d(s) + \ell_{(s,u)} = 1$



$d'(v) = d(s) + \ell_{(s,v)} = 2$

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s	u	v	x	y	z
0	1	2			

shortest path distances from s

$d'(x) = d(s) + \ell_{(s,x)} = 4$ $d'(x) = d(u) + \ell_{(u,x)} = 2$

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Dijkstra's Algorithm (Pseudocode)

<u>Dijsktra (G, ℓ)</u>

Let S be the set of explored nodes, A be a list of distances.

Initially $S = \{s\}$ and $d(s) = 0$, $A[s] = d(s) = 0$

While $S \neq V$

Select a node $v \in V - S$ connected via an edge with at least one node in S for which

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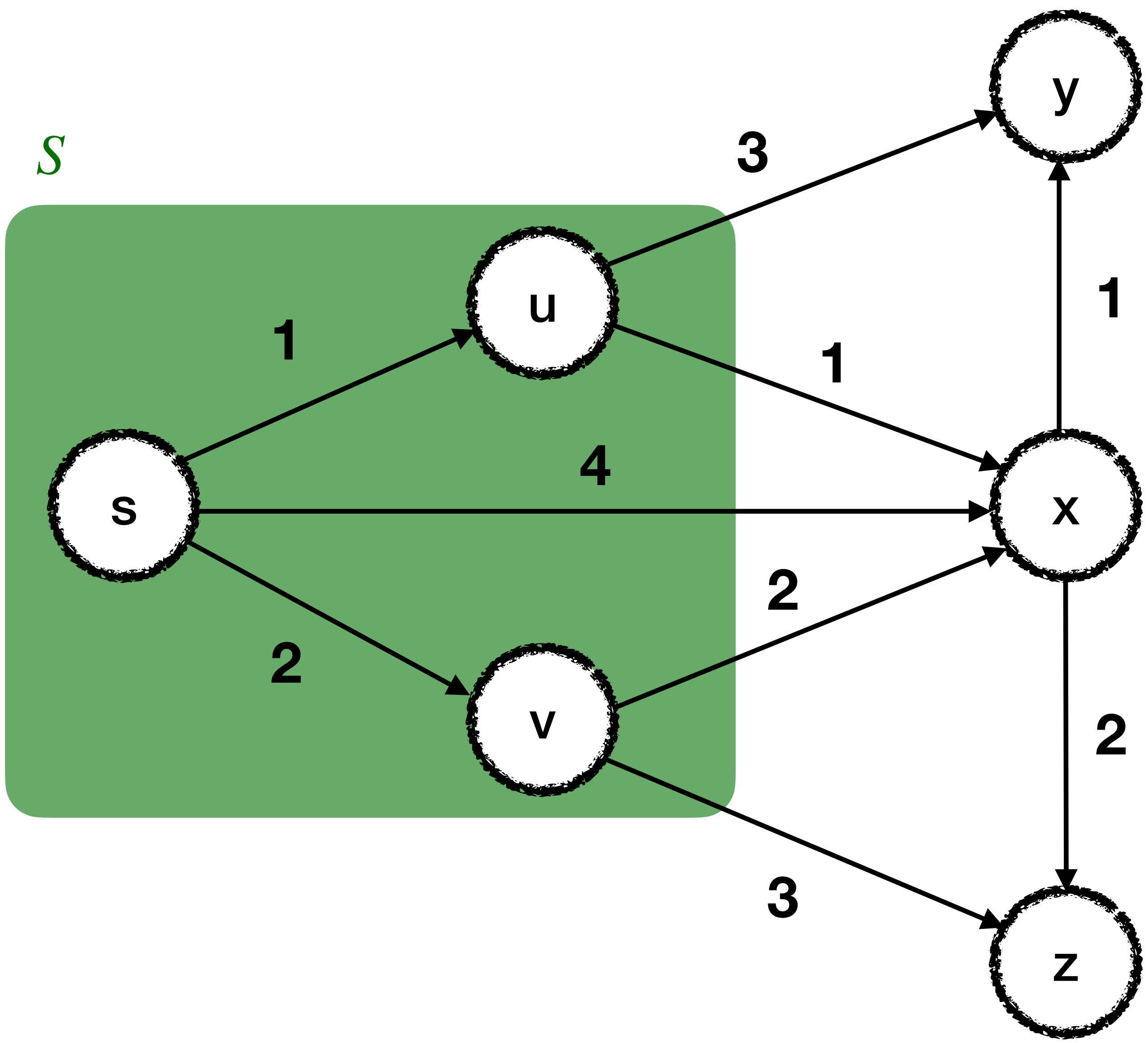

s	u	v	x	y	z
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shortest path distances from s

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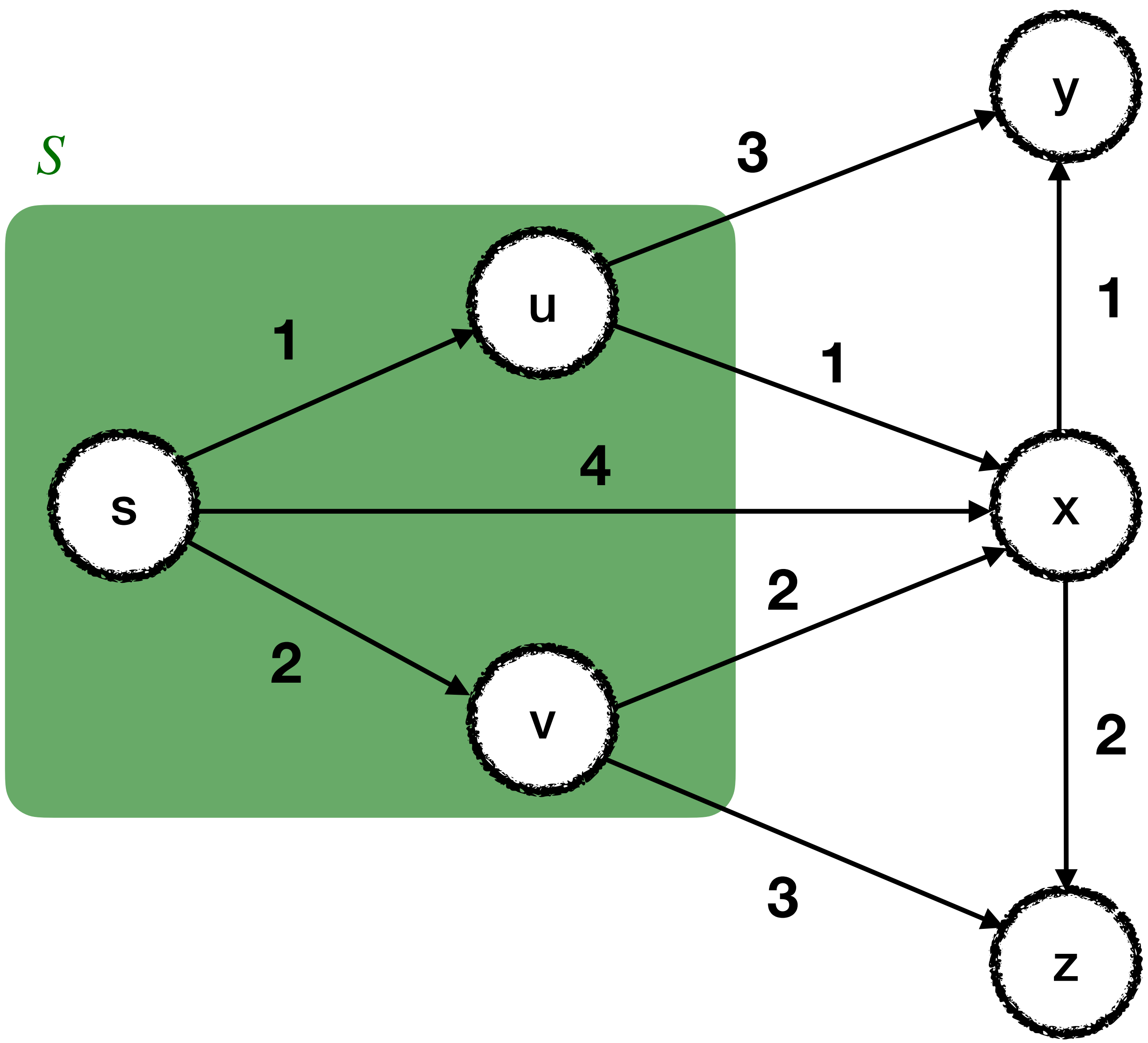

s	u	v	x	y	z
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shortest path distances from s

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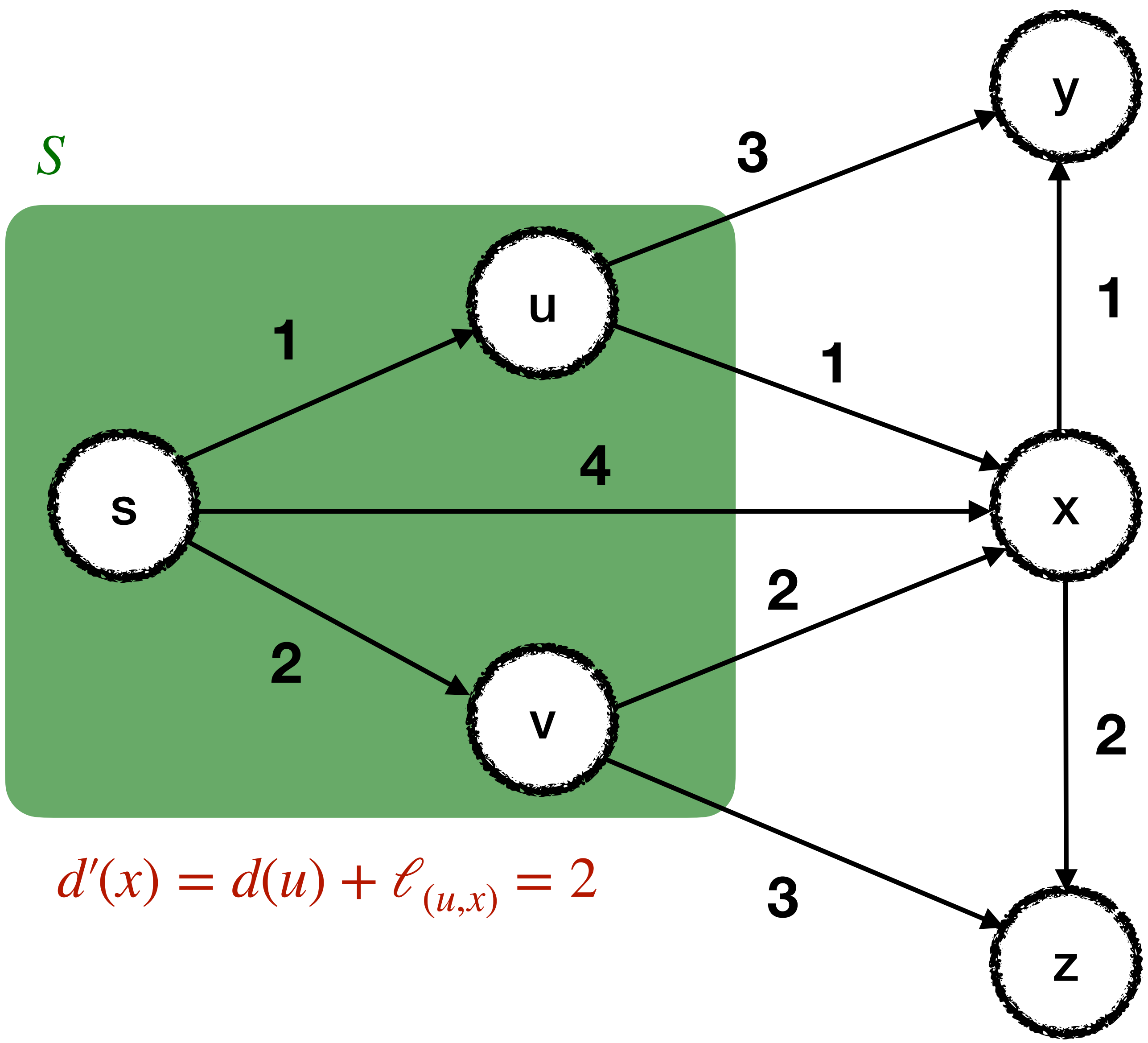

$d'(x) = d(u) + \ell_{(u,x)} = 2$

s	u	v	x	y	z
0	1	2			

shortest path distances from s

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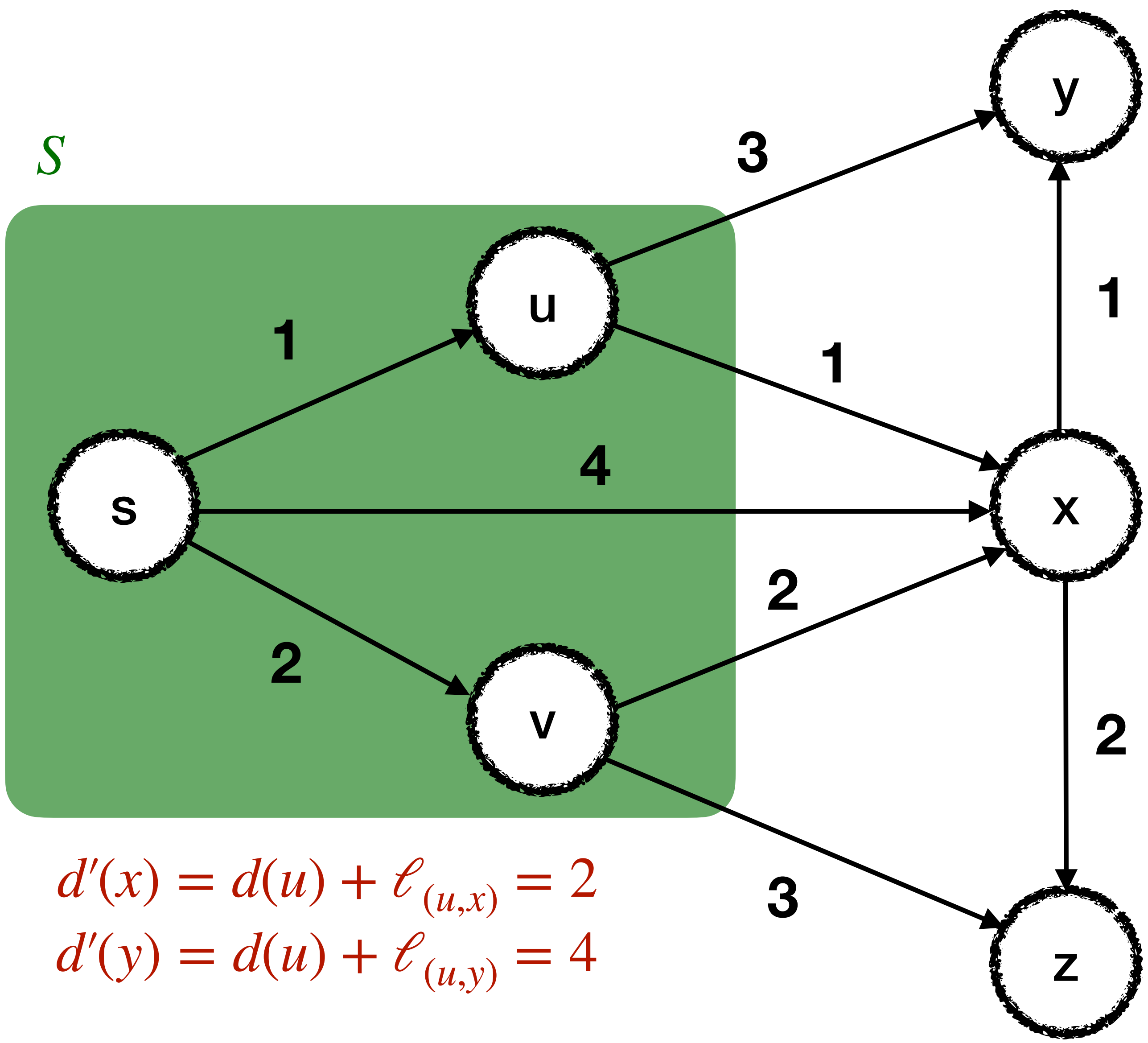
>
> Add v to S and define $d(v) = d'(v)$
>
> Let $A[v] = d(v)$

Running Example (KT Figure 5.7)



$d'(x) = d(u) + \ell_{(u,x)} = 2$

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shortest path distances from s

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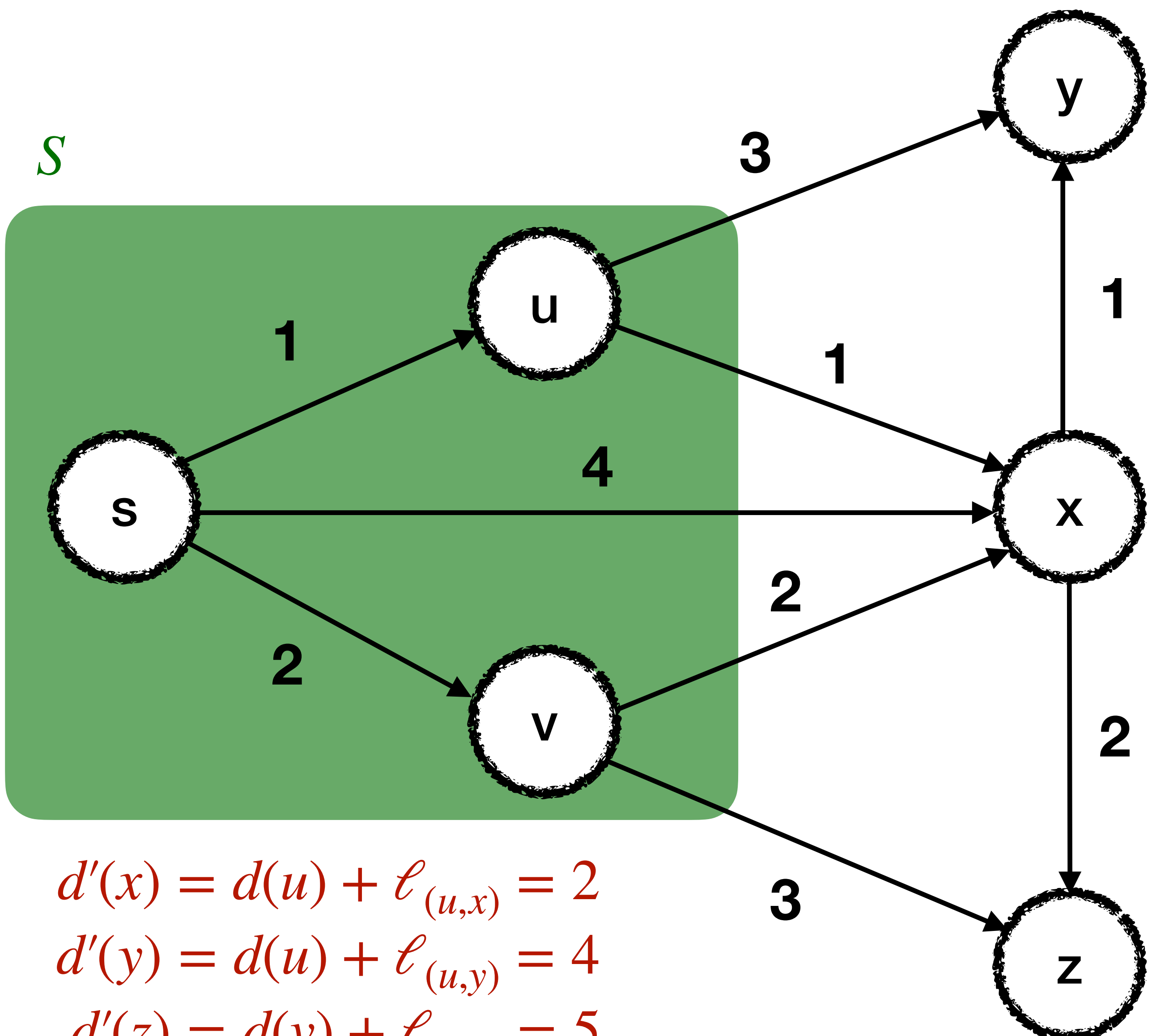

$d'(x) = d(u) + \ell_{(u,x)} = 2$

$d'(y) = d(u) + \ell_{(u,y)} = 4$

$d'(z) = d(v) + \ell_{(v,z)} = 5$

s	u	v	x	y	z
0	1	2			

shortest path distances from s

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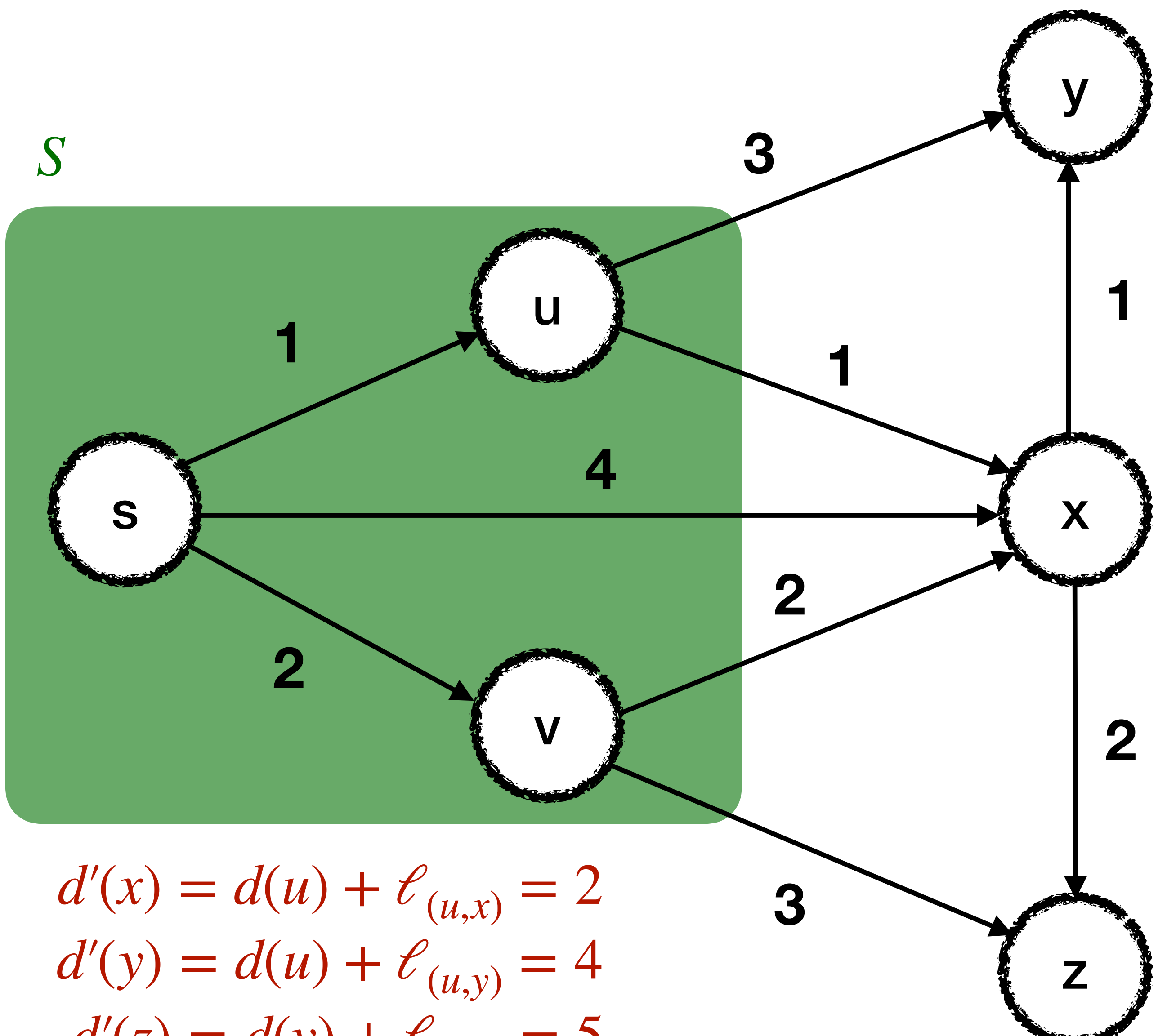
Running Example (KT Figure 5.7)



$$d'(x) = d(u) + \ell_{(u,x)} = 2$$
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s	u	v	x	y	z
0	1	2			

shortest path distances from s

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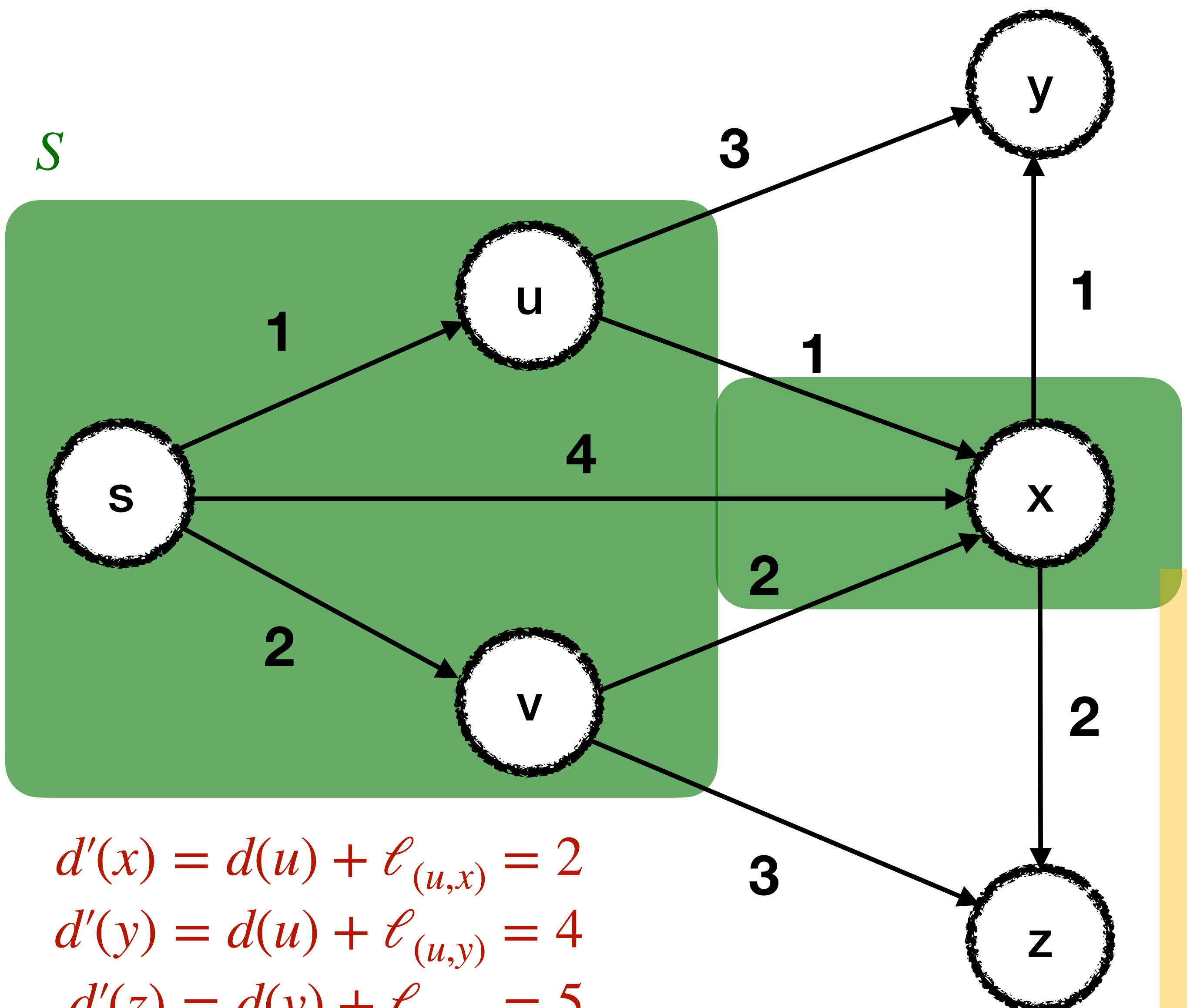

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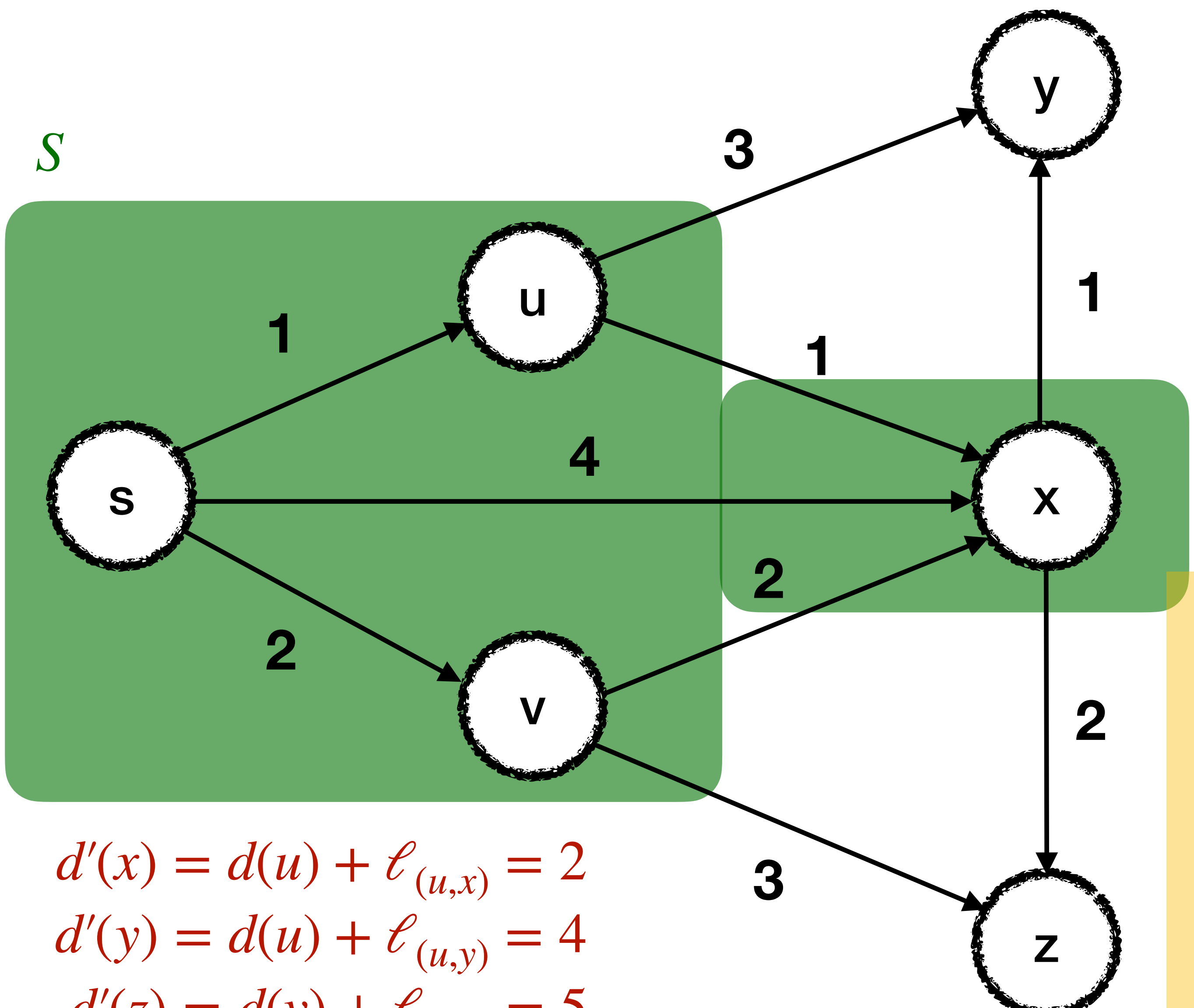

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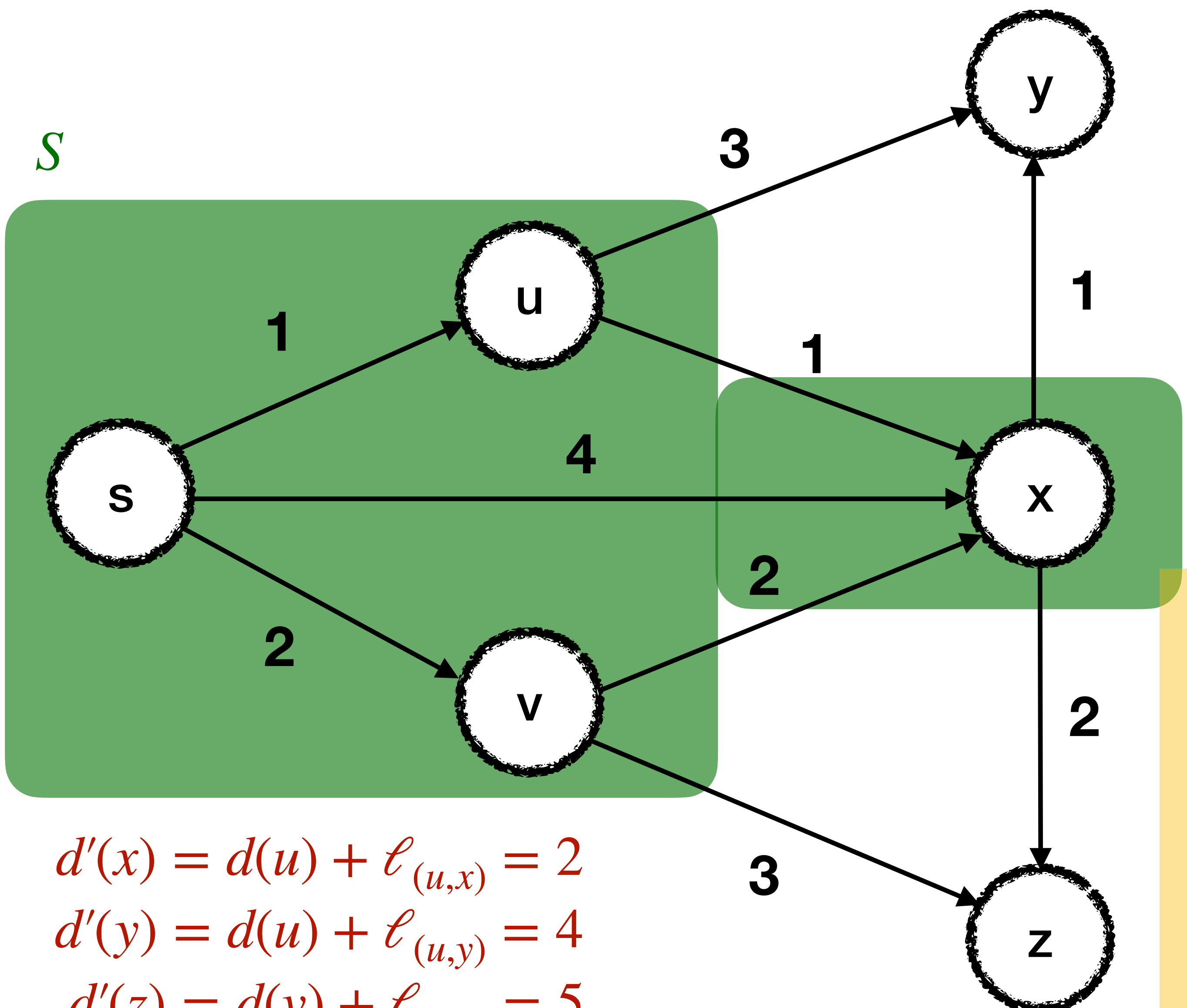

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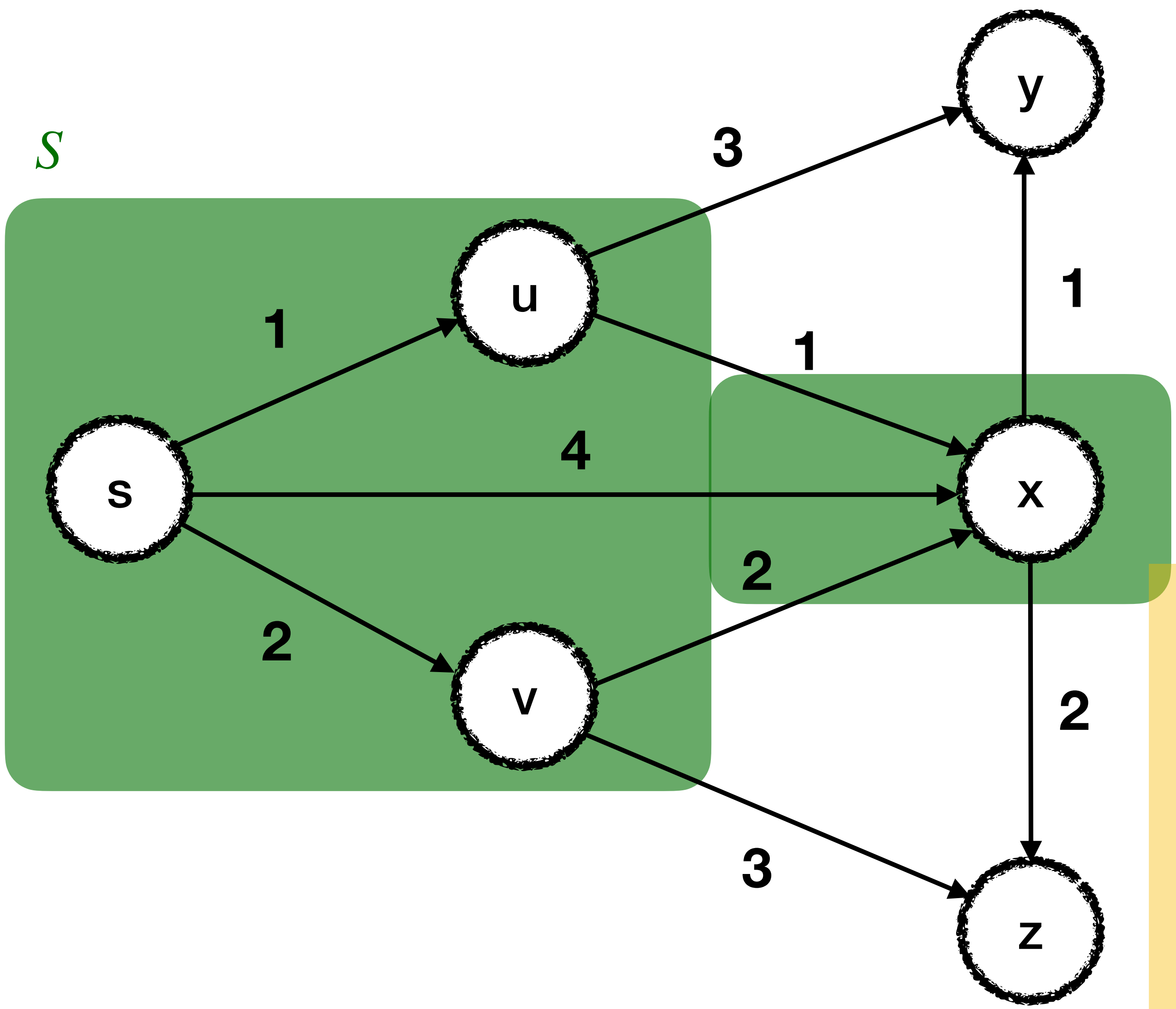

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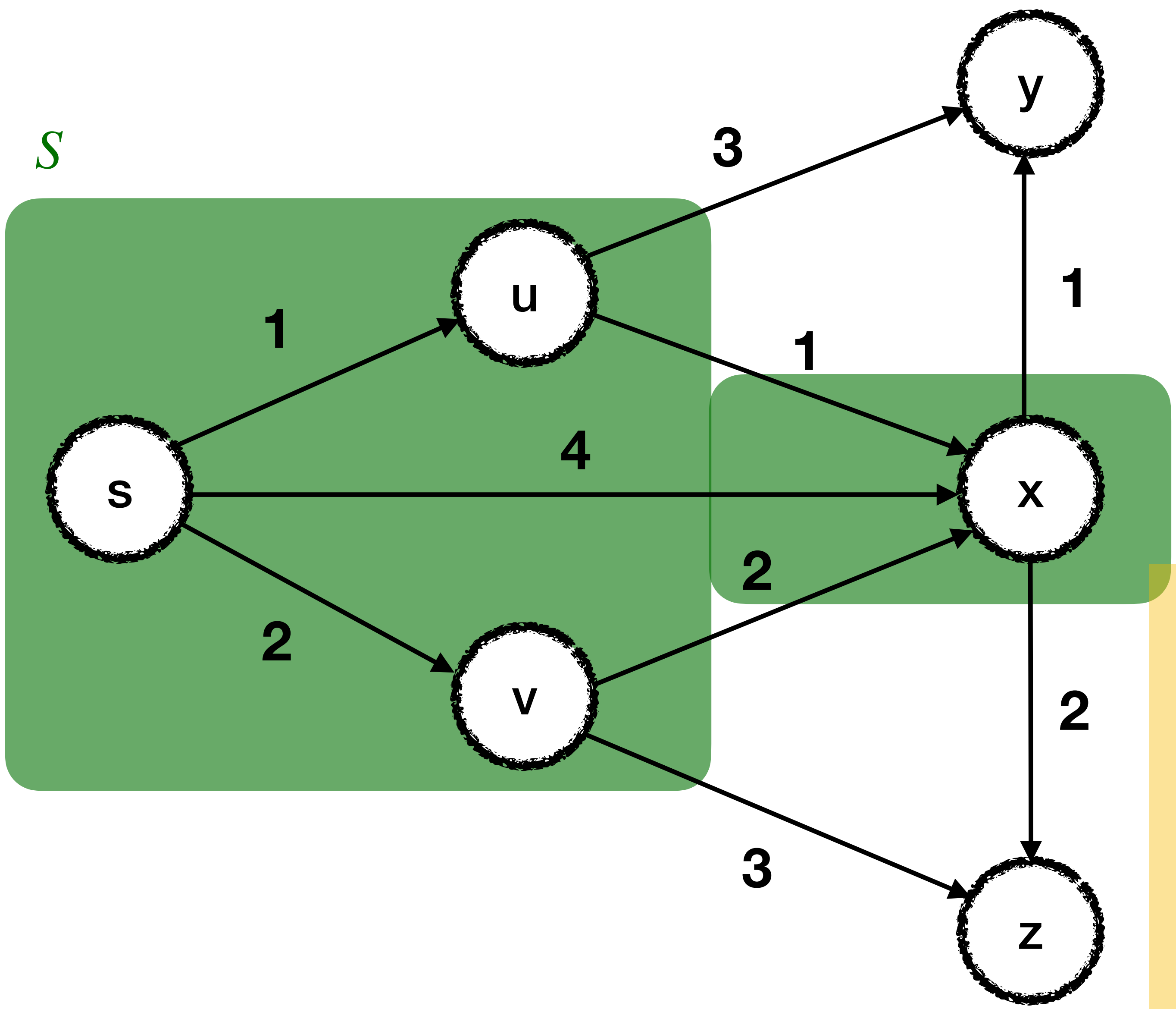

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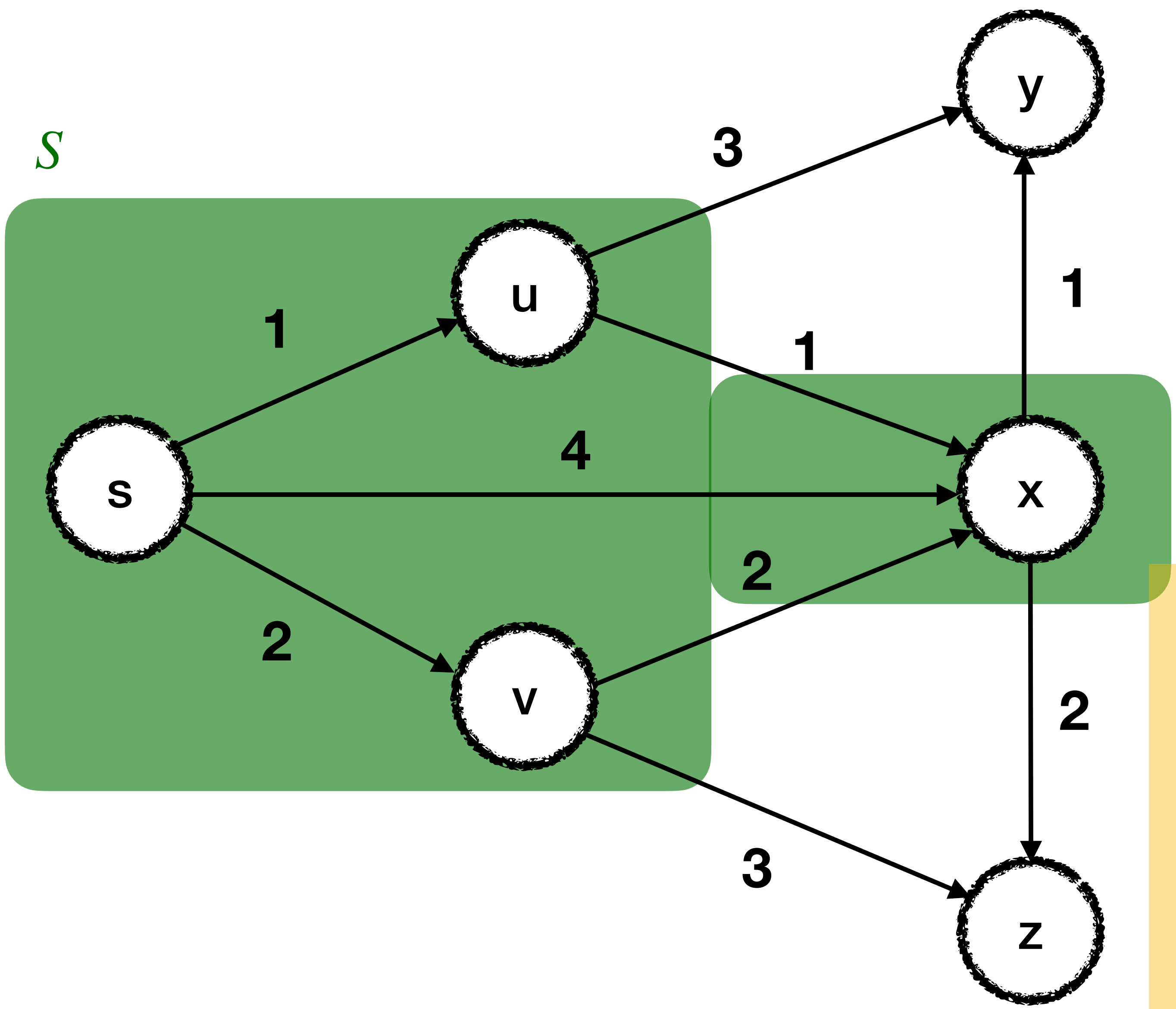

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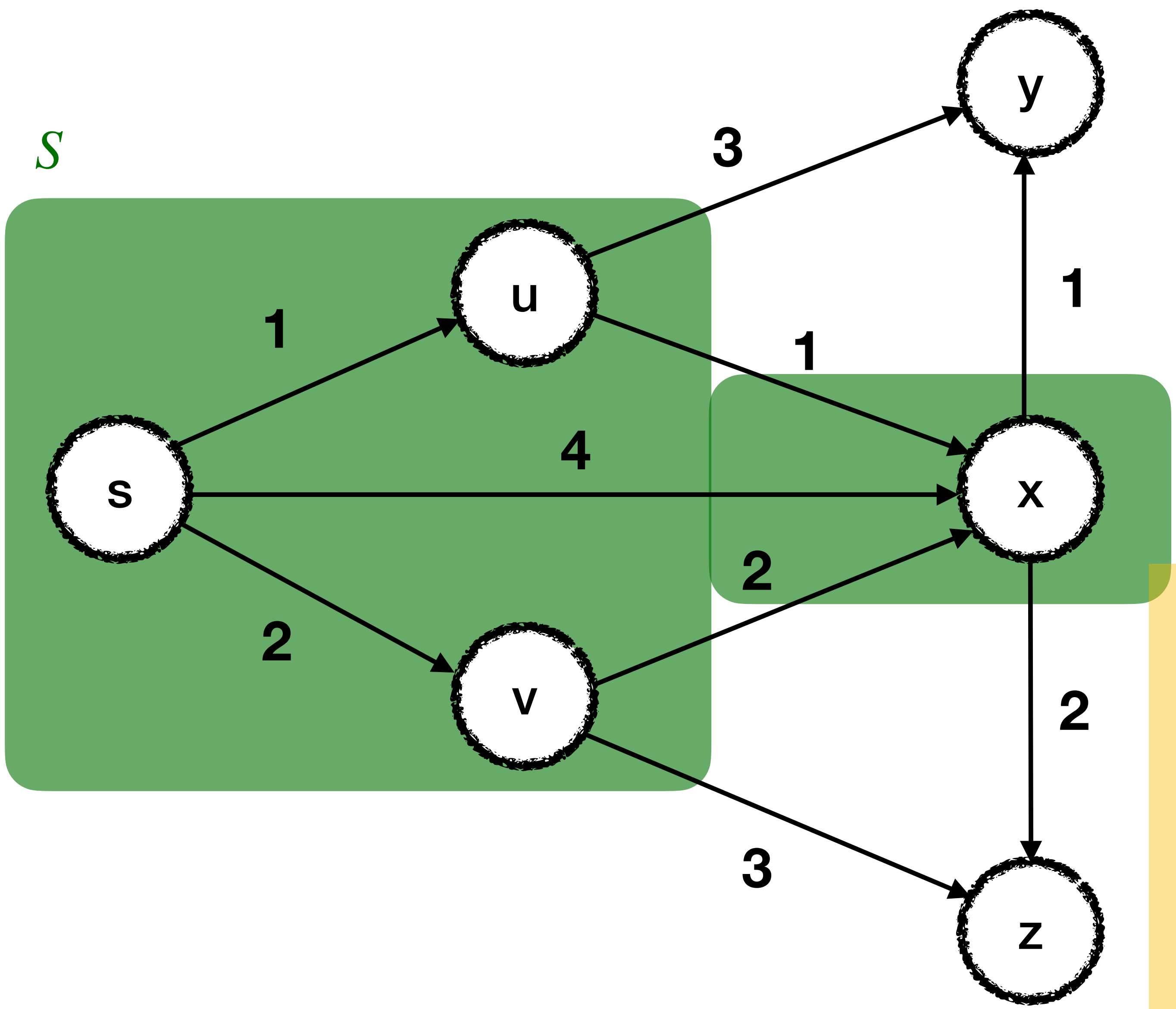

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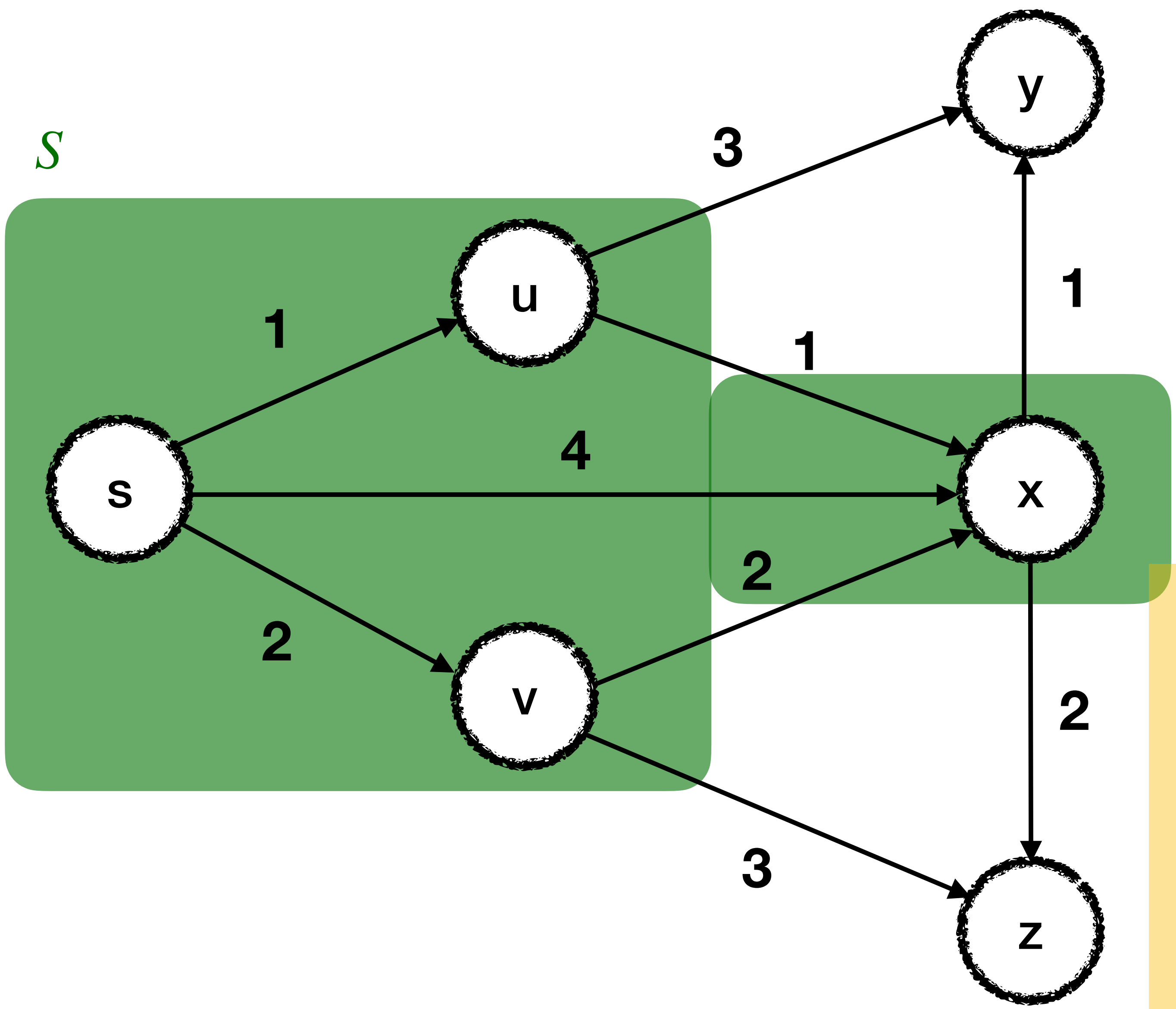

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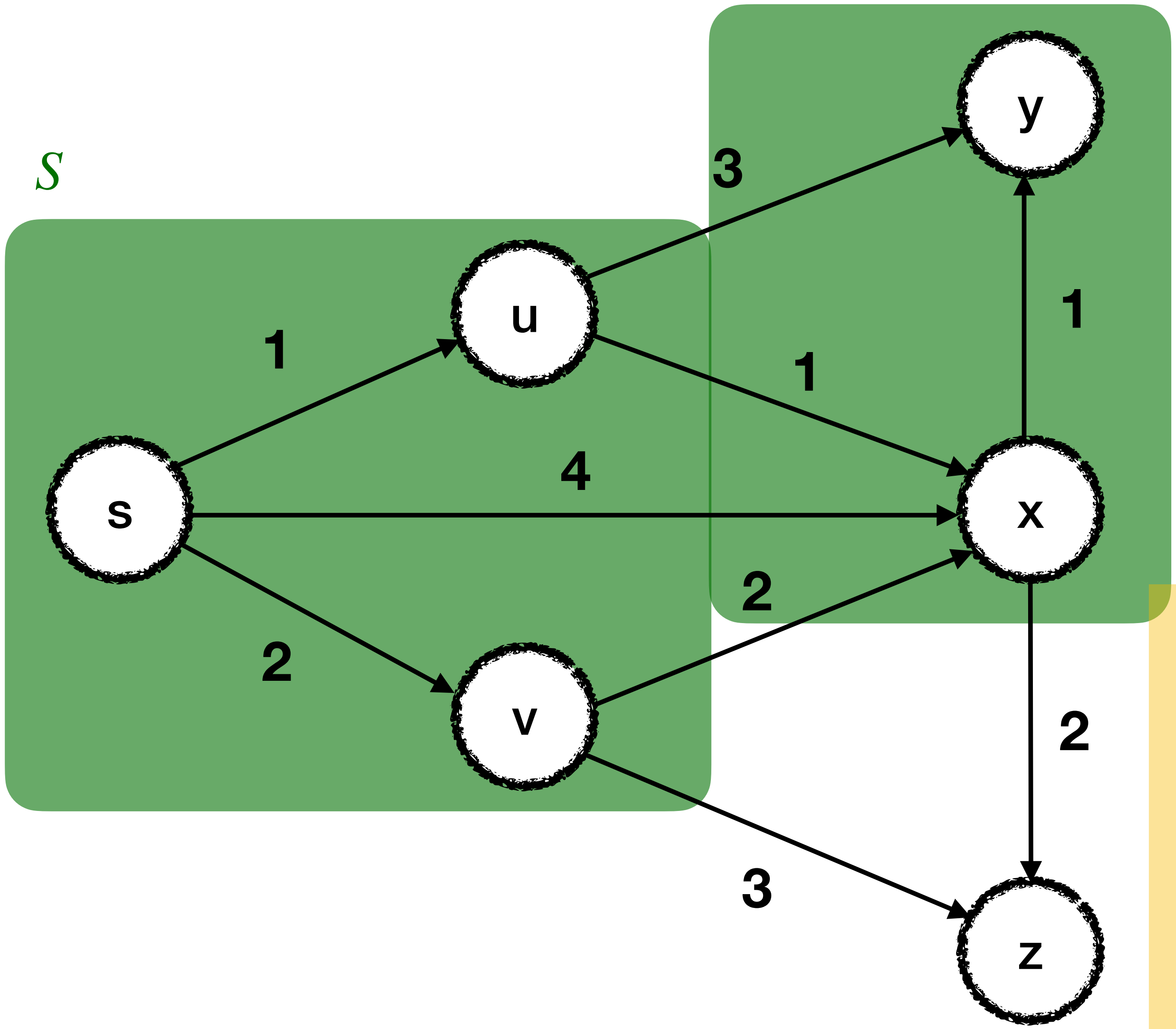

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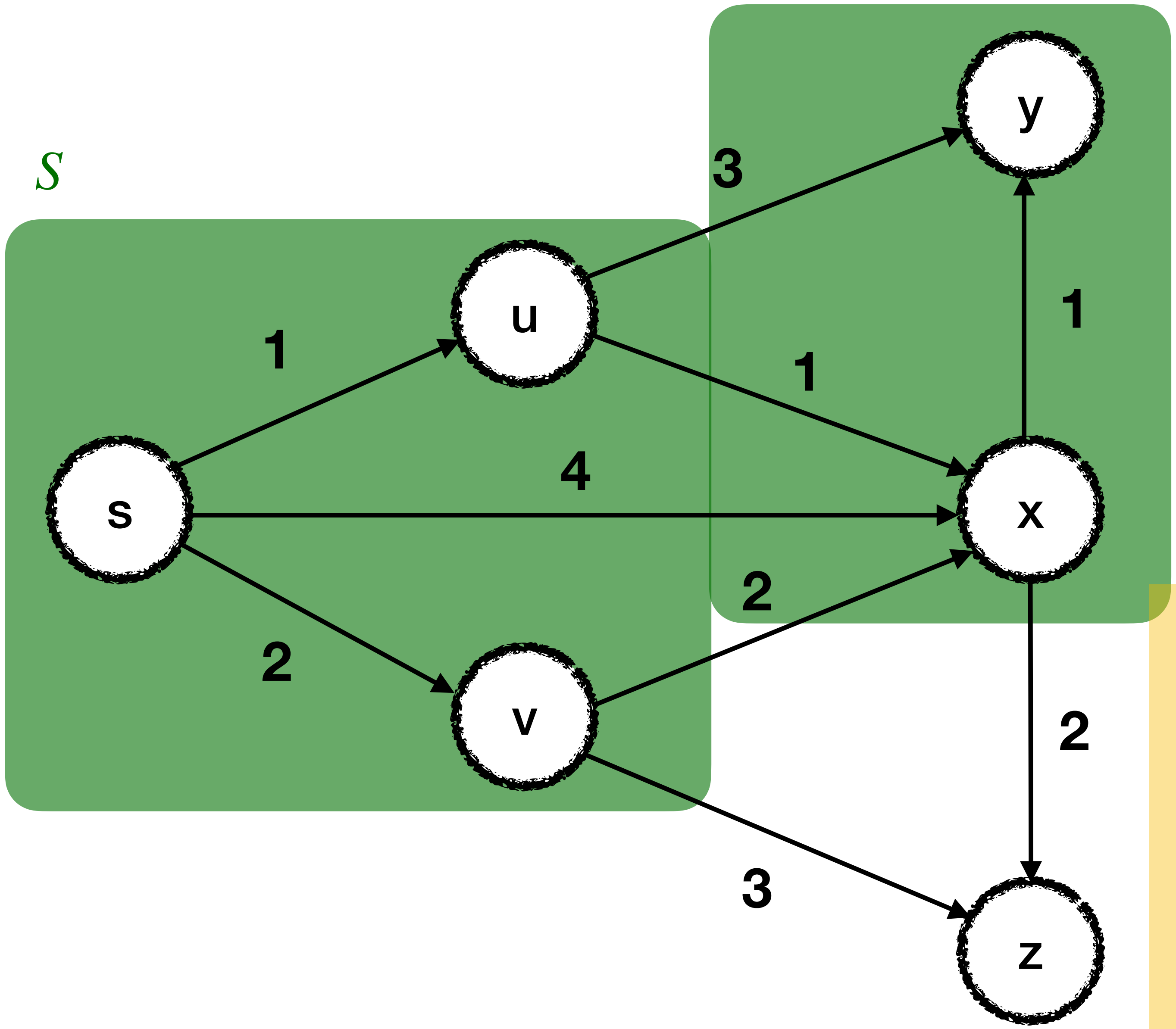

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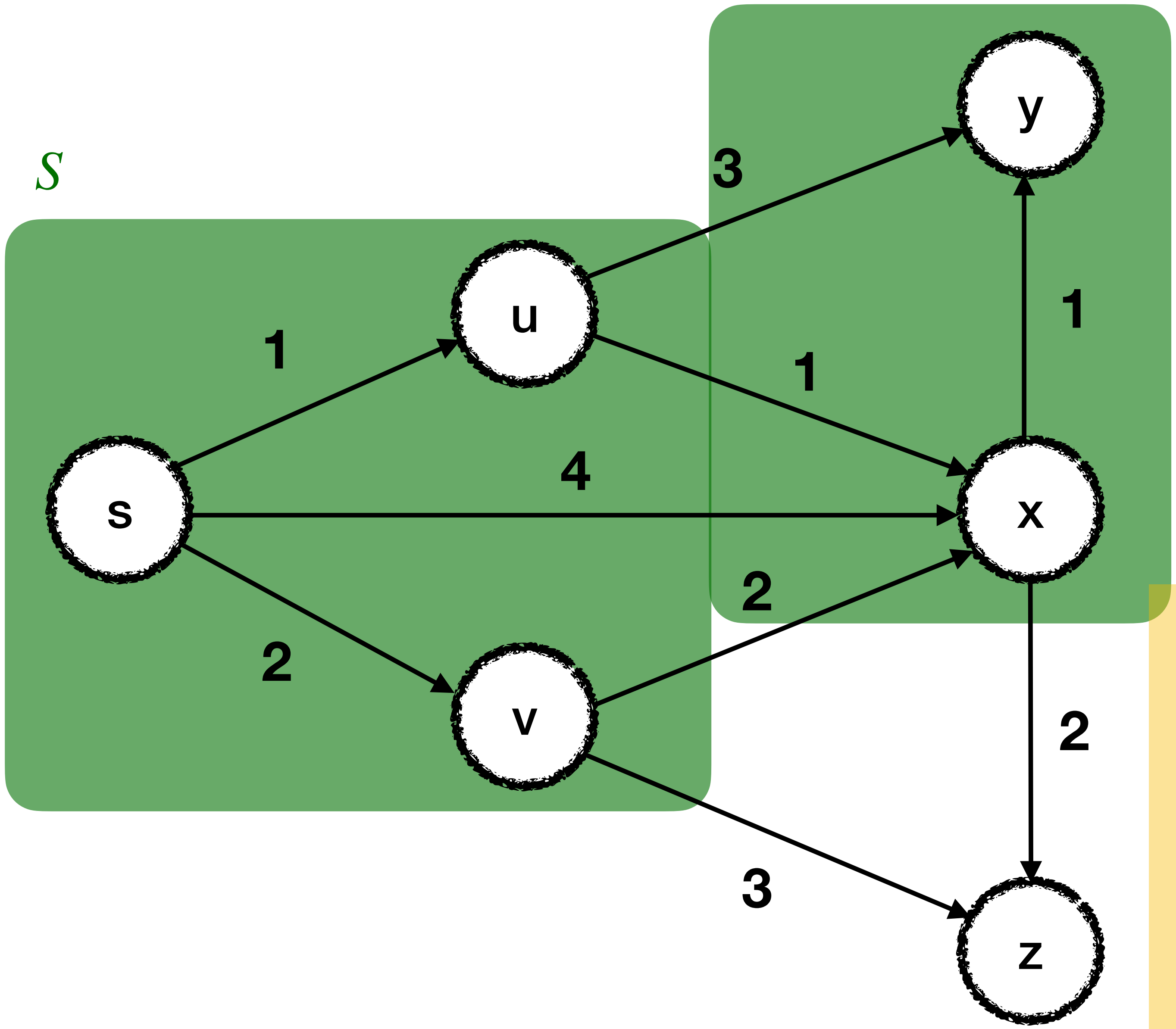

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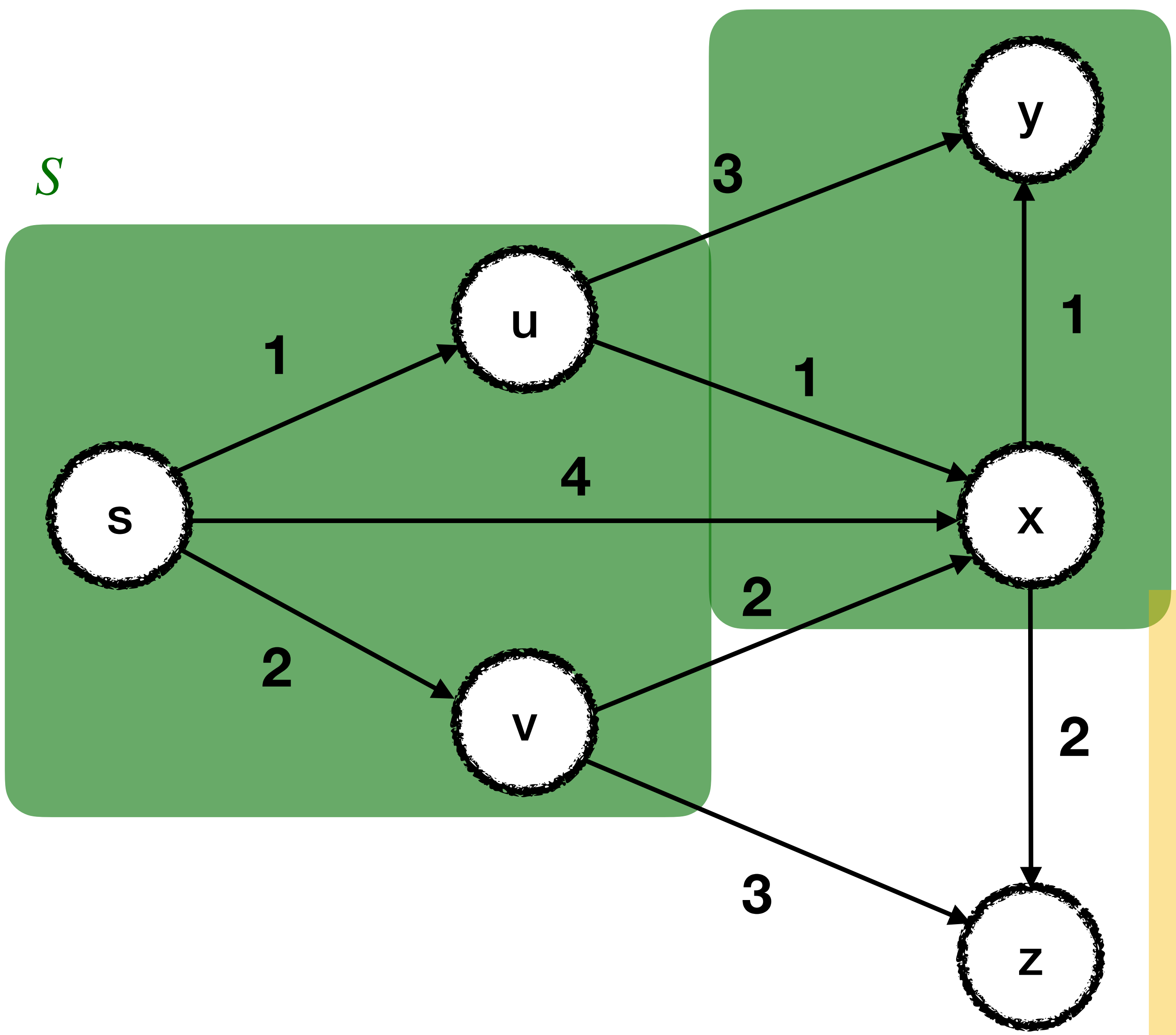

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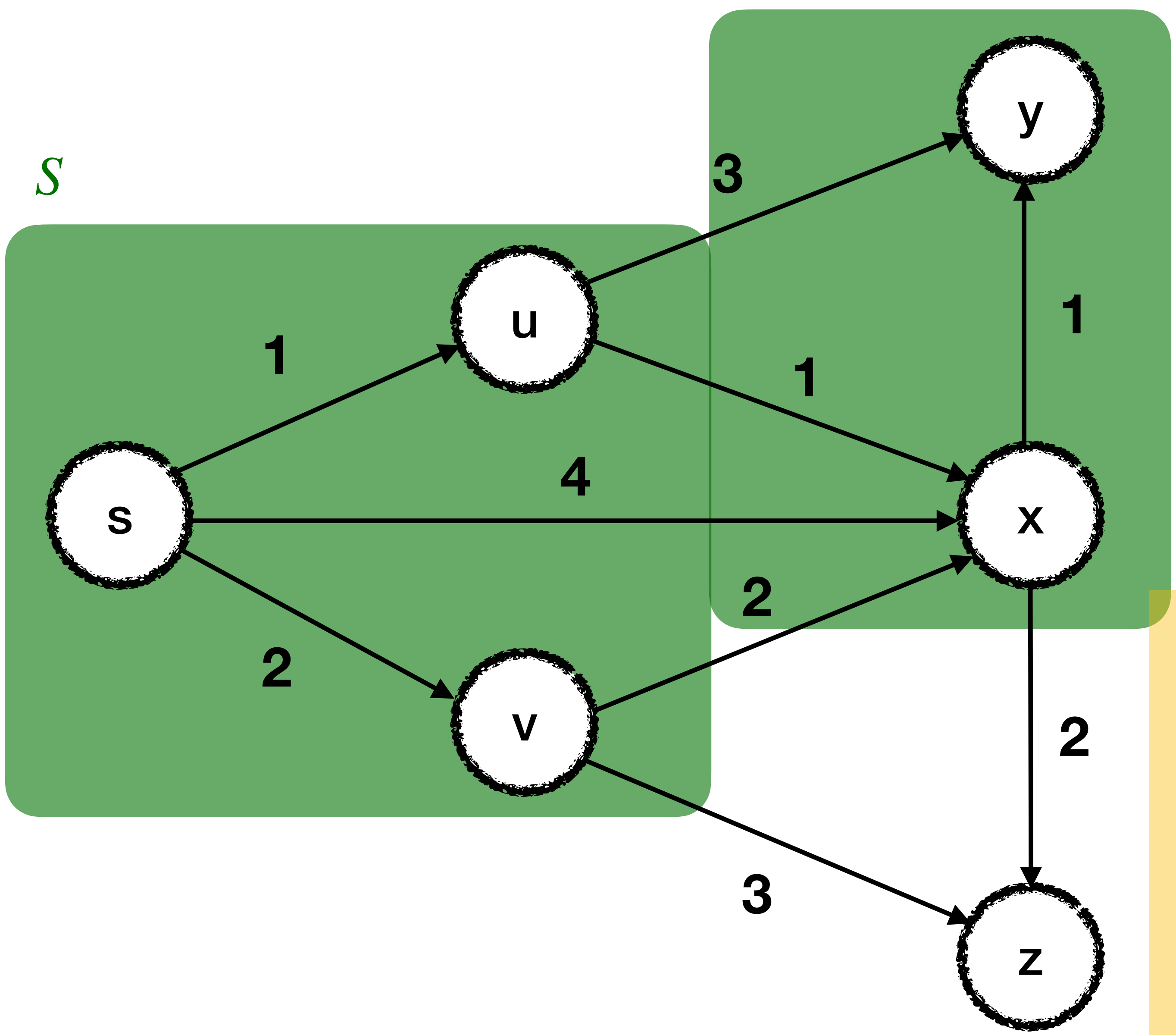

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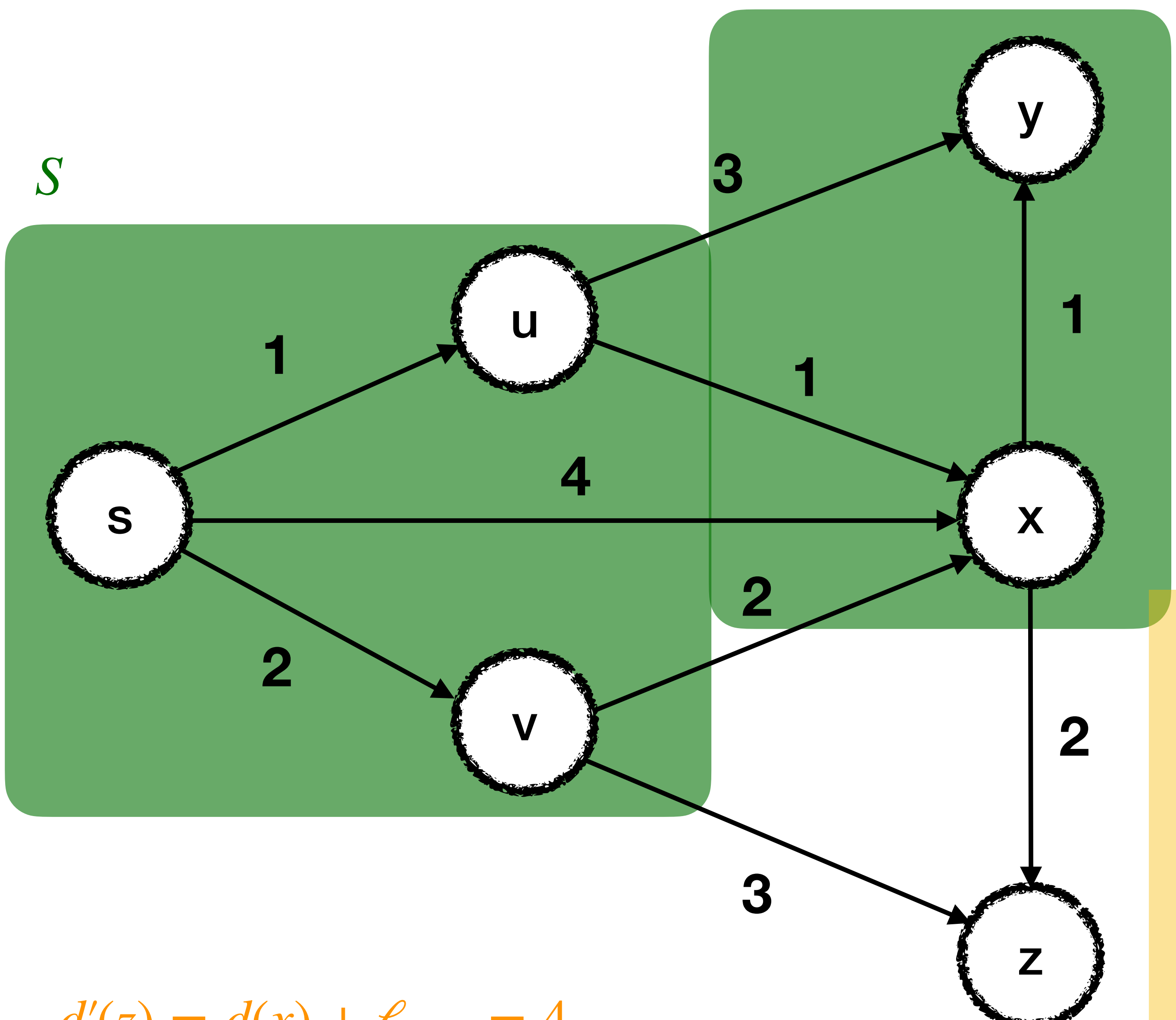

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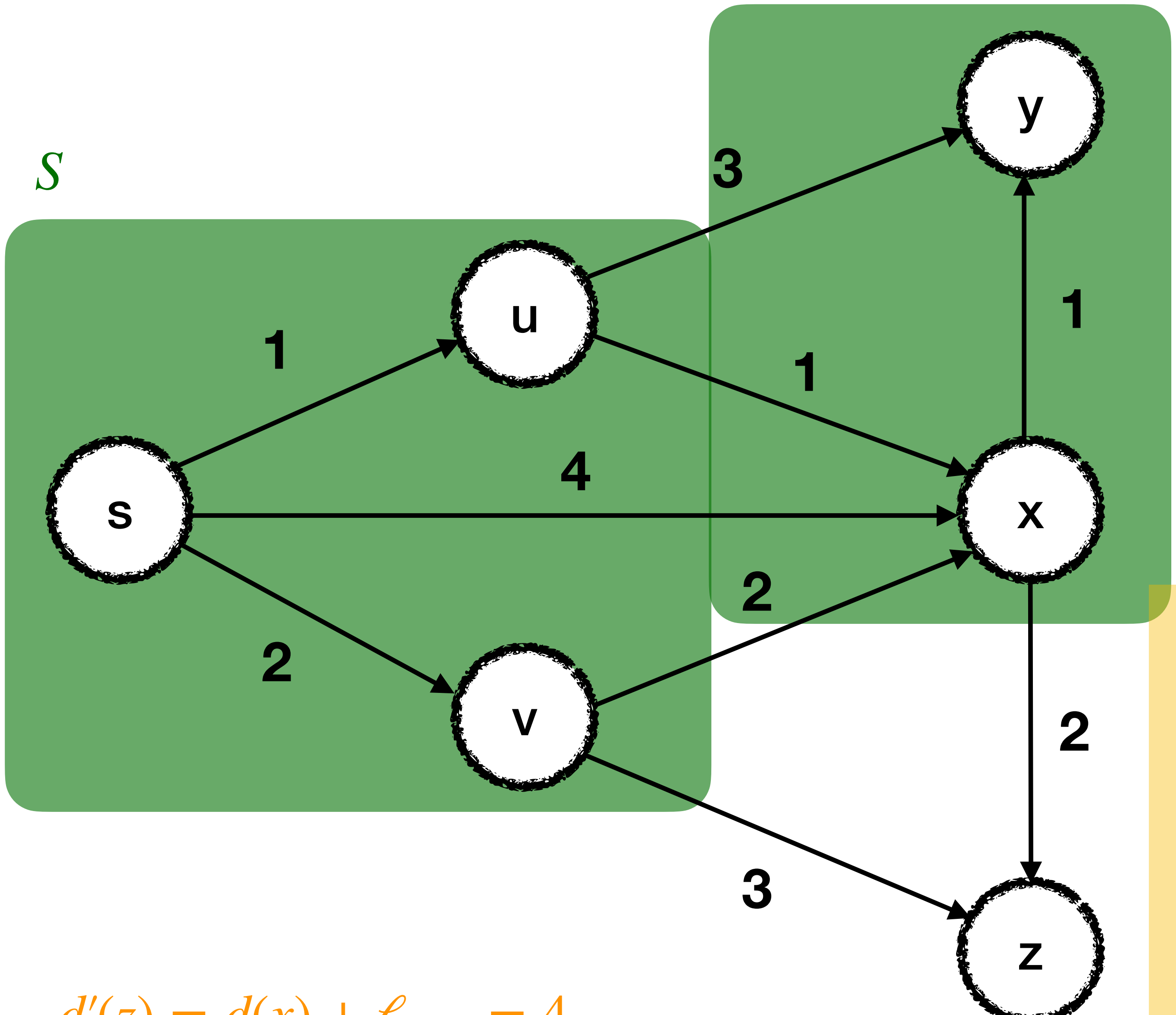

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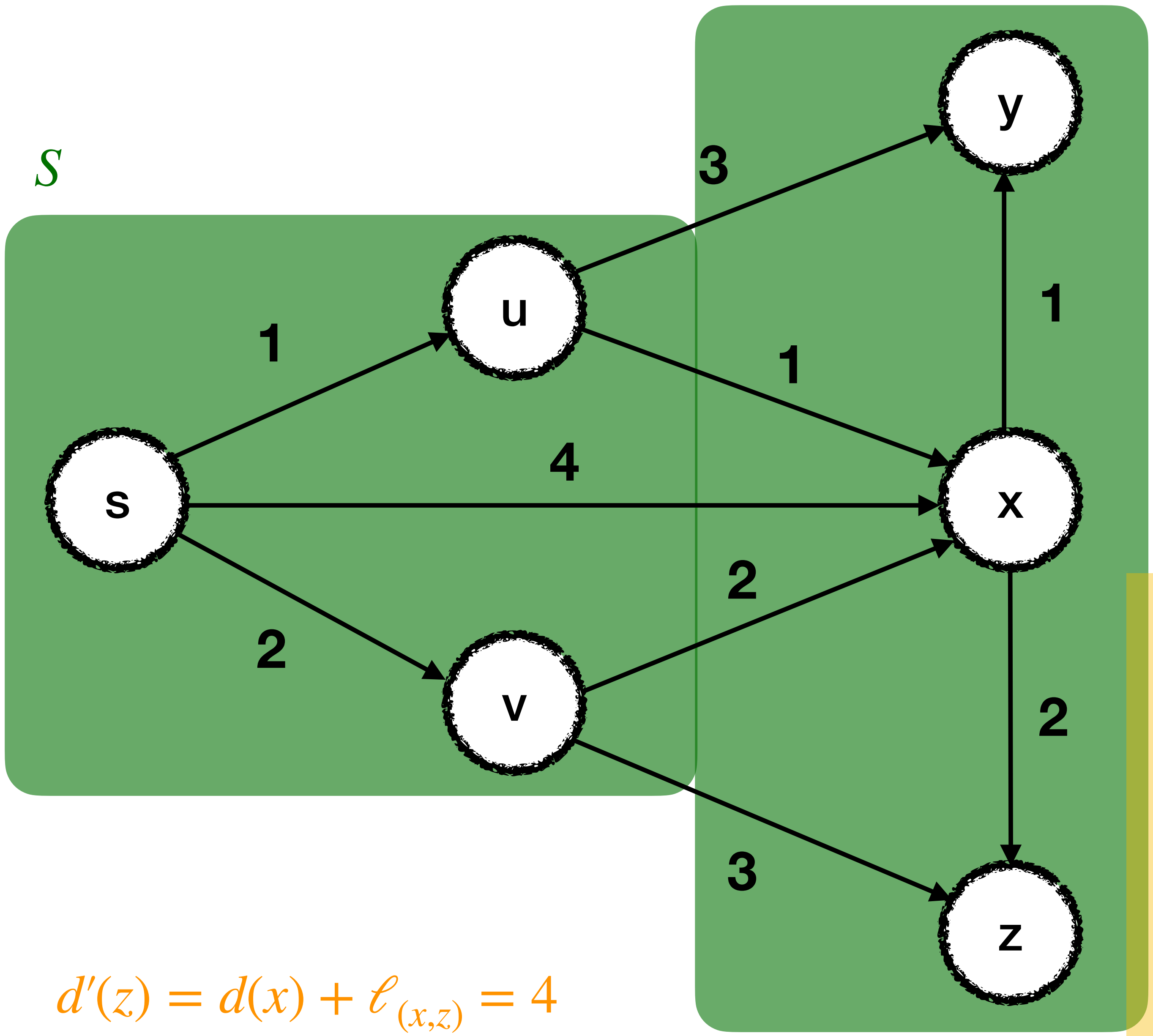

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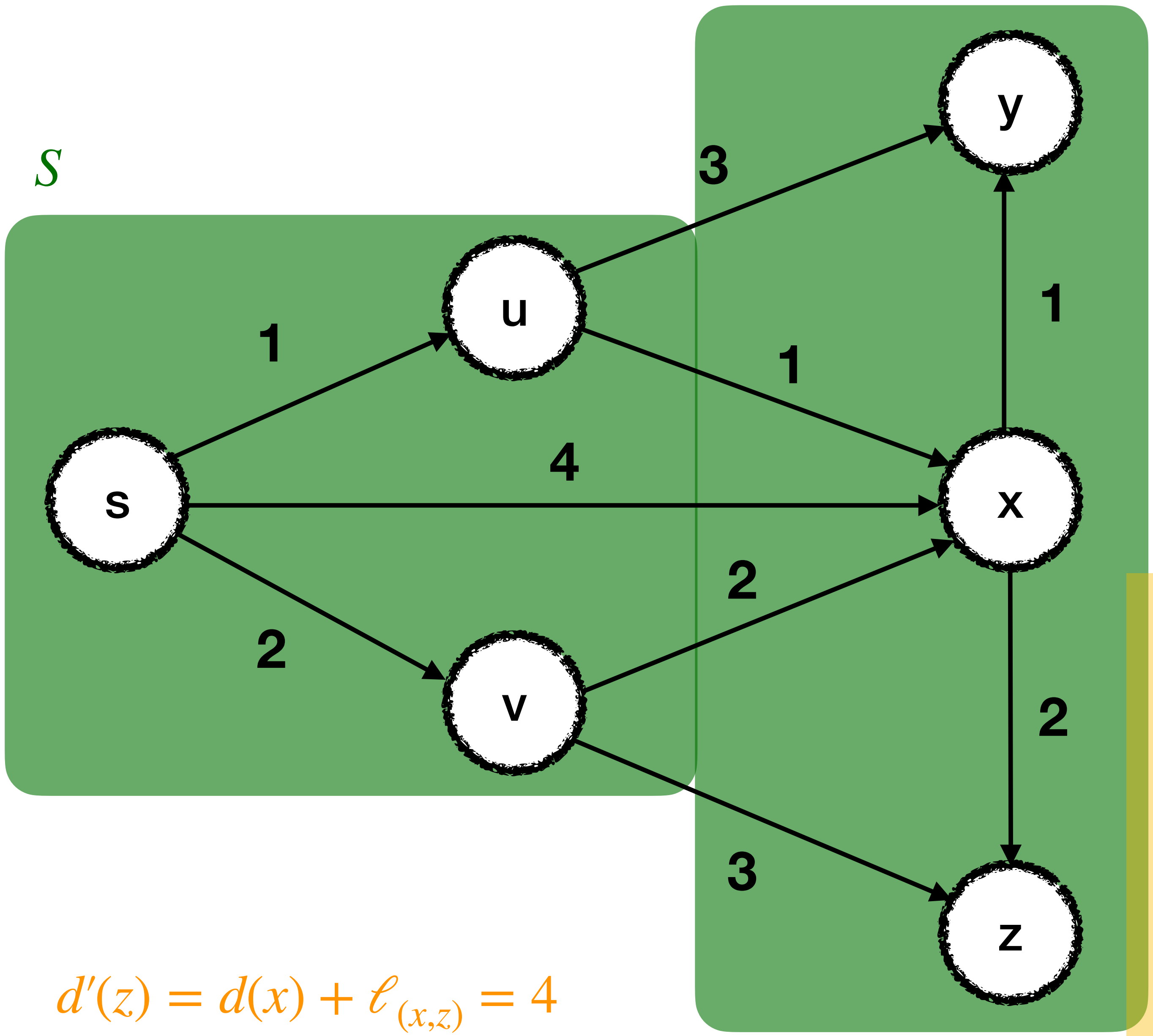

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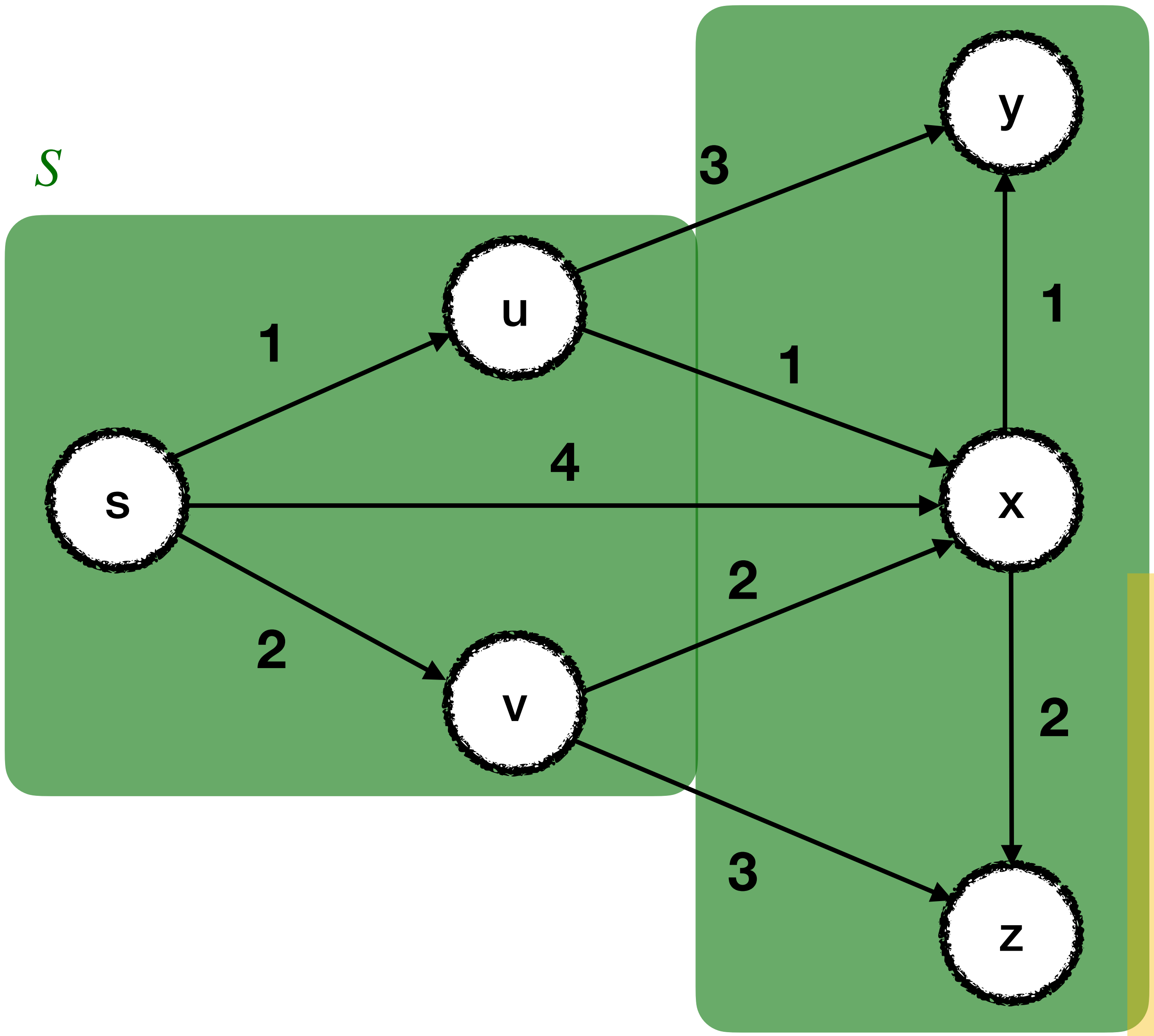

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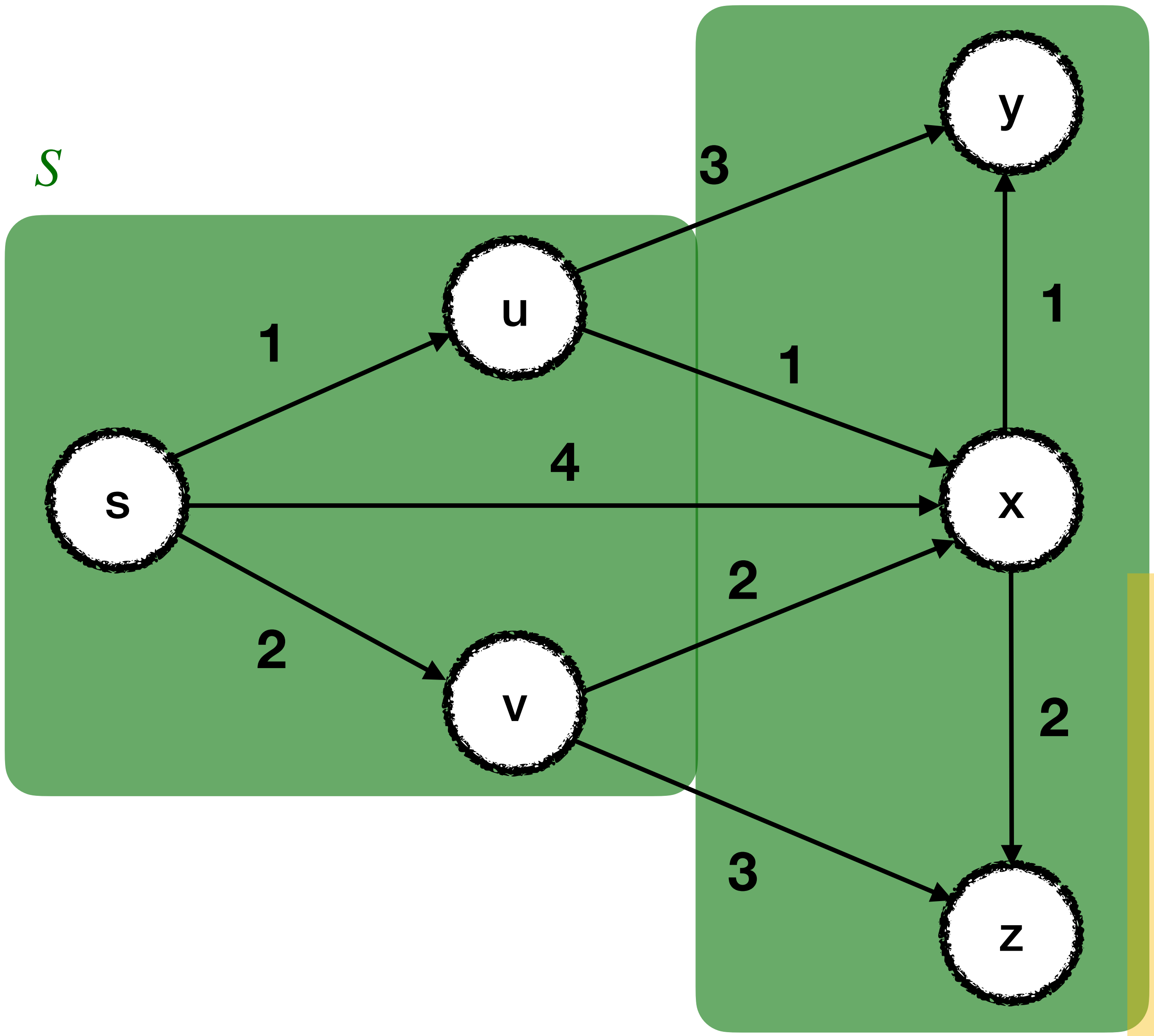

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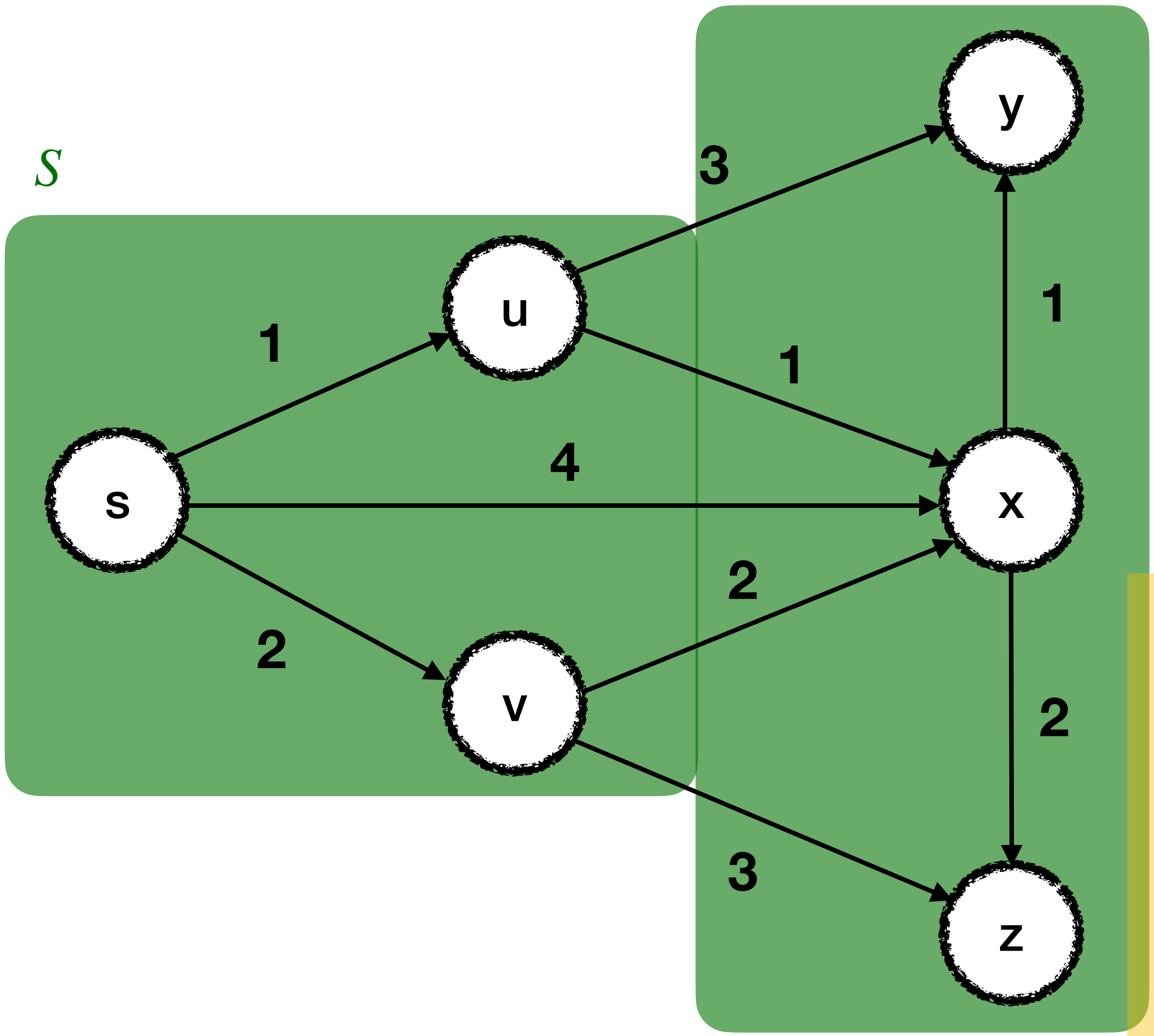

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This is only a list of shortest path lengths, not the paths themselves!

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- This is enough to recursively recover the path P_v : P_v is just P_u + (u,v). In turn, P_u is P_w + (w,u), where w is the node from which we explored u, and so on.

Correctness

Theorem: Consider the set S at any point in the execution of the algorithm. For each $u \in S$, the path P_u is a shortest s-u path.

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This is enough to prove correctness. Why?

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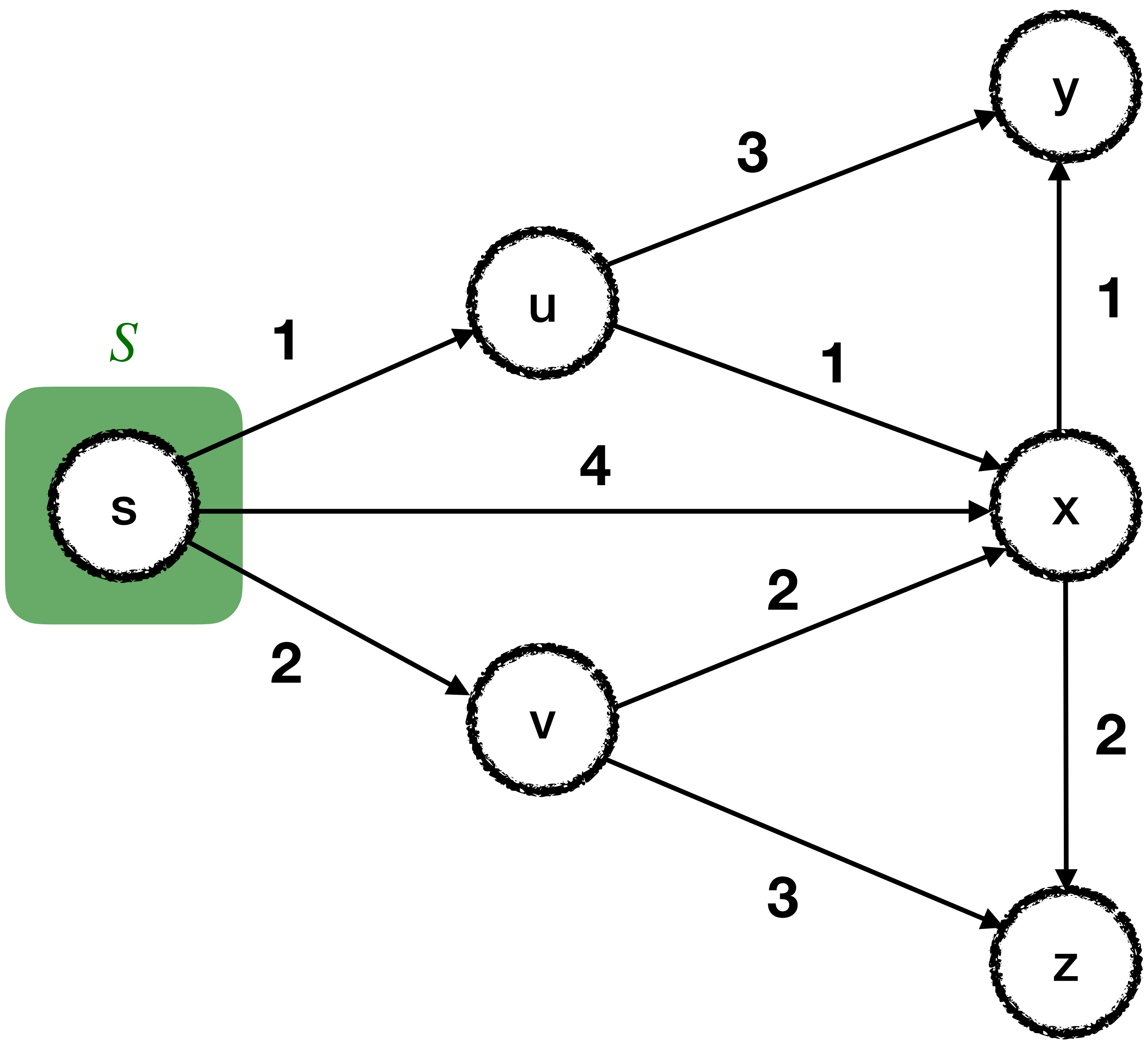
Proof by induction on the size of S.

Base Case: $|S| = 1$

Running Example (KT Figure 5.7)



s	u	v	x	y	z
0					

$\underbrace{\hspace{6em}}$

shortest path distances from s

Correctness

Theorem: Consider the set S at any point in the execution of the algorithm. For each $u \in S$, the path P_u is a shortest s-u path.

Proof by induction on the size of S.

Base Case: $|S| = 1$, $S = \{s\}$, $d(s) = 0$, trivially shortest path.

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Induction Step: We will prove that it holds for $|S| = k + 1$

Adding a node

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Let v^* be the node added to S. Let (u^*, v^*) be the final edge on the path P_{v^*}.

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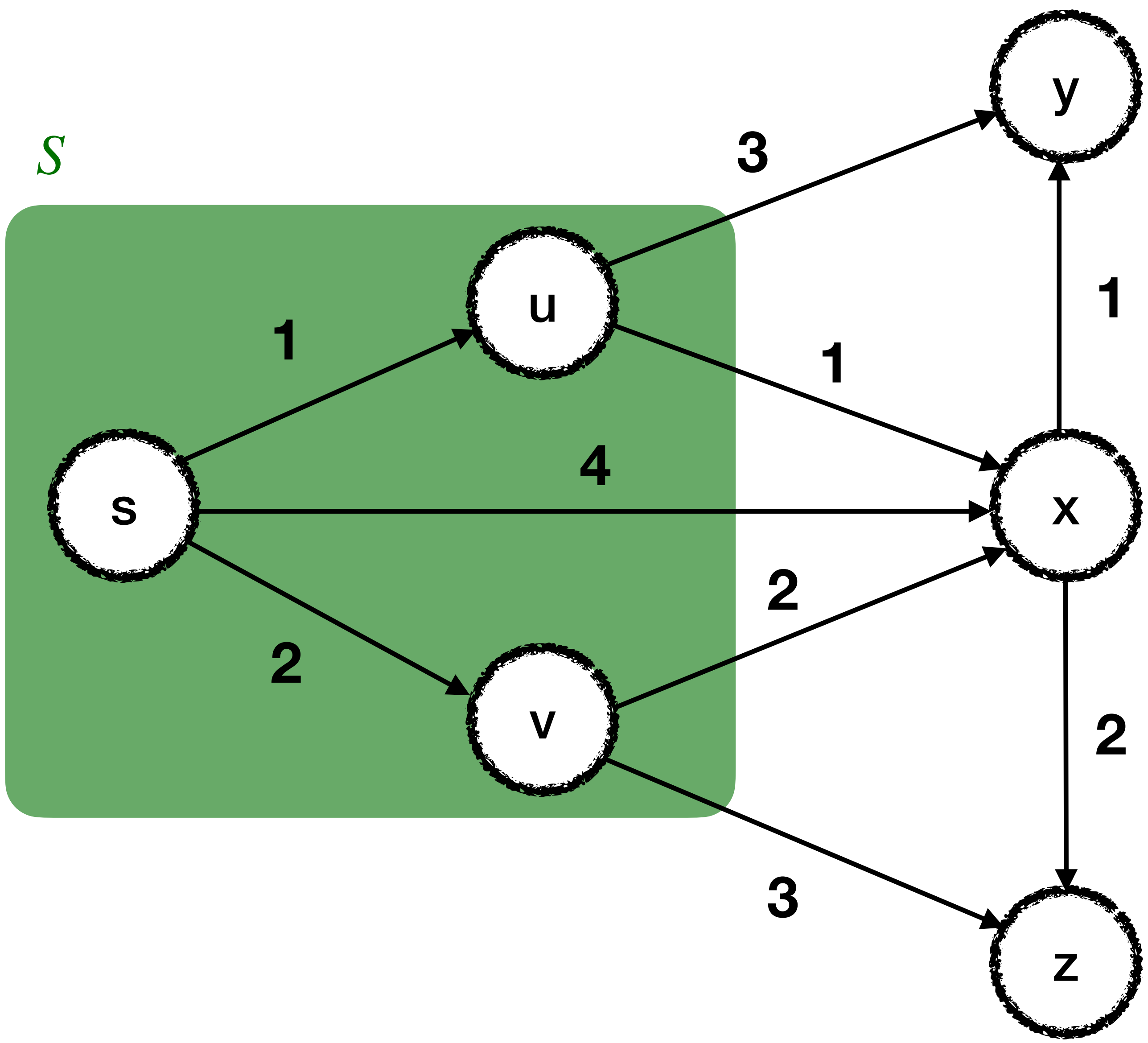

s	u	v	x	y	z
0	1	2			

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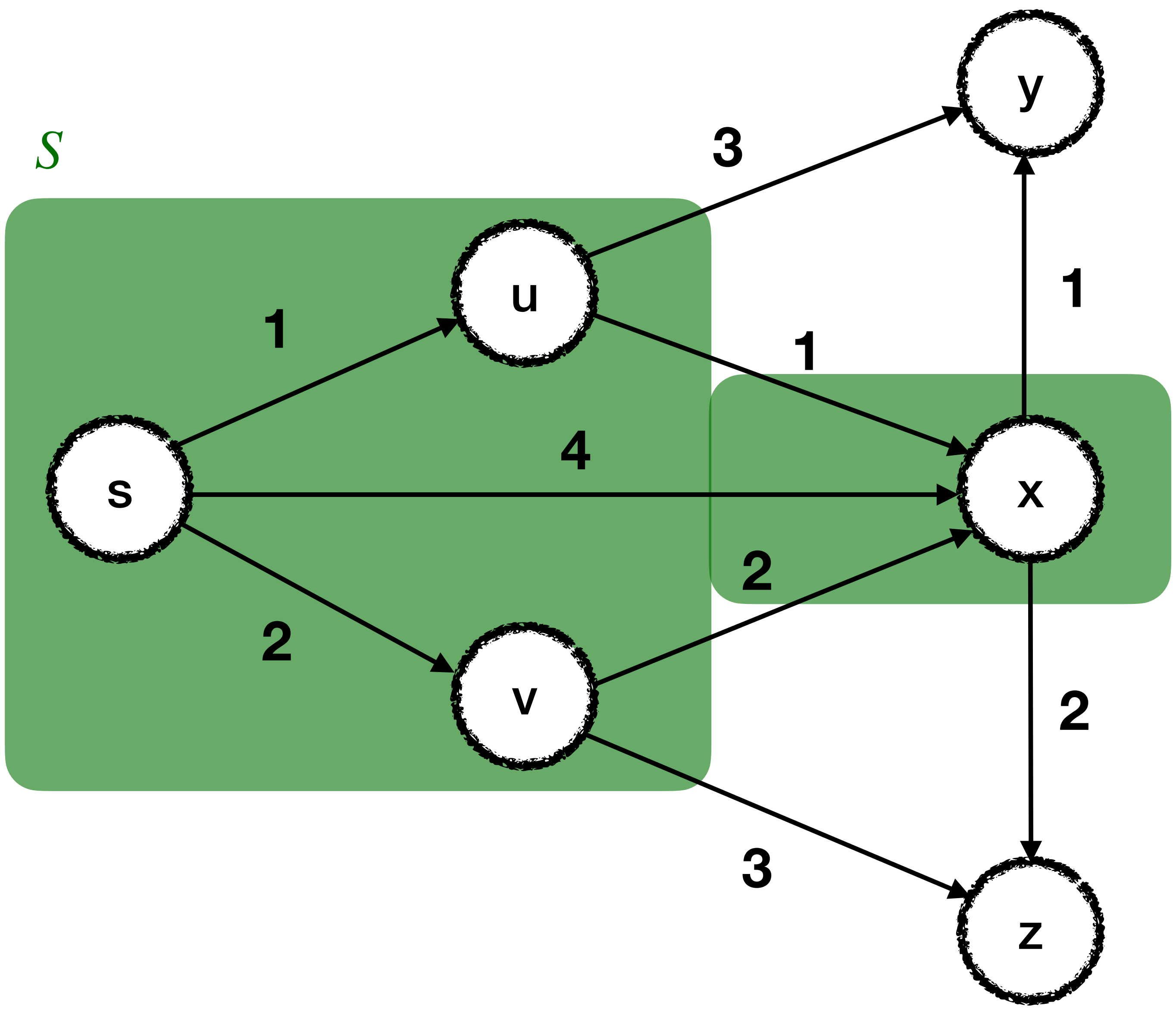

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That means that there exists some other path P that is shorter than P_{v^*}.

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Assume by contradiction that P_{v*} is *not* a shortest s-$v*$ path.

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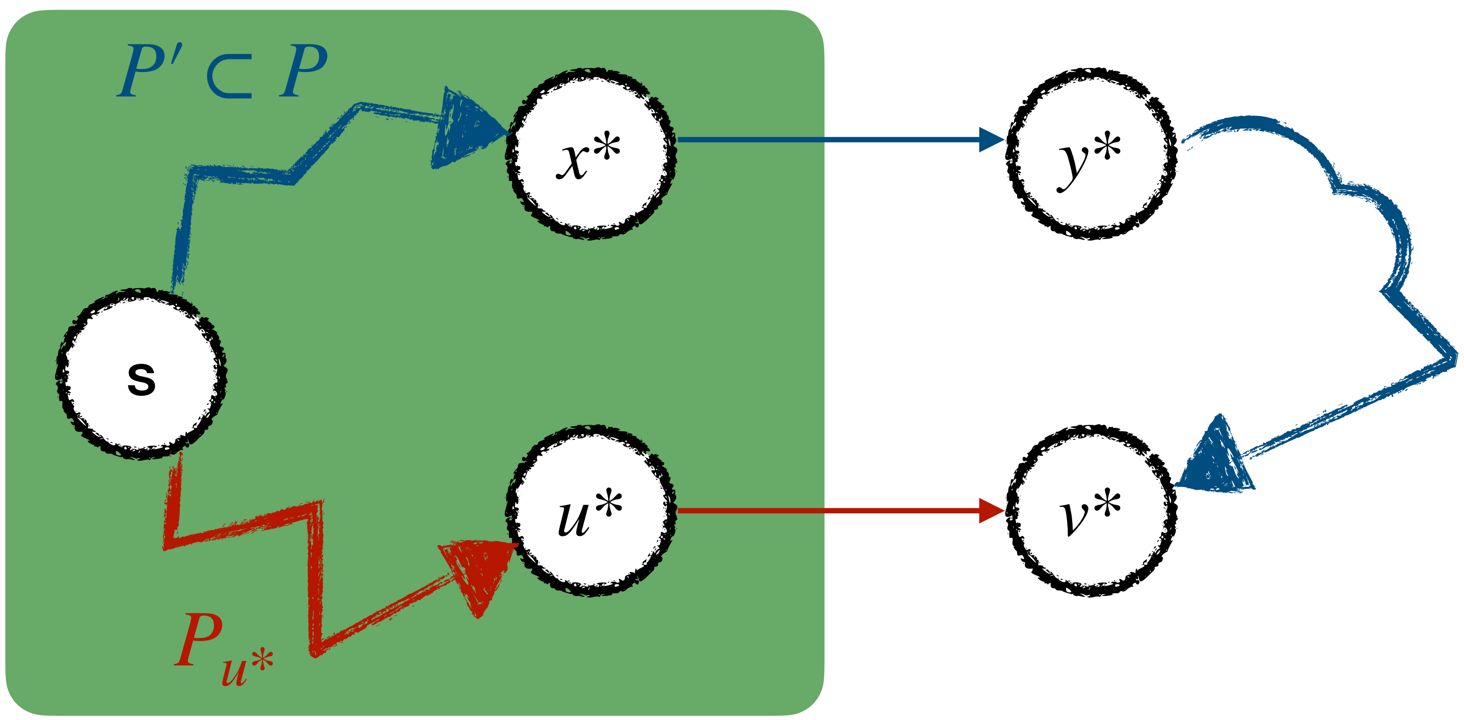
Pictorially



Let v^* be the node added to S. Let (u^*, v^*) be the final edge on the path P_{v^*}.

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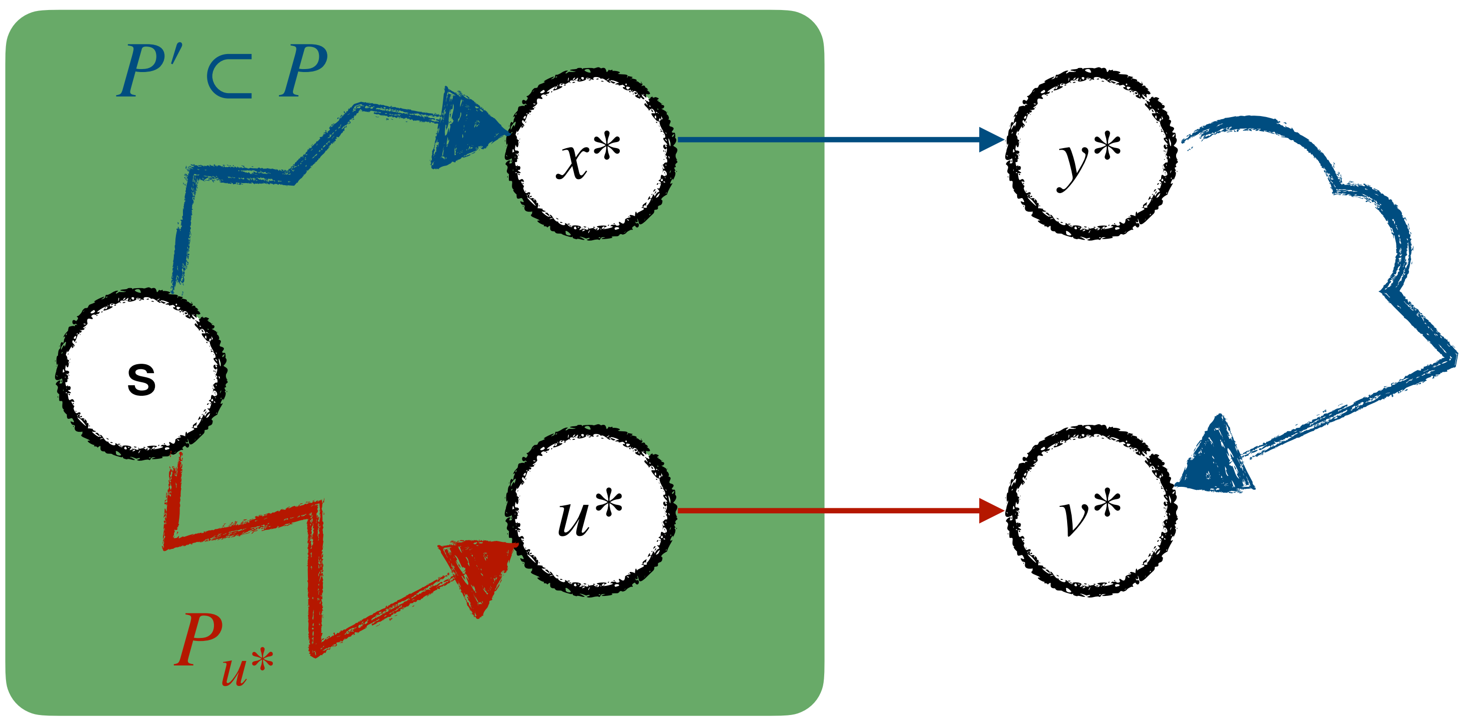
Correctness



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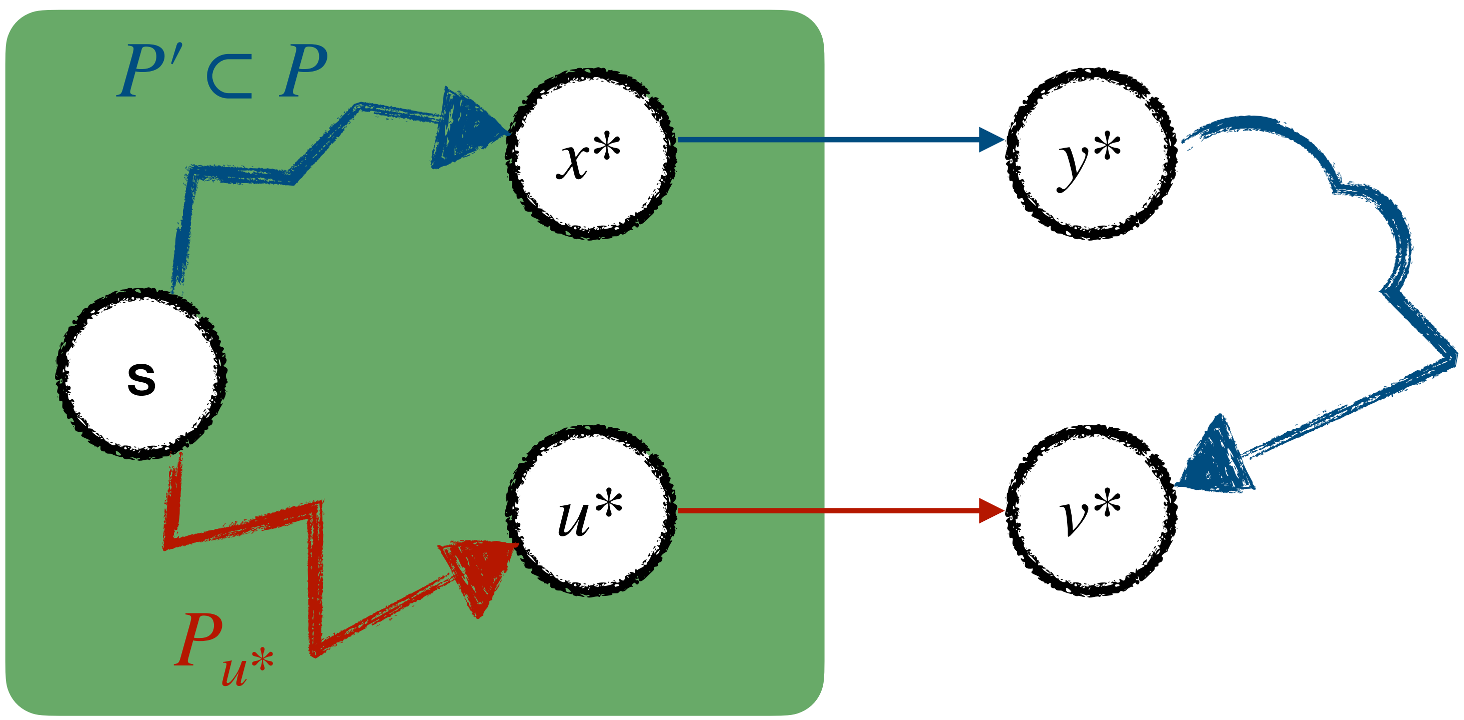
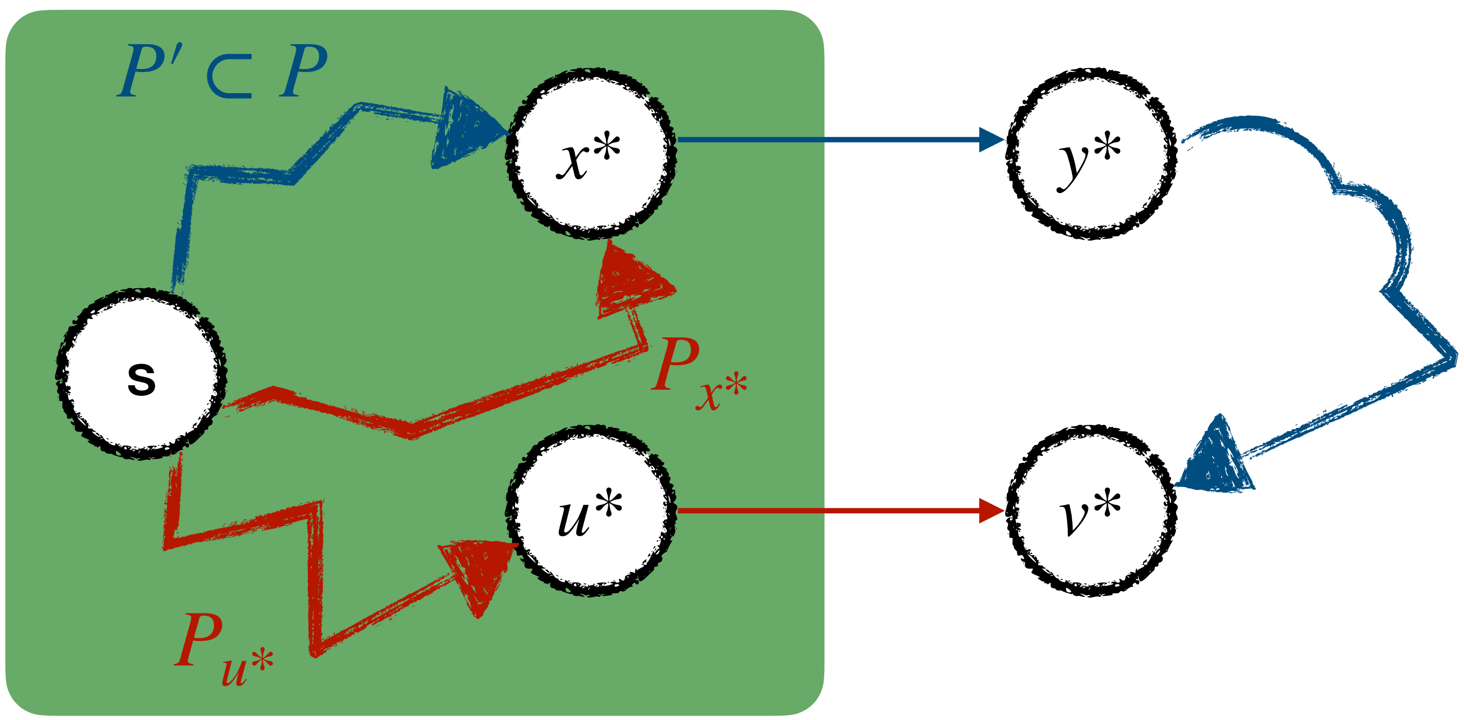


We know that the s-x^* path P_{x^*} is a shortest path. Why?

Correctness



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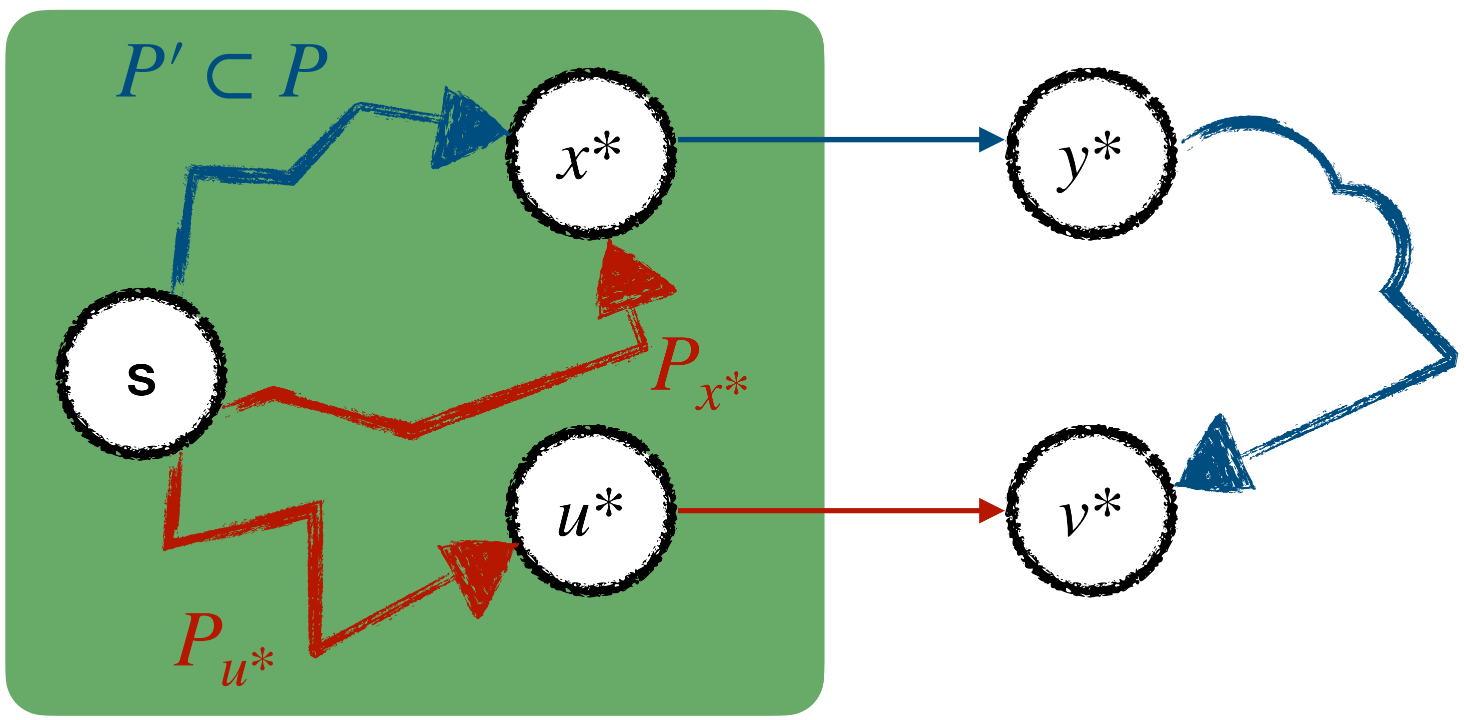

We now that the s-x^* path P_{x^*} is a shortest path. Why?
By the induction hypothesis.

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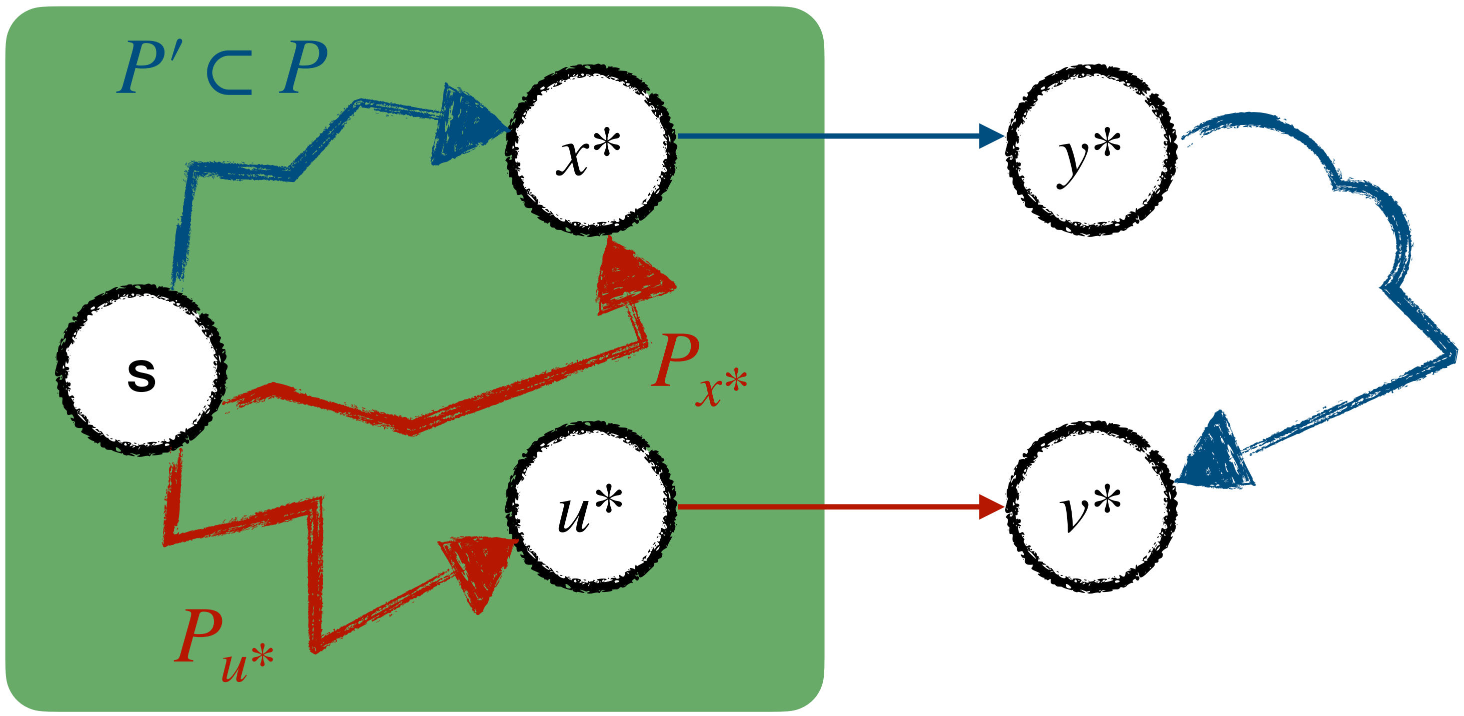

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This means that $\ell(P') \geq \ell(P_{x^*}) = d(x^*)$

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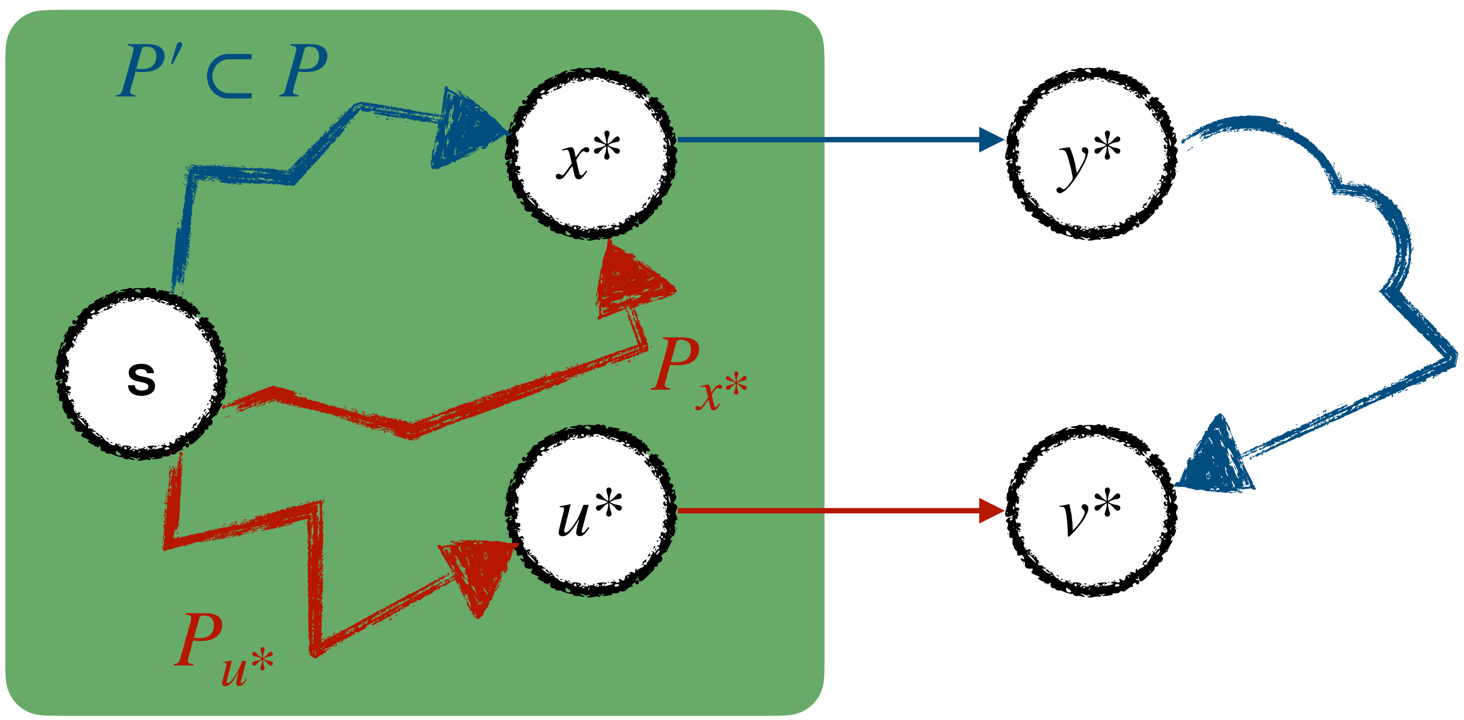

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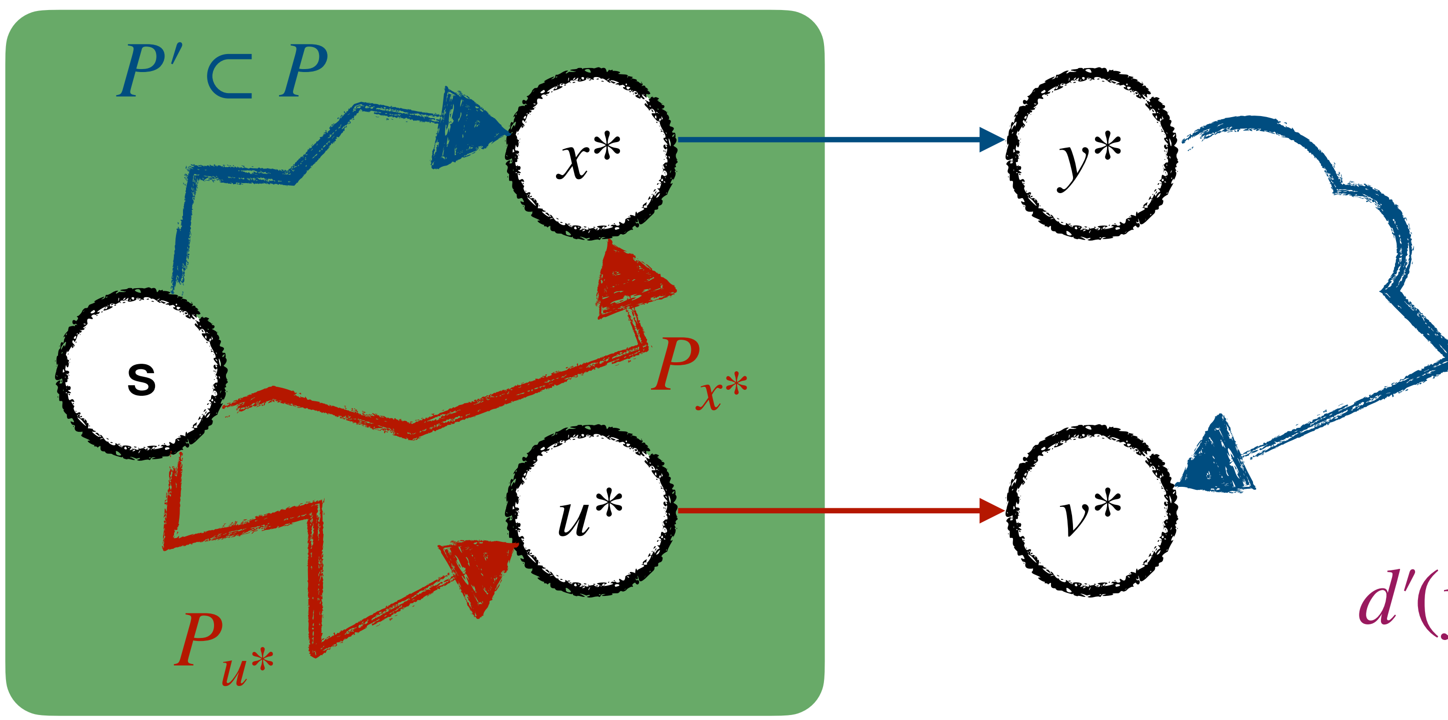
Therefore $\ell(P) \geq \ell(P') + \ell(x^*, y^*) \geq \ell(x^*, y^*) + d(x^*) \geq d'(y^*)$

Correctness



$$d'(y^*) = \min_{e=(u,y^*):u \in S} d(u) + \ell_e$$

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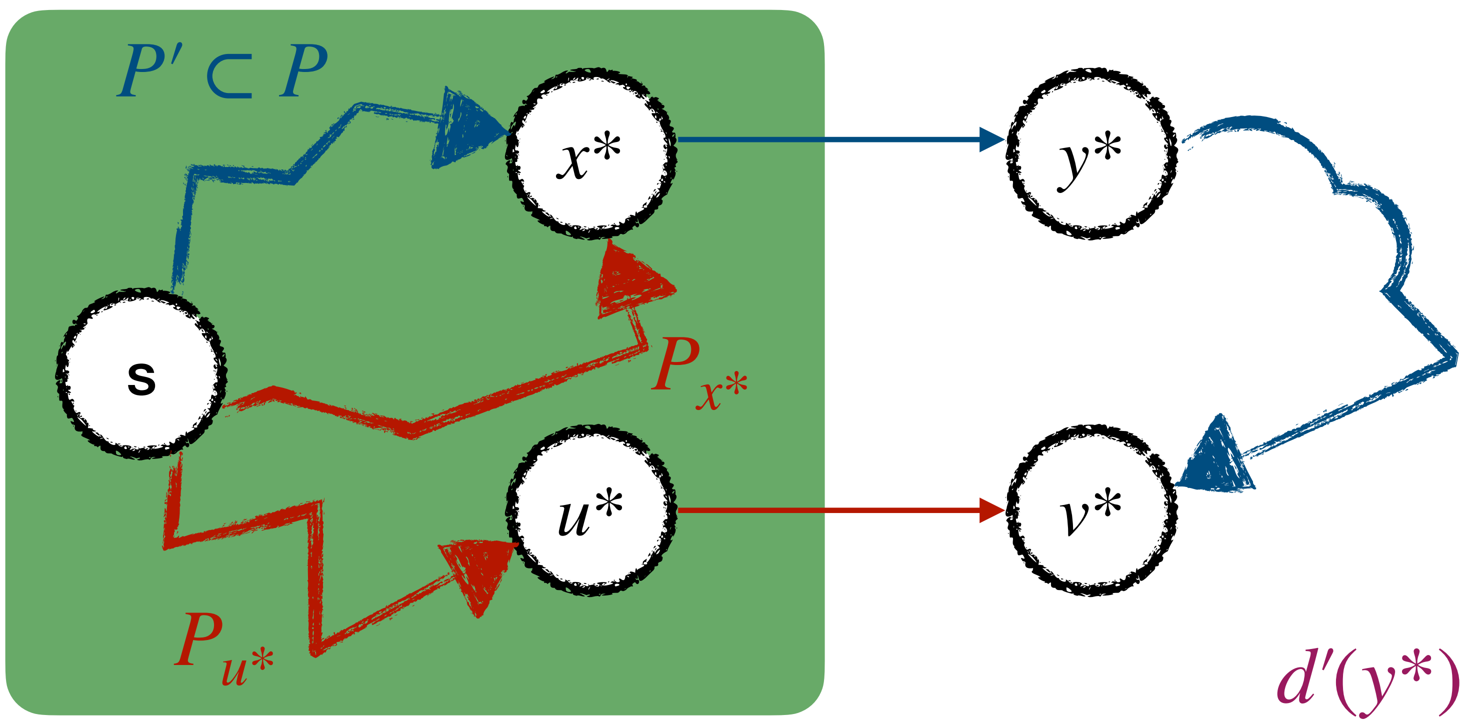

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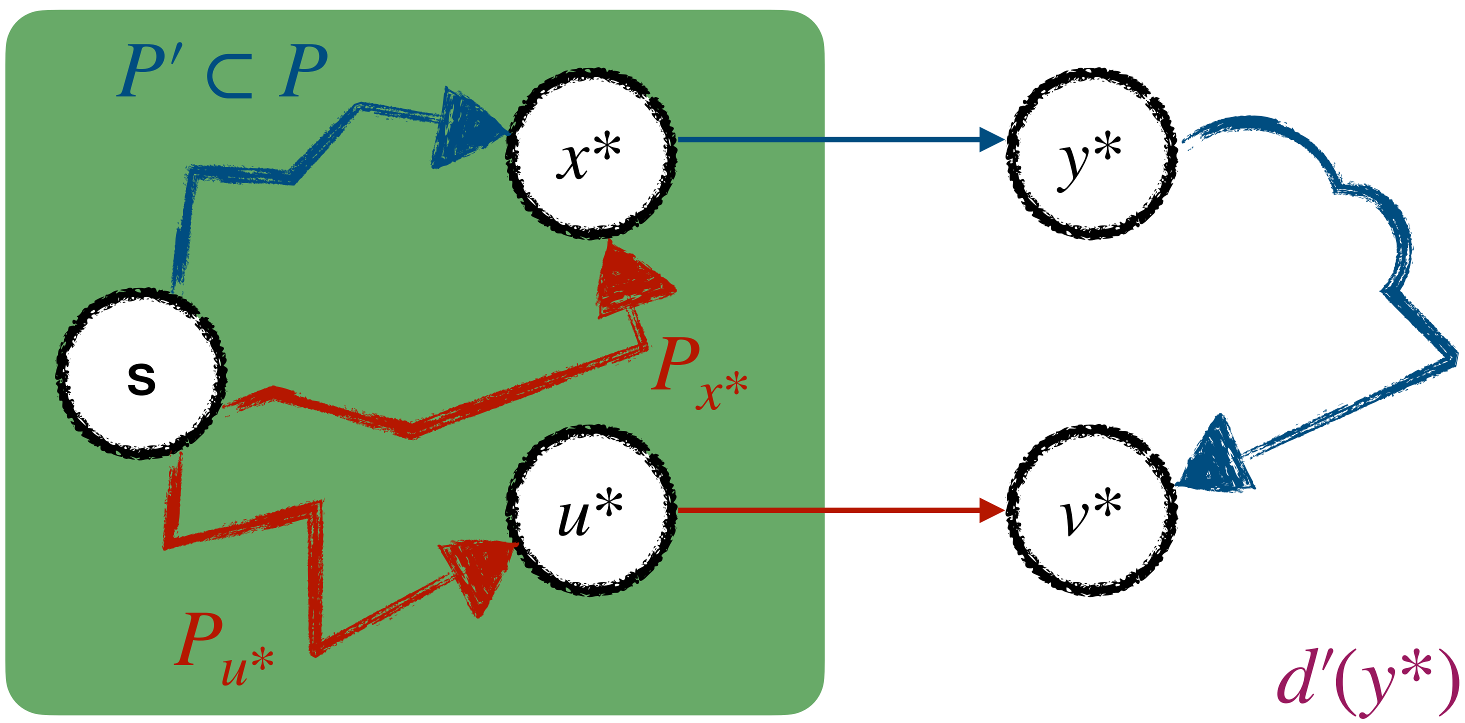

$$d'(y^*) = \min_{e=(u,y^*):u \in S} d(u) + \ell_e$$

Therefore $\ell(\,P\,) \geq d'(\,y^*\,)$

At the same time, we know that $d'(\,y^*\,) \geq d'(\,v^*\,) = \ell(\,P_{v^*}\,)$.
Why?

Dijkstra's Algorithm (Pseudocode)

Dijsktra (G, ℓ)

Let S be the set of explored nodes, A be a list of distances.

Initially $S = \{s\}$ and $d(s) = 0$, $A[s] = d(s) = 0$

While $S \neq V$

Select a node $v \in V - S$ connected via an edge with at least one node in S such that

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Add v to S and define $d(v) = d'(v)$

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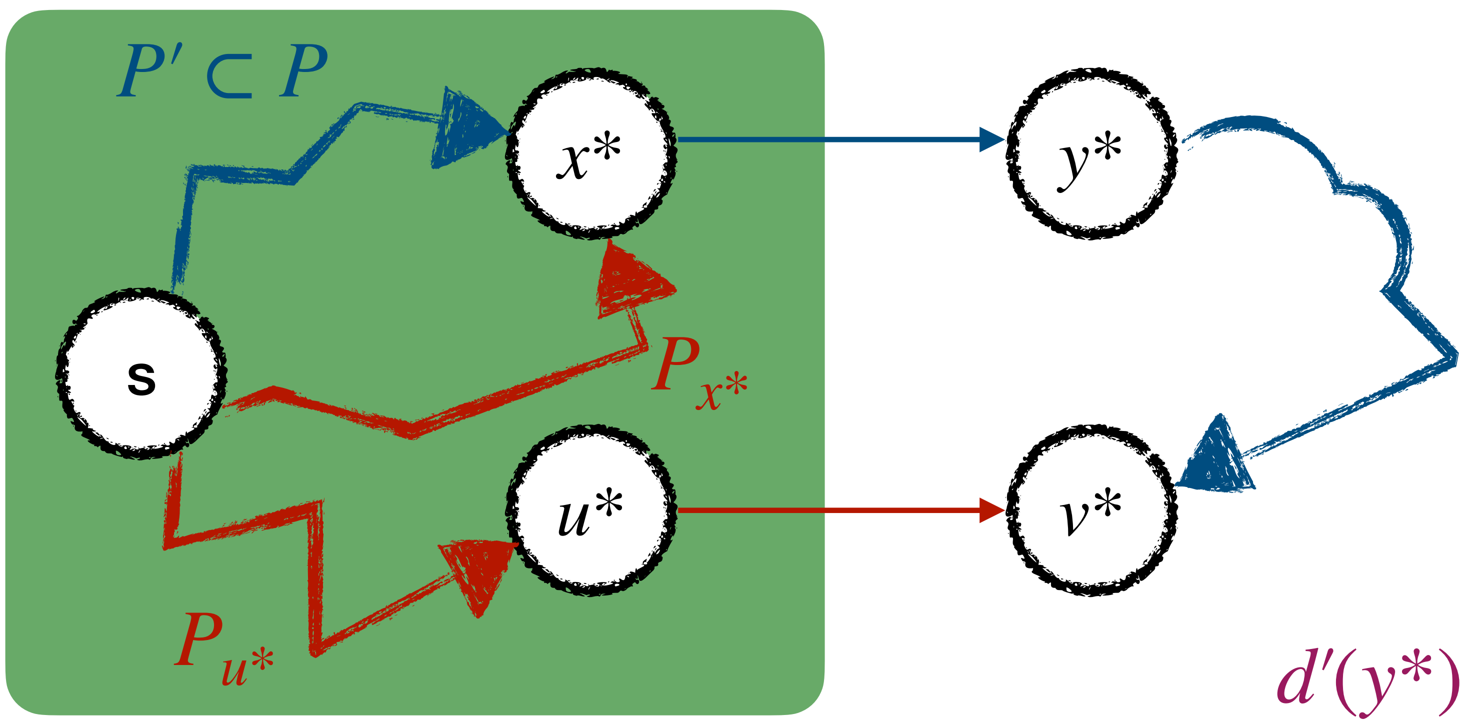

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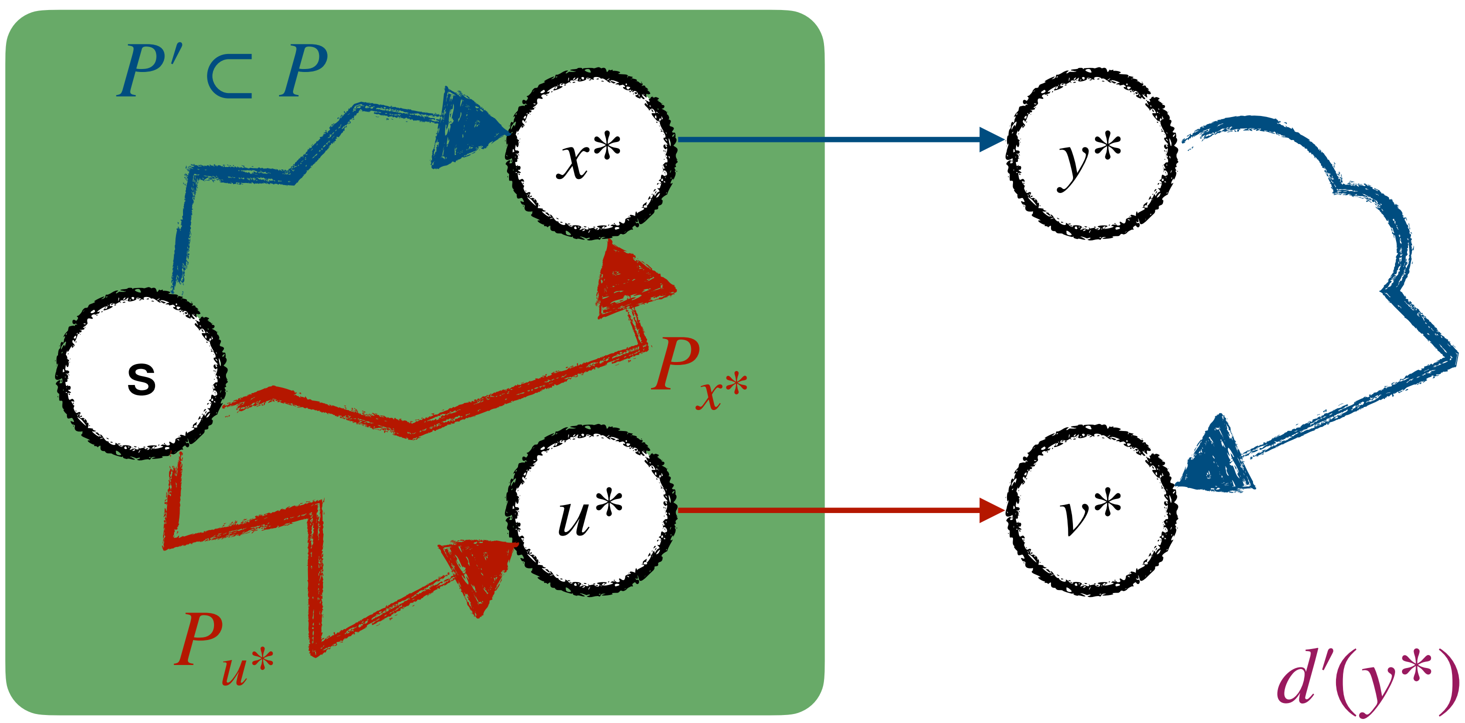
Why? Because v^* was chosen by Dijkstra's Algorithm.

Putting it together



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That implies that $\ell(\,P\,) \geq \ell(\,P_{v^*}\,)$, a contradiction.

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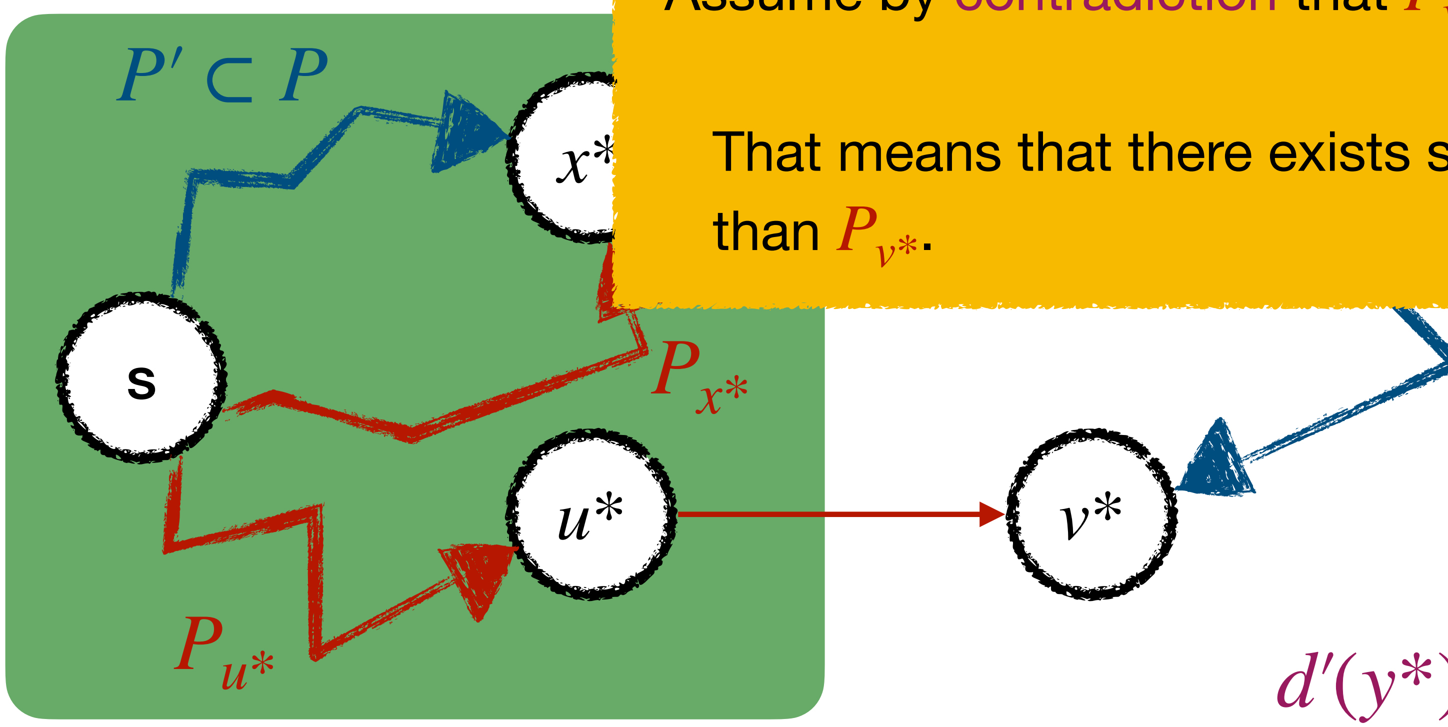

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Running Time

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Here, consider every node v outside S, and then consider all edges between S and v.

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Overall: $O(nm)$.

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$|E| = m$

Let $A[v] = d(v)$

Overall: $O(nm)$. **Not terrible, not great.**

Running Time

Lets look at the pseudocode.

That was somewhat naive. Can we do better?

Priority Queues

Tue 31 Oct	Lecture 12	Aris	**Lecture 12 - Heap operations and Priority Queues** **Lecture 12 - Heap operations and Priority Queues (.key file, better animations)** Recording from previous years (caution: different lecturer, different slides, presents all heap operations in Lecture 11 and Heapsort in Lecture 12).	CLRS 6.5 KT 2.5 (uses min-heap) Roughgarden 10.2, 10.5 (caution: RG uses min-heap and the terms "Heapify" for Build-Max-Heap and implements Min-Heapify as part of "ExtractMin").

- The values denote *priorities*.

- For Max-Priority Queues, the elements with the largest values are those with the highest priority.

Priority Queues

- Priority queue: A data structure that maintains
  - A set of elements S.
  - Each with an associated value, key(v).
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Priority Queue Operations

- Insert(Q, v) inserts a new item v in the priority queue.
- FindMin(Q) finds the element with the maximum priority (the smallest value) in the priority queue and returns it (but does not remove it).
- ExtractMin(Q) finds the element with the maximum priority (smallest value) in the priority queue, returns it, and deletes it from the queue.

Priority Queue Operations

- ExtractMin(Q) finds the element with the maximum priority (smallest value) in the priority queue, returns it, and deletes it from the queue.

- ChangeKey(Q, v, a) changes the key value of element v to key(v) $= a$.

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How do we get the node v to be added to S?

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Issue: Once we add nodes to S, we need to update the values $d'(v)$, i.e., the keys in the priority queue.

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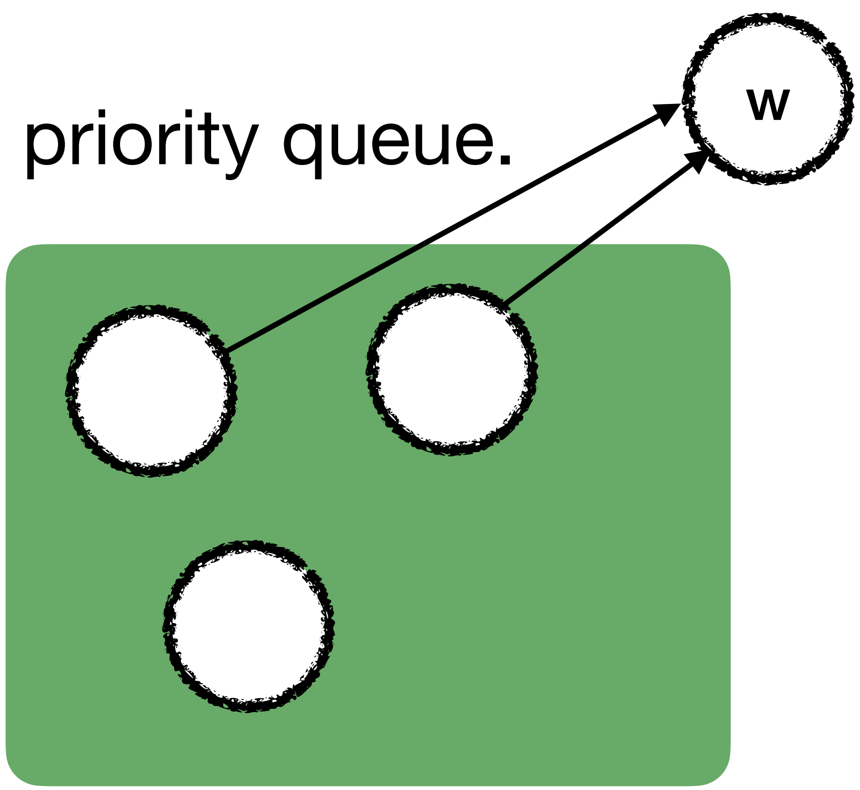

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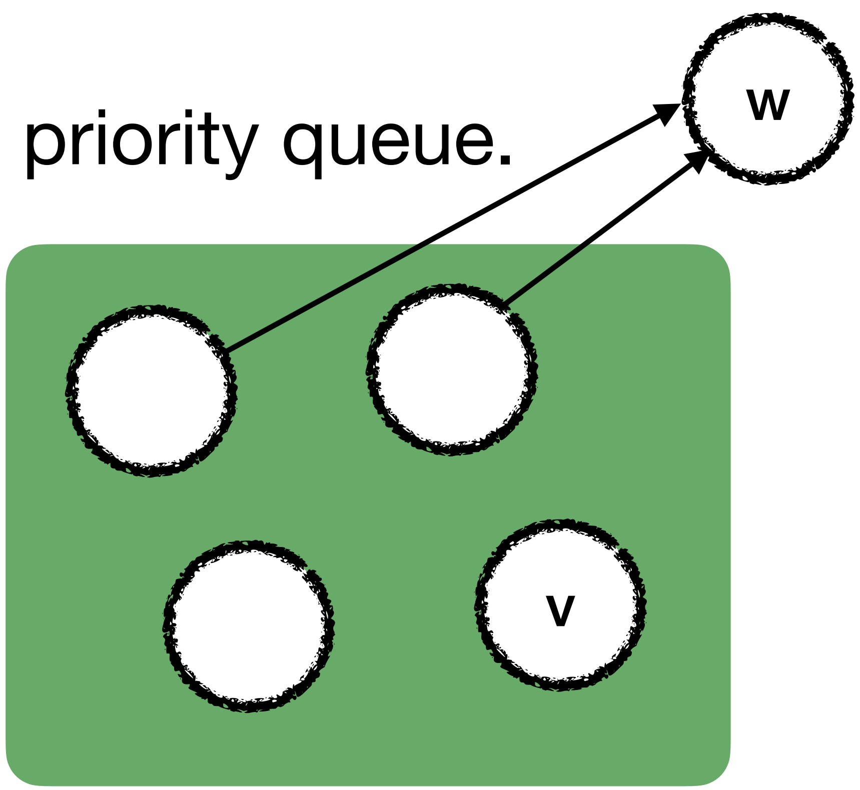

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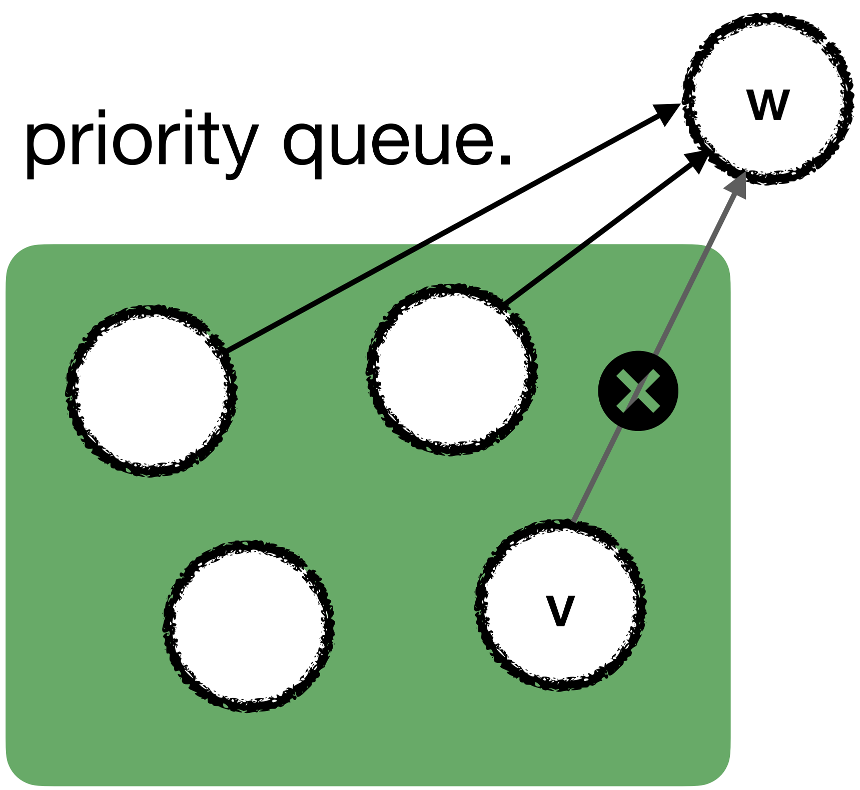

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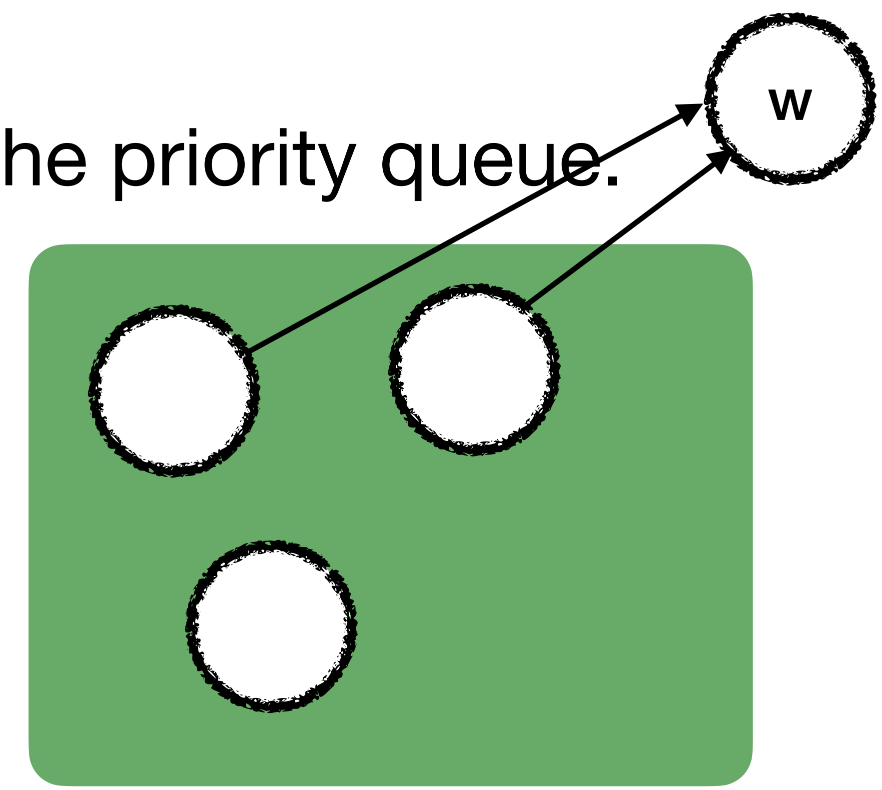

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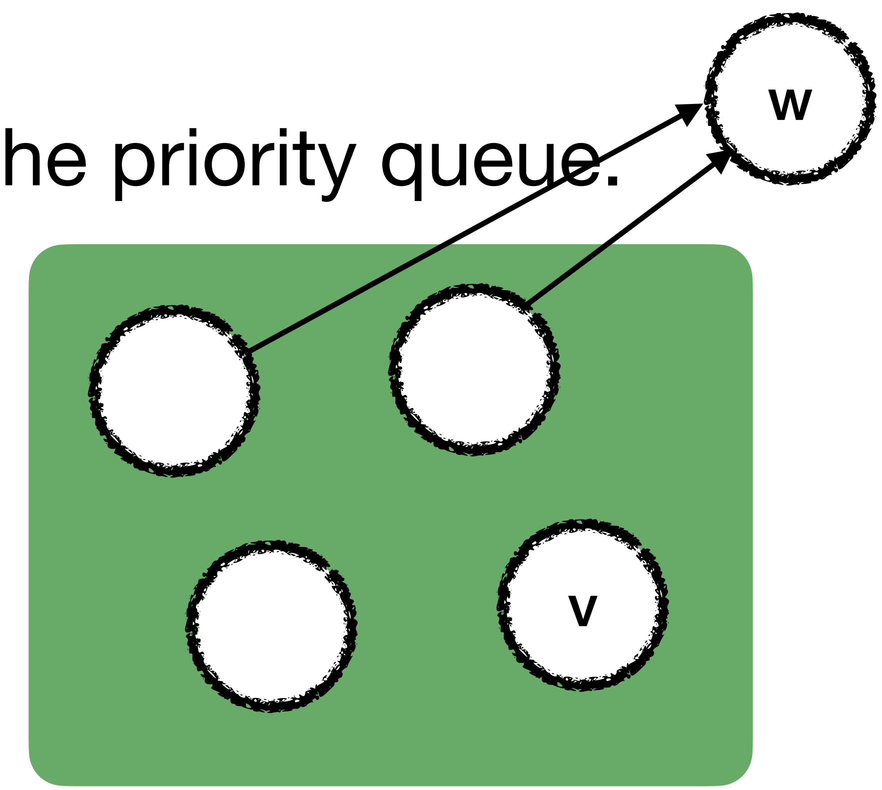

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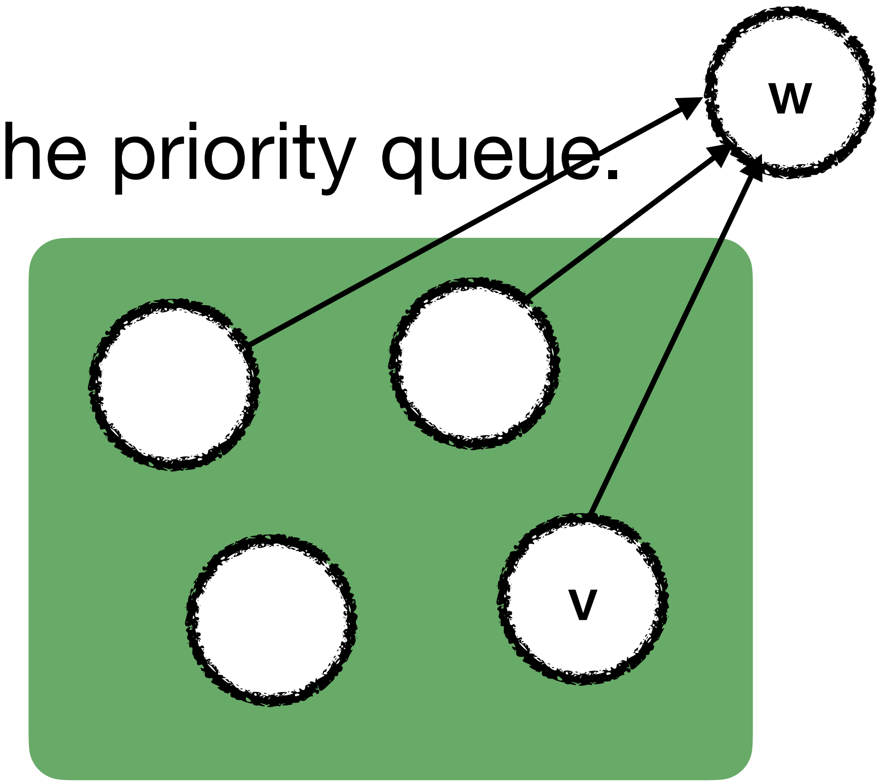

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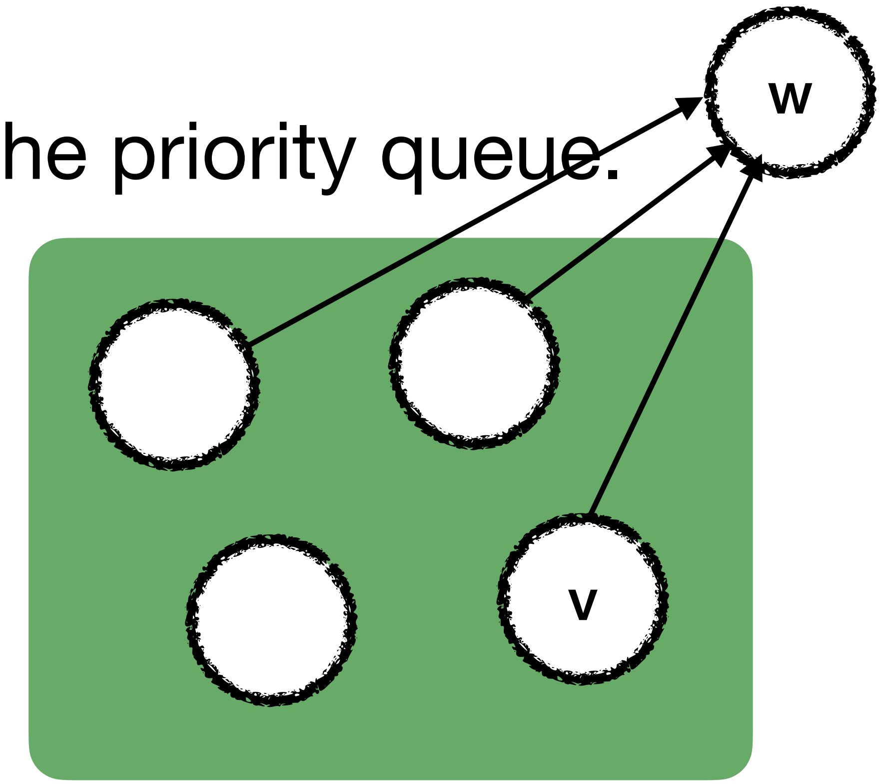

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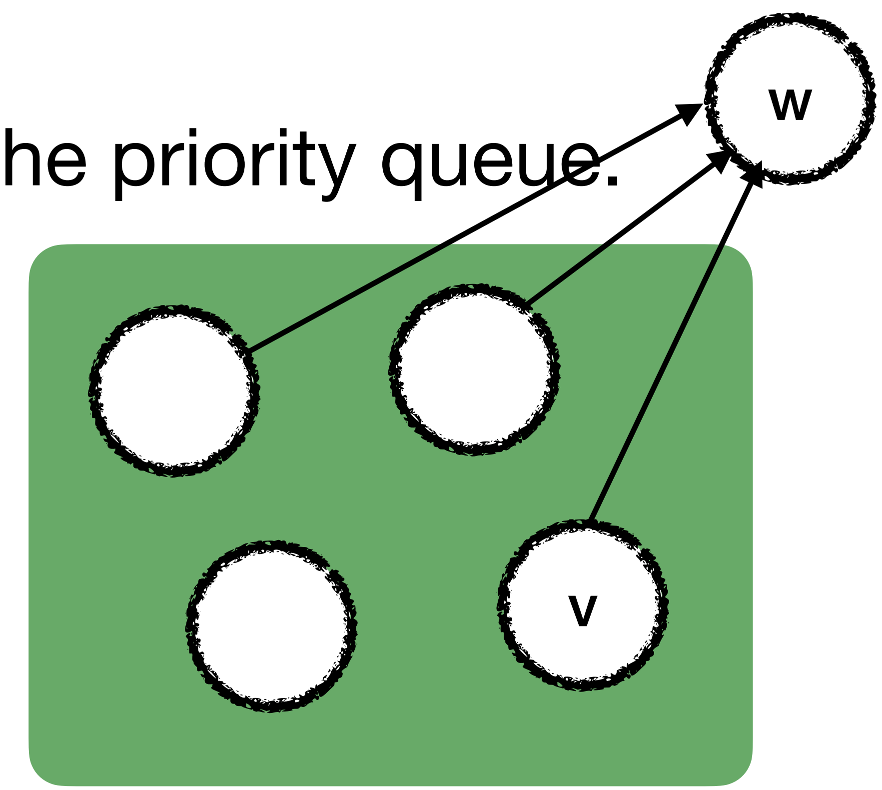

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ChangeKey(Q, v, a) **At most once per edge!**

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Overall: $O(m \log n)$

Reading

Kleinberg and Tardos Chapter 5.4. (or 4.4. in the online weird version). Slides follow this religiously.

Roughgarden 9.2., 9.3.

CLRS 24.3.

You can also find visualisers online and play around with them, e.g., https://www.cs.usfca.edu/~galles/visualization/Dijkstra.html and the more general https://visualgo.net/en/sssp?slide=1