Introduction to Algorithms and Data Structures

Dynamic Programming - Weighted Interval Scheduling

Dynamic Programming

- An technique for solving optimisation problems.
- Term attributed to Bellman (1950s).
    - “Programming” as in “Planning” or “Optimising”.

Dynamic Programming

- The paradigm of dynamic programming:
  - Given a problem P, define a sequence of subproblems, with the following properties:
    - The subproblems are ordered from the smallest to the largest.
    - The largest problem is our original problem P.
    - The optimal solution of a subproblem can be constructed from the optimal solutions of sub-sub-problems. (*Optimal Substructure*).
  - Solve the subproblems from the smallest to the largest. When you solve a subproblem, store the solution (e.g., in an array) and use it to solve the larger subproblems.

Recall: Interval Scheduling

- A set of requests {*1*, *2*, … , *n*}.
  - Each request has a starting time s(*i*) and a finishing time f(*i*).
  - Alternative view: Every request is an interval [s(*i*), f(*i*)].
- Two requests *i* and *j* are compatible if their respective intervals do not overlap.
- Goal: Output a schedule which maximises the number of compatible intervals.

Weighted Interval Scheduling

- A set of requests {*1*, *2*, … , *n*}.
  - Each request has a starting time s(*i*), a finishing time f(*i*), and a value v(*i*).
  - Alternative view: Every request is an interval [s(*i*), f(*i*)] associated with a value v(*i*).
- Two requests *i* and *j* are compatible if their respective intervals do not overlap.
- Goal: Output a schedule which maximises the total value of compatible intervals.

Greedy Approaches

- Which one of the following Greedy Algorithms might have a chance to work?
  - Earliest starting time.
  - Smallest interval.
  - Minimum number of conflicts.
  - Earliest finishing time.

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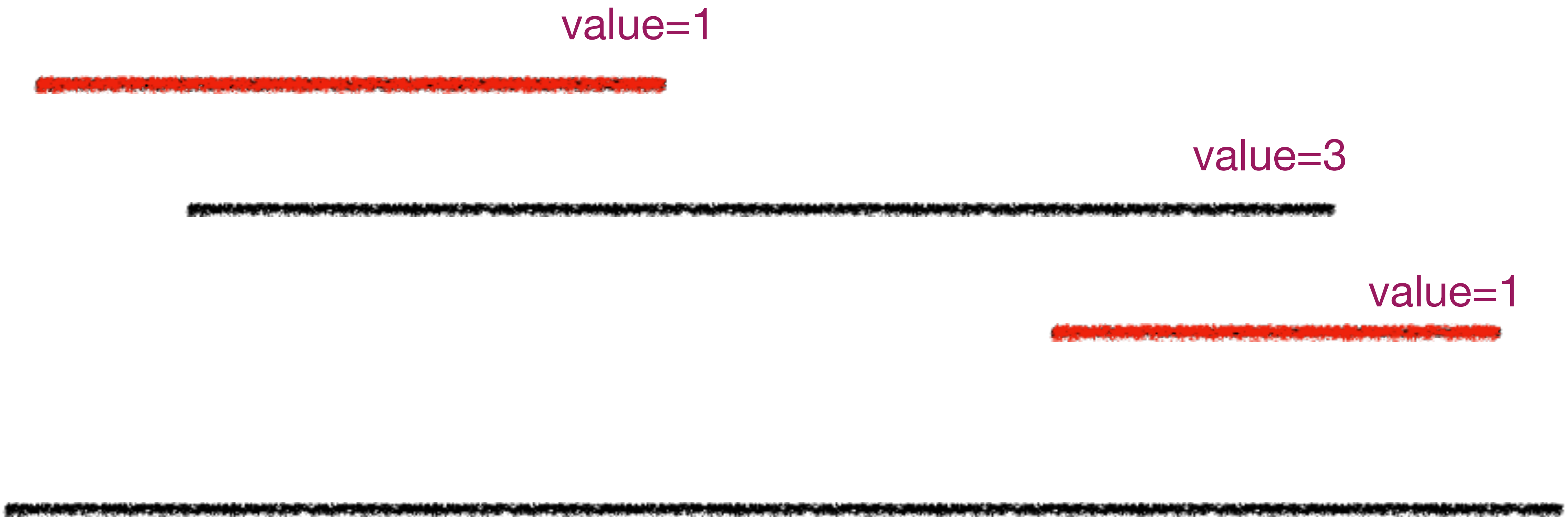
Does it work?

No approach that ignores the values can work!



value=1

value=3

value=1

Greedy Approaches

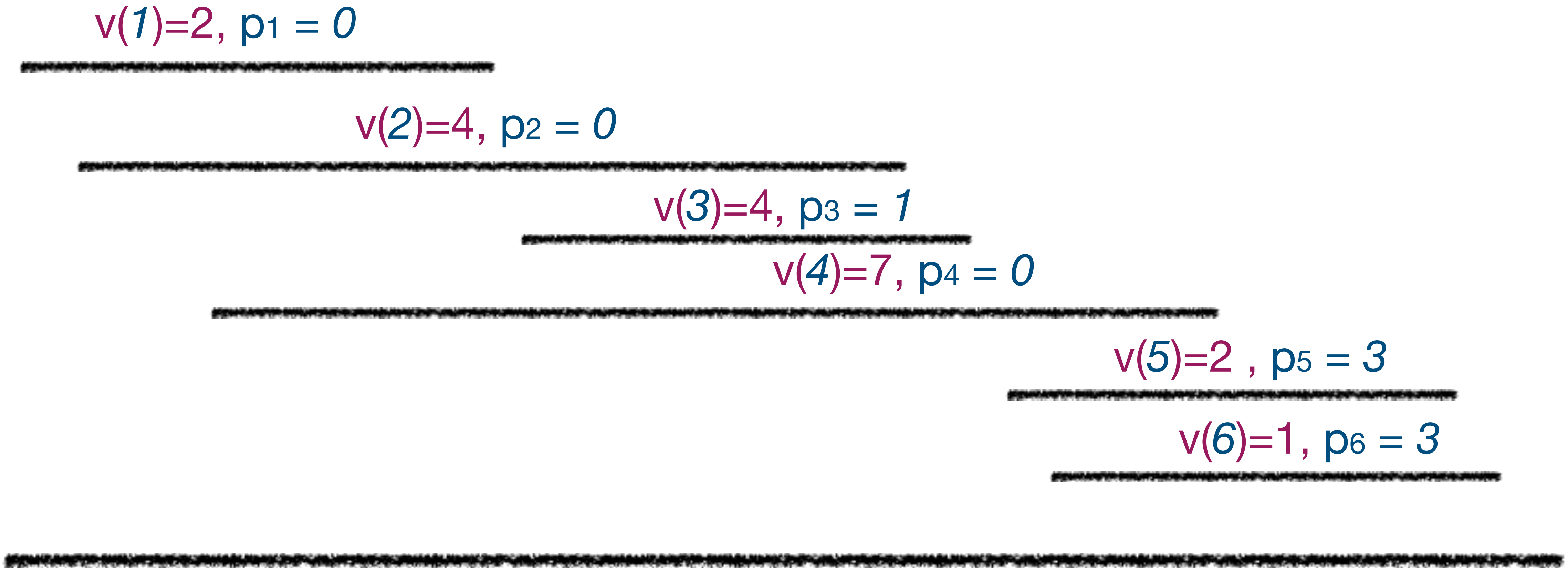
- Which one of the following Greedy Algorithms might have a chance to work?
  - Earliest starting time.
  - Smallest interval.
  - Minimum number of conflicts.
  - Earliest finishing time.
  - Largest value.

Does it work?

A view of the input

- Consider the intervals in sorted order of non-decreasing finishing time, i.e., $f(1) \leq f(2) \leq \ldots \leq f(n)$.
- For an interval $j = (s(j), f(j))$, let p_j be the largest index $i < j$ such that intervals i and j are disjoint.
    - i.e., i is the last interval in the ordering that ends before j begins.
    - if no such interval exists, define $p_j = 0$.

Example



$v(1)=2$, $p_1 = 0$

$v(2)=4$, $p_2 = 0$

$v(3)=4$, $p_3 = 1$

$v(4)=7$, $p_4 = 0$

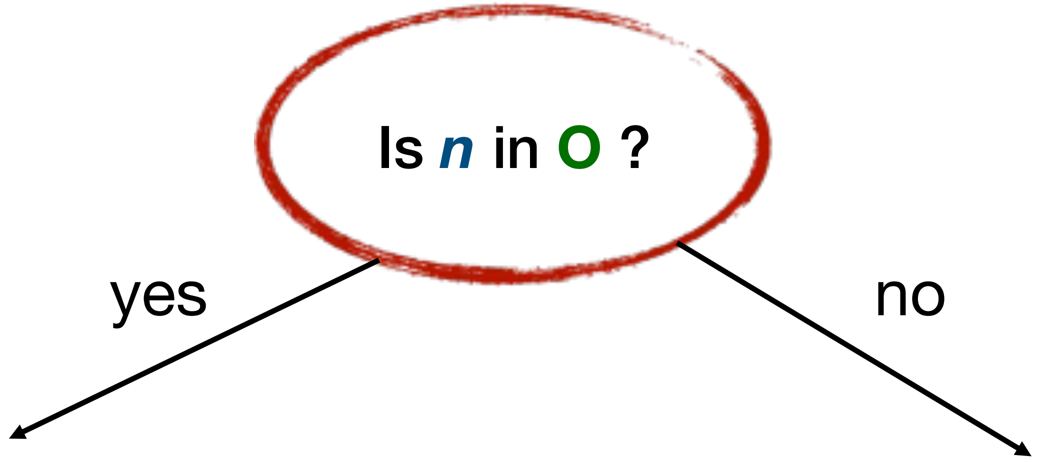
$v(5)=2$, $p_5 = 3$

$v(6)=1$, $p_6 = 3$

Step-by-step?

- Let O be the optimal schedule.
- Fact: O either contains interval *n* or not.

Building up a solution



Is *n* in O ?

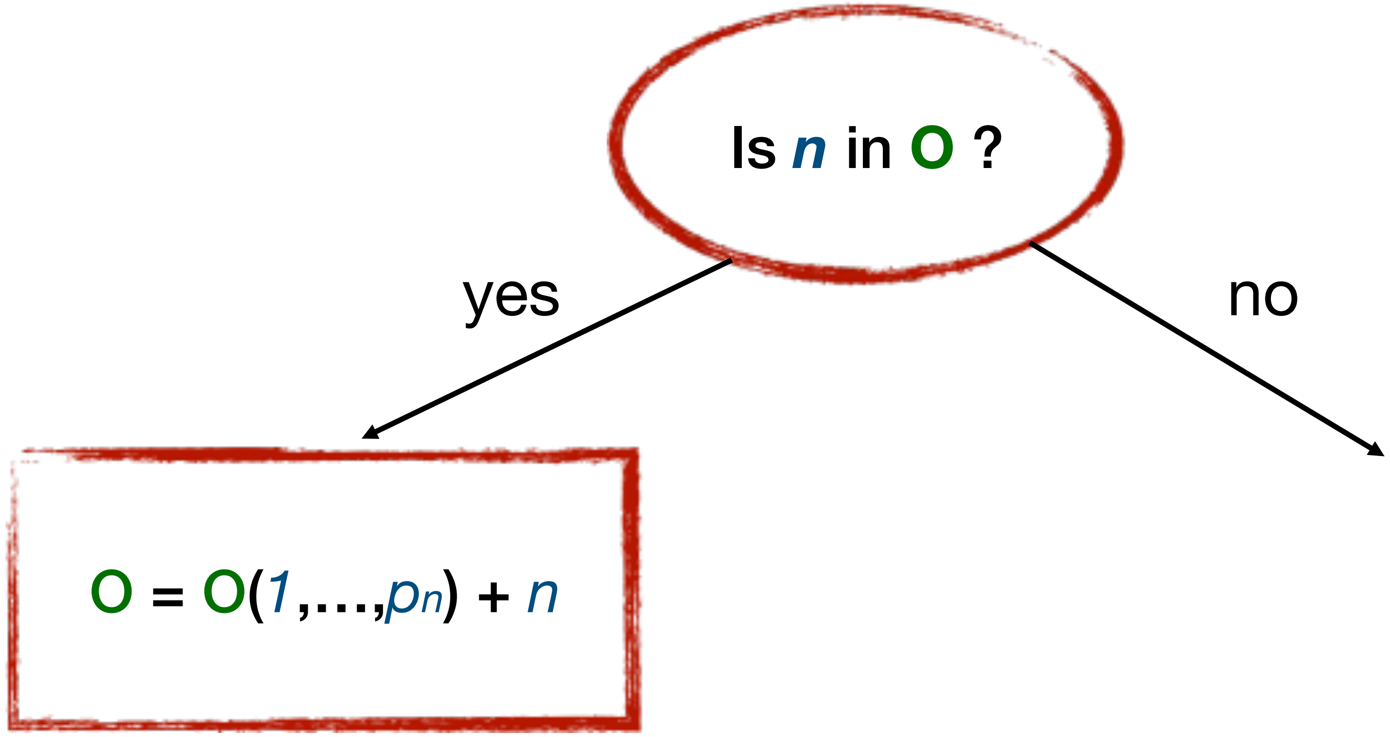
yes

no

If *n* is in O

- What does that mean for the other intervals?
- Any interval that overlaps with *n* cannot be in O.
- Any interval *j* > p_n cannot be in O.
- O contains an optimal solution O' of the subproblem {*1*, *2*, …, p_n } (why?)
  - Because otherwise we could replace O with O' ∪ {*n*} and obtain a better solution.
- Lets use O(*i*, …, *j*) to denote the optimal solution on (sorted) intervals *i*, … , *j*.

Building up a solution



Is n in O ?

yes

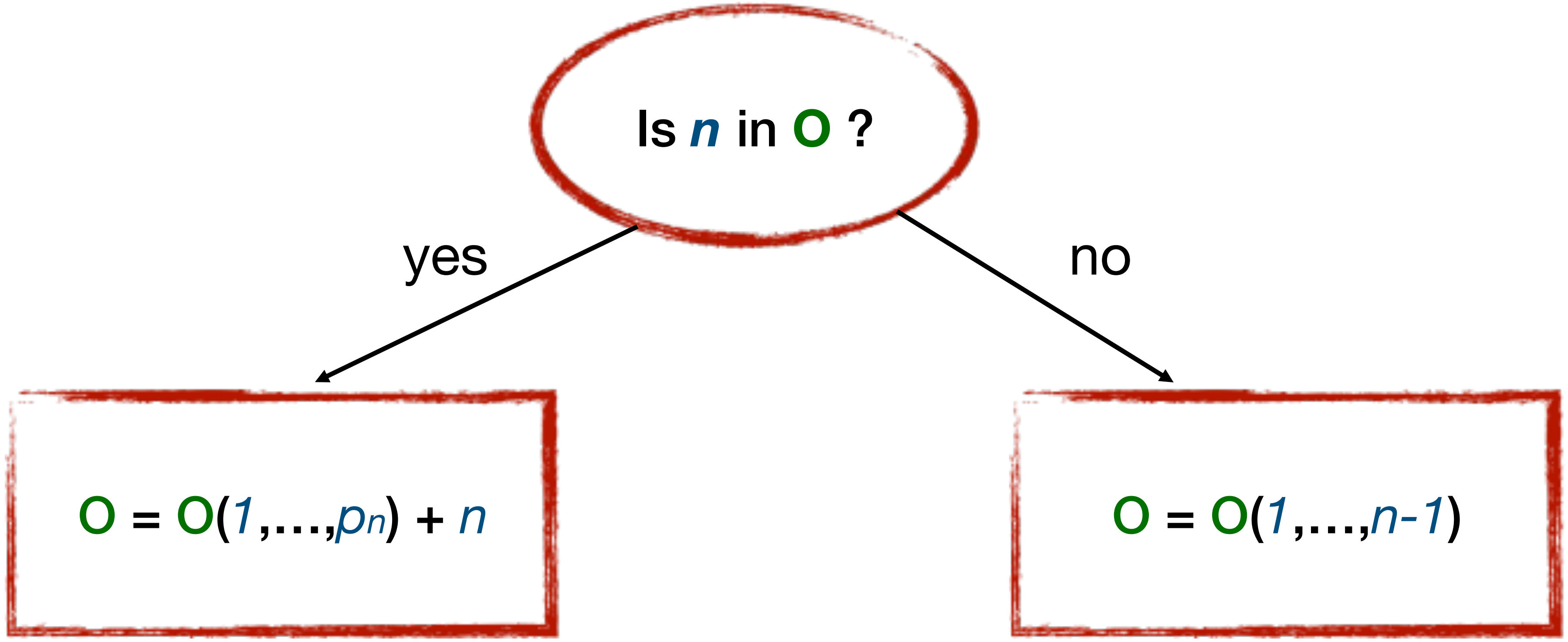
no

$O = O(1,\dots,p_n) + n$

If n is not in O

- Then $\mathrm{O} = \mathrm{O}(1, \ldots, n-1)$
- Same argument: Since n is not chosen, all intervals $1, \ldots, n-1$ are “free” to be chosen.
- Not picking the optimal schedule for them would violate the optimality of O.

Building up a solution



Is n in O ?

yes

no

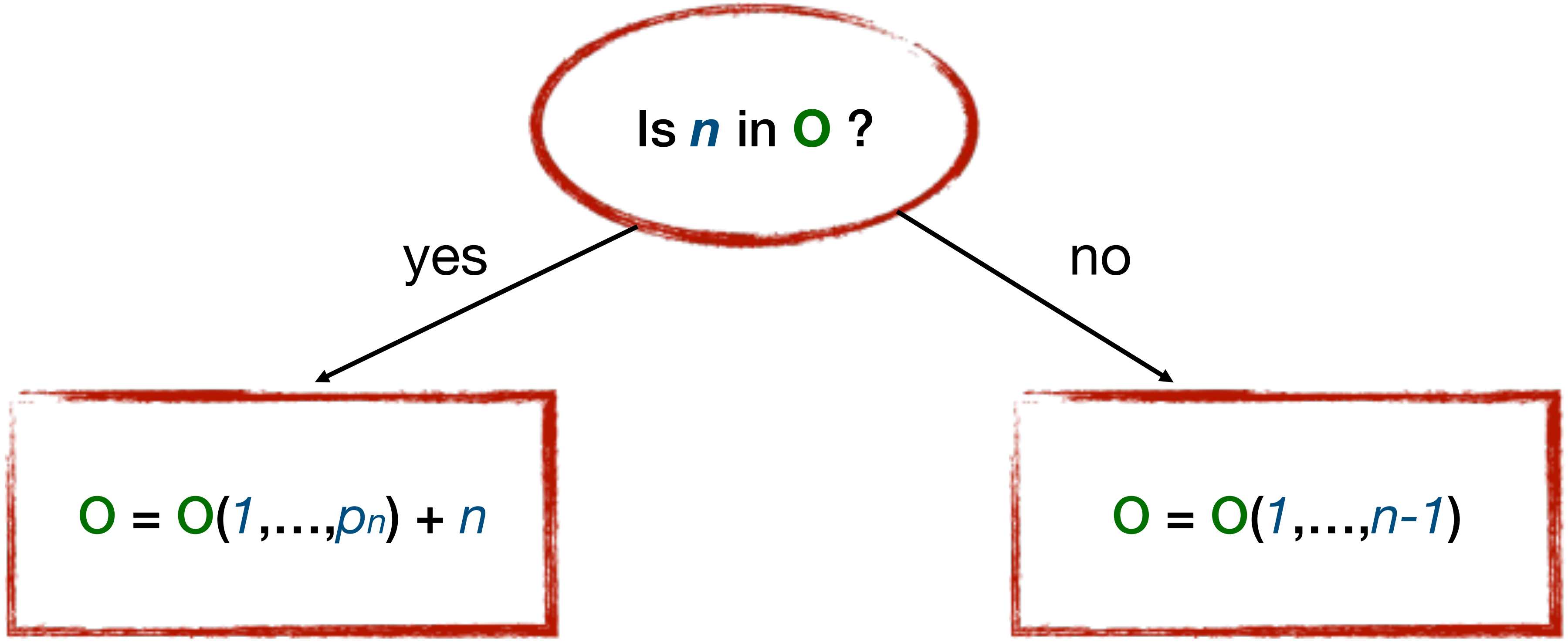
$O = O(1,\ldots,p_n) + n$

$O = O(1,\ldots,n-1)$

Building up a solution

- So, in order to find O, it suffices to look at smaller problems and find $O(1, \dots, j)$ for some j.
- Let O_j be a shorthand for $O(1, \dots, j)$ and let $OPT(j)$ be its total value.
- Define $OPT(0) = 0$.
- Then, $O = O_n$ with value $OPT(n)$.

Building up a solution



Is n in O ?

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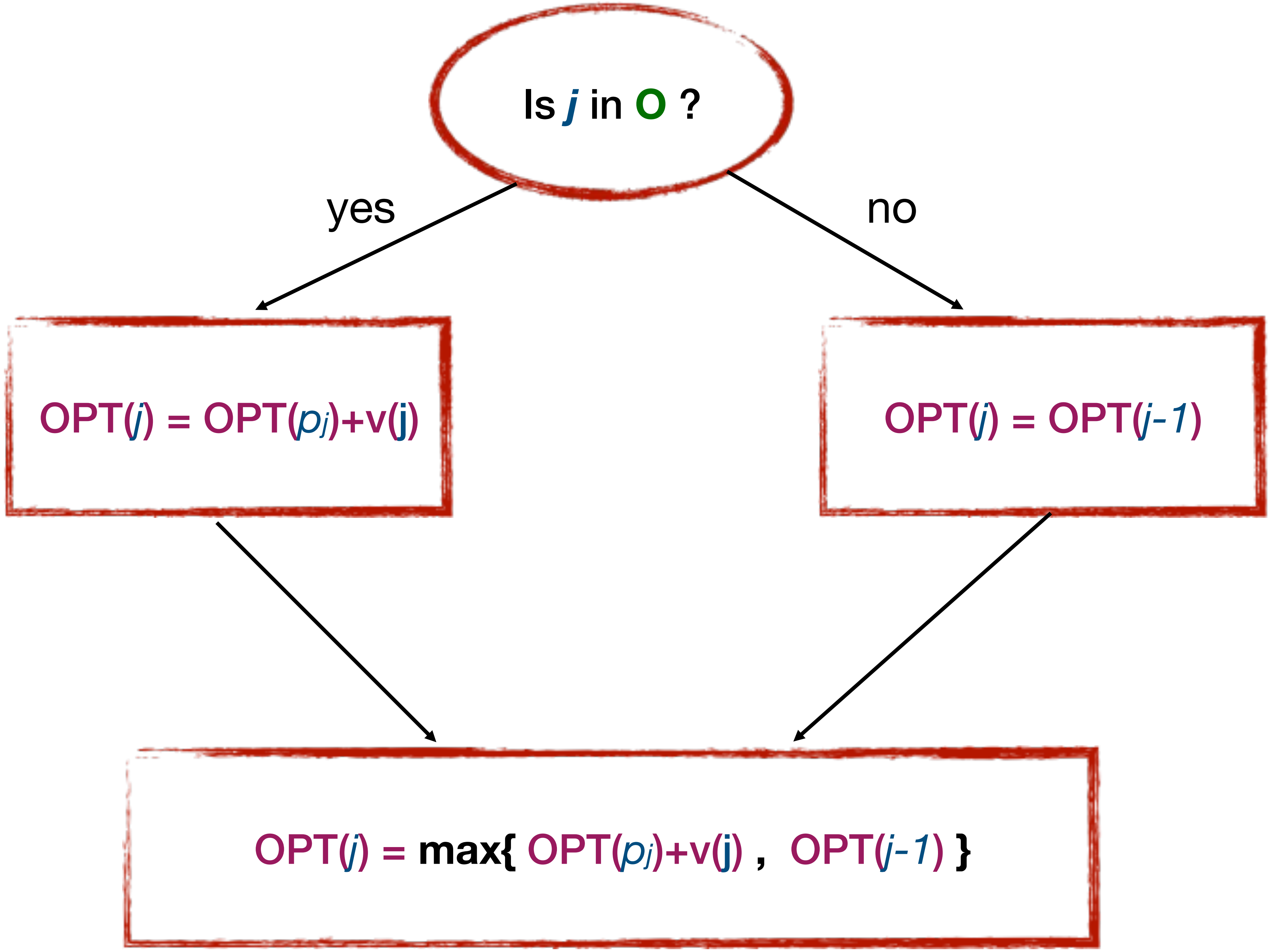
Generalising



Is j in **O** ?

yes

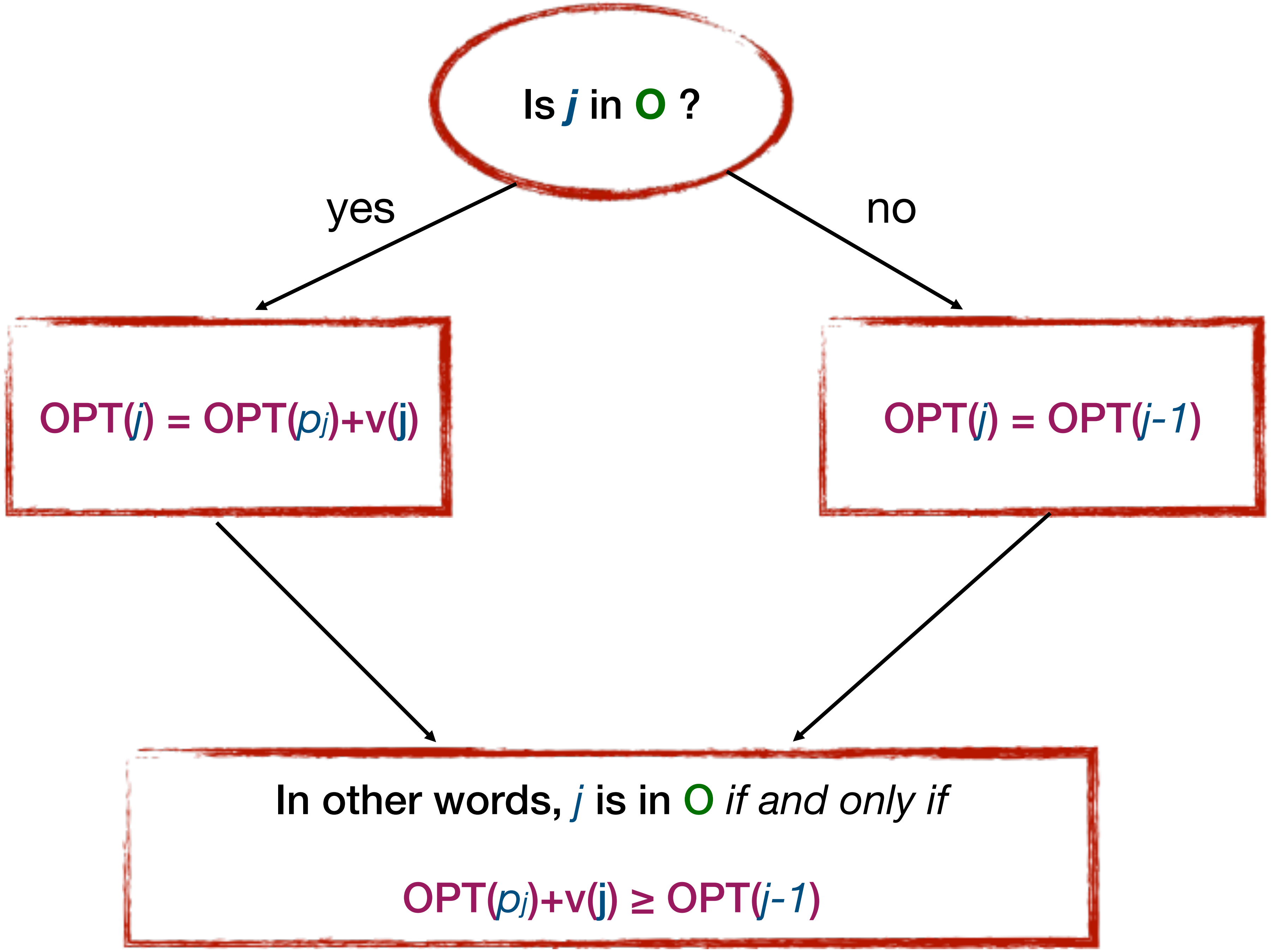
no

$\mathrm{OPT}(j) = \mathrm{OPT}(p_j) + v(\mathbf{j})$

$\mathrm{OPT}(j) = \mathrm{OPT}(j-1)$

$\mathrm{OPT}(j) = \max\{\, \mathrm{OPT}(p_j) + v(\mathbf{j}) \,,\ \mathrm{OPT}(j-1) \,\}$

Generalising



Is j in O ?

yes

no

$\mathrm{OPT}(j) = \mathrm{OPT}(p_j)+v(\mathbf{j})$

$\mathrm{OPT}(j) = \mathrm{OPT}(j-1)$

In other words, j is in O *if and only if*

$\mathrm{OPT}(p_j)+v(\mathbf{j}) \geq \mathrm{OPT}(j-1)$

Building up a solution

$$\mathrm{OPT}(j) = \max\{\, \mathrm{OPT}(p_j)+v(j)\ ,\ \ \mathrm{OPT}(j-1)\,\}$$

- What does this look like?
- Assume that there was an algorithm that inputed $\{1, \ldots, j\}$ and outputted $\mathrm{OPT}(j)$.
- It’s a recurrence relation!

Building up a solution

- What does this look like?
- Assume that there was an algorithm that inputed $\{1, \ldots, j\}$ and outputted OPT(j).
- It's a recurrence relation!

```
ComputeOpt(j)
    If j = 0 then
        Return 0
    Else
        Return max{v(j) + ComputeOpt(p_j) , ComputeOpt(j-1)}
    EndIf
```

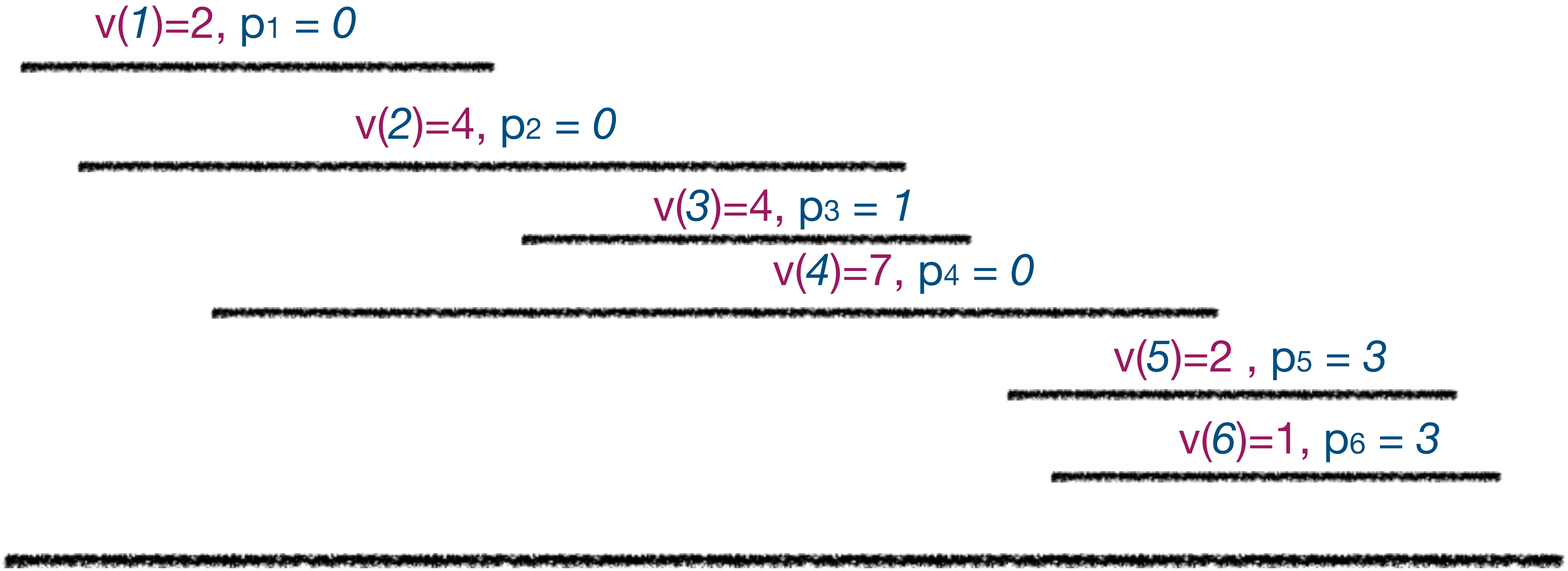
Correctness

- **ComputeOPT**(j) correctly computes OPT(j) for each j=1, …n
- Proof by induction:
  - Base Case: OPT(0) = 0 by definition.
  - Inductive step: Assume that it is true for all $i < j$. (inductive hypothesis).

    Return **max{**v(j) + **ComputeOpt**(p_j) **, ComputeOpt**(j-1)**}**

    OPT(j) = **max{** OPT(p_j)+v(j) **,** OPT(j-1) **}**

Example



$v(1)=2,\ p_1 = 0$

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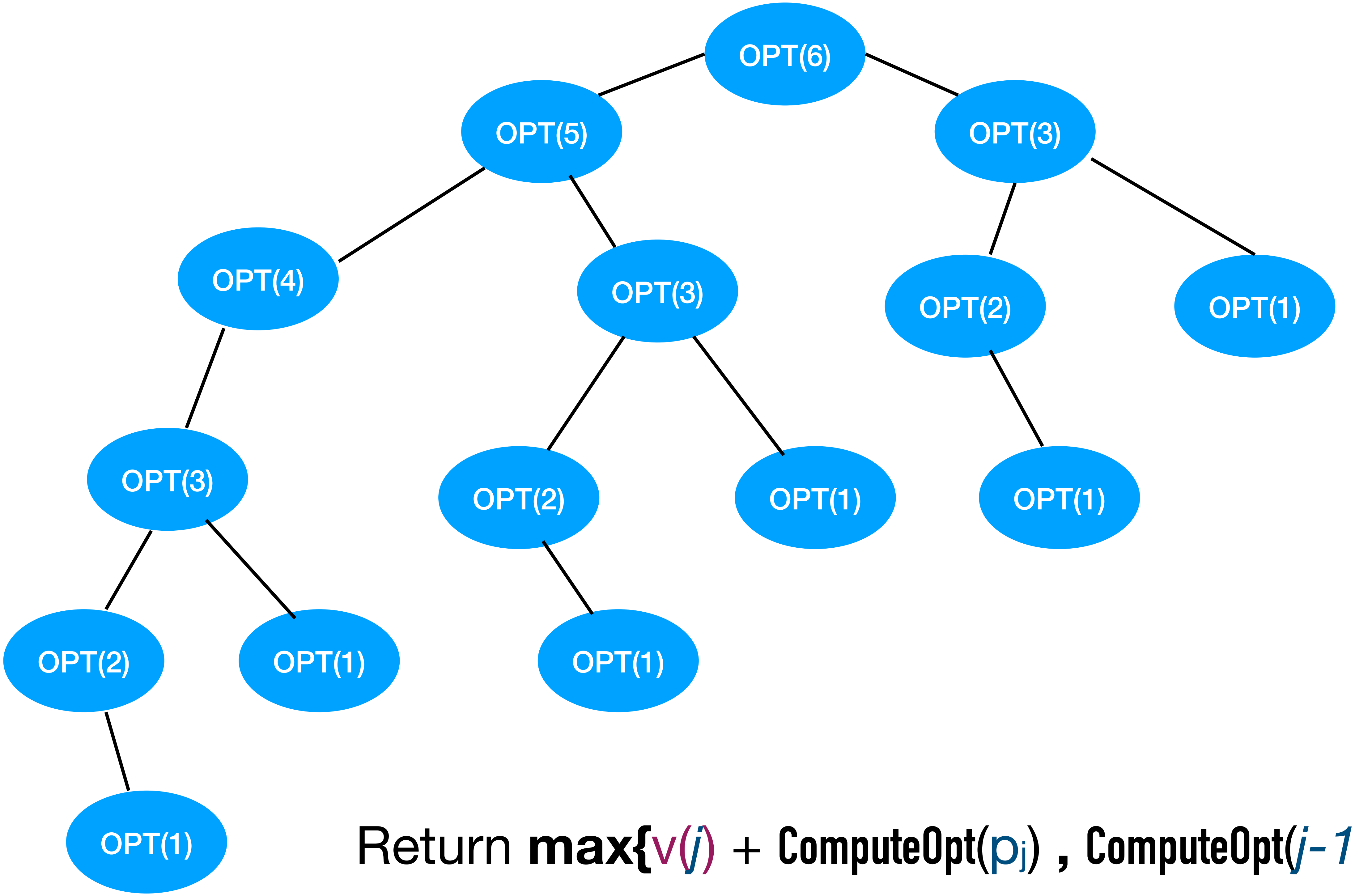
$v(5)=2\ ,\ p_5 = 3$

$v(6)=1,\ p_6 = 3$

Example



$p_6 = 3$

$p_5 = 3$

$p_4 = 0$

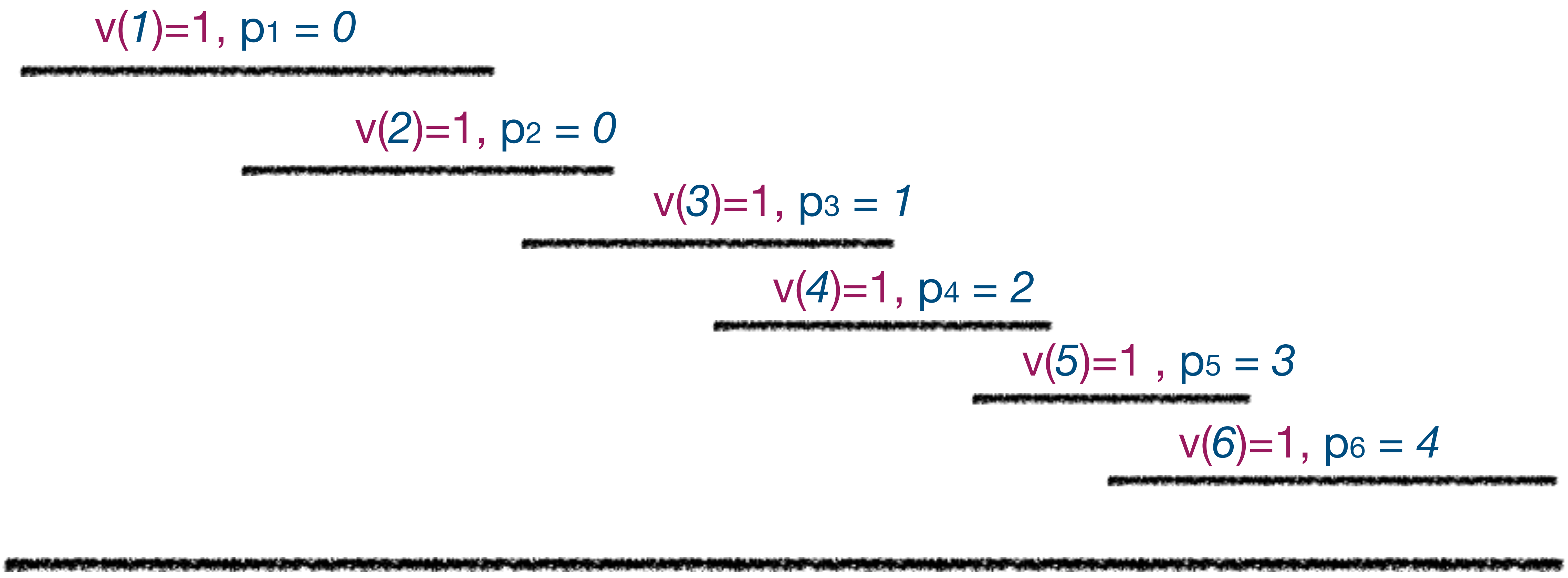
$p_3 = 1$

$p_2 = 0$

$p_1 = 0$

Return **max{**$v(j)$ + **ComputeOpt**(p_j) **, ComputeOpt**($j-1$)**}**

Another example



$v(1)=1$, $p_1 = 0$

$v(2)=1$, $p_2 = 0$

$v(3)=1$, $p_3 = 1$

$v(4)=1$, $p_4 = 2$

$v(5)=1$, $p_5 = 3$

$v(6)=1$, $p_6 = 4$

ComputeOpt(6) requires **ComputeOpt**(5) and **ComputeOpt**(4)

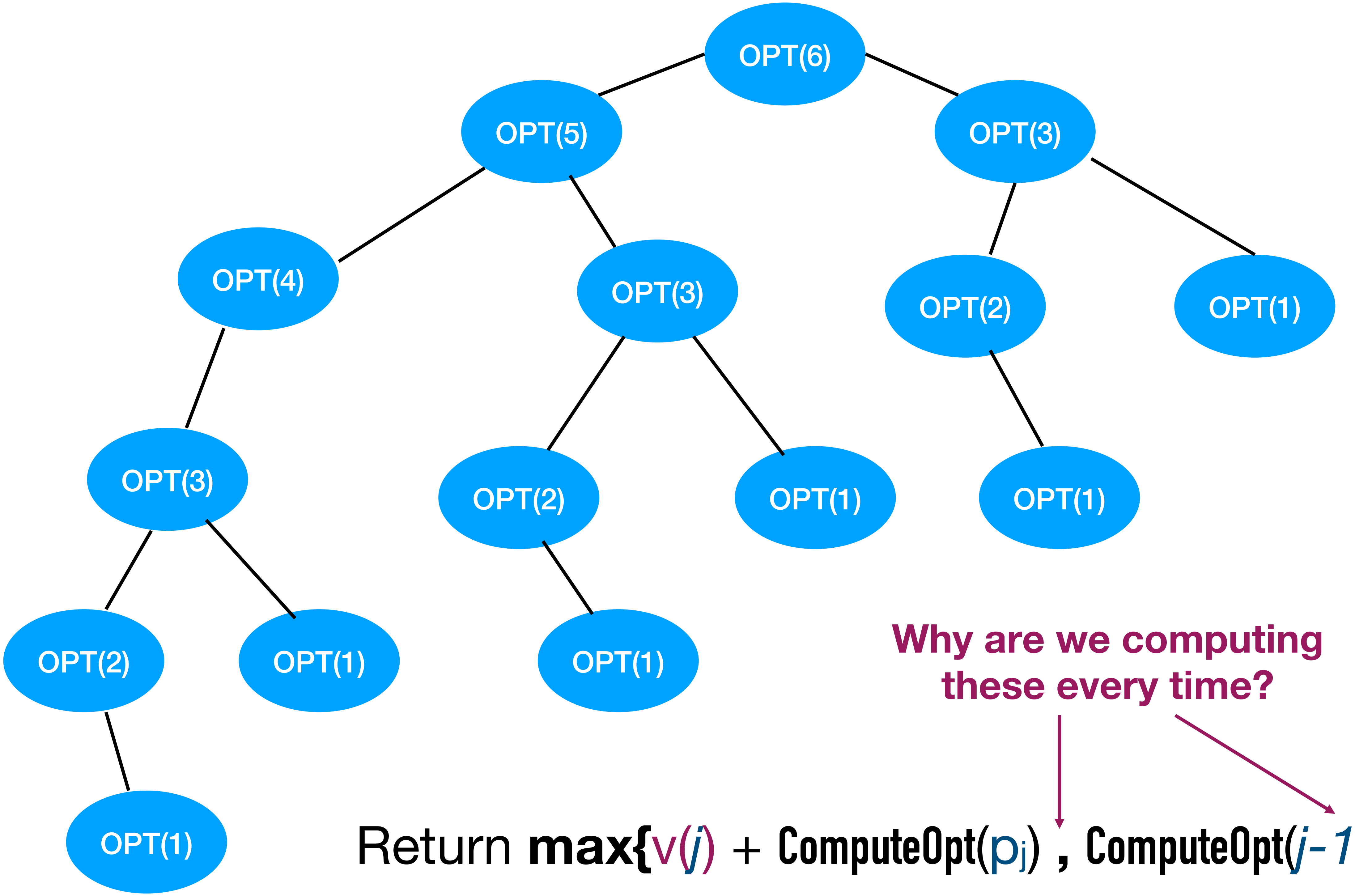
ComputeOpt(5) requires **ComputeOpt**(4) and **ComputeOpt**(3)

ComputeOpt(4) requires **ComputeOpt**(3) and **ComputeOpt**(2)

Running time

- What is the running time of the algorithm?
- A problem of size j requires solving problems of sizes $j-1$ and $j-2$.
- Do you know any numbers for which $F(n) = F(n-1) + F(n-2)$?
  - Fibonacci numbers.
- The nth Fibonacci number is approximately $\phi^n/\sqrt{5}$
- The running time of our algorithm is $\Omega(2^n)$!

Example



$p_6 = 3$

$p_5 = 3$

$p_4 = 0$

$p_3 = 1$

$p_2 = 0$

$p_1 = 0$

Why are we computing these every time?

Return **max{**$v(j)$ + **ComputeOpt**(p_j) **, ComputeOpt**($j-1$)**}**

Memoization

- Compute **ComputeOpt**(*j*) once for every *j*.
- Store it in an accessible place to use again in the future.
- Keep an array M[*0*, … ,*n*].
    - Initially M[j] = “empty” for all *j*.
    - When **ComputeOpt**(*j*) is calculated, M[*j*] = **ComputeOpt**(*j*)

A more clever implementation

```
M-ComputeOpt($j$)

    If $j$=0 then
        Return 0

    Else if M[$j$] is not empty then
        Return M[$j$]

    Else
        M[$j$] = max{v($j$) + M-ComputeOpt($p_j$) , M-ComputeOpt($j-1$)}
        Return M[$j$]

    EndIf
```

Running time

- In each call of **M-ComputeOpt**, there is a constant number of operations, besides the recursive calls. So the running time is bounded by the number of recursive calls.
- The two recursive calls only happen when M[j] is empty.
- But when they happens, M[j] is no longer empty.
- So the recursively calls only happen O(n) times.
- The running time of **M-ComputeOpt** is O(n), assuming we are given the intervals as sorted by their finishing times, otherwise O($n \log n$), to sort them first.

So our algorithm …

- … solved the main problem by solving subproblems of smaller sizes,
- stored the solutions to the smaller problems in an array,
- recalled them from the array every time they needed to used. (memoization).
- Anything else?

What does M-ComputeOpt(n) actually find?

```
M-ComputeOpt($j$)

    If $j$=0 then
        Return 0

    Else if M[$j$] is not empty then
        Return M[j]

    Else
        M[$j$] = max{v($j$) + M-ComputeOpt($p_j$) , M-ComputeOpt($j-1$)}
        Return M[$j$]

    EndIf
```

It finds the value of the optimal schedule O.

Is that what we were looking for?

Weighted Interval Scheduling

- A set of requests {$1, 2, \dots, n$}.
  - Each request has a starting time $s(i)$, a finishing time $f(i)$, and a value $v(i)$.
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From values to schedules

In other words, j is in O *if and only if*

$$OPT(p_j)+v(j) \geq OPT(j-1)$$

```
FindSolution($j$)

  If $j$=0 , no solution

  Else
    If $v(j) + M(p_j) \geq M(j-1)$ then
       Output $j$ together with FindSolution($p_j$)
    Else
       Output FindSolution($j-1$)
    EndIf
  End If
```

This can be done in O(n) time.

Dynamic Programming vs Divide and Conquer

- DP is an optimisation technique and is only applicable to problems with optimal substructure.
- DP splits the problem into parts, finds solutions to the parts and joins them.
  - The parts are not significantly smaller and are overlapping.
- In DP, the subproblem dependency can be represented by a DAG.

- DQ is not normally used for optimisation problems.
- DQ splits the problem into parts, finds solutions to the parts and joins them.
  - The parts are significantly smaller and do not normally overlap.
- In DQ, the subproblem dependency can be represented by a tree.