Introduction to Algorithms and Data Structures

Dynamic Programming - Subset Sum and Knapsack

The subset sum problem

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Goal: Select a subset S of the items such that

$\sum_{i \in N} w_i \leq W$ and $\sum_{i \in N} w_i$ is maximised.

To be more precise

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How are these inputs represented by a computer?

Representation

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Binary representation: $5_{10} = 101_2 \rightarrow 101$

Unary representation: $5_{10} \rightarrow 11111$

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Each item i has a non-negative weight w_i *given in binary representation.*

We are given a bound W *given in binary representation.*

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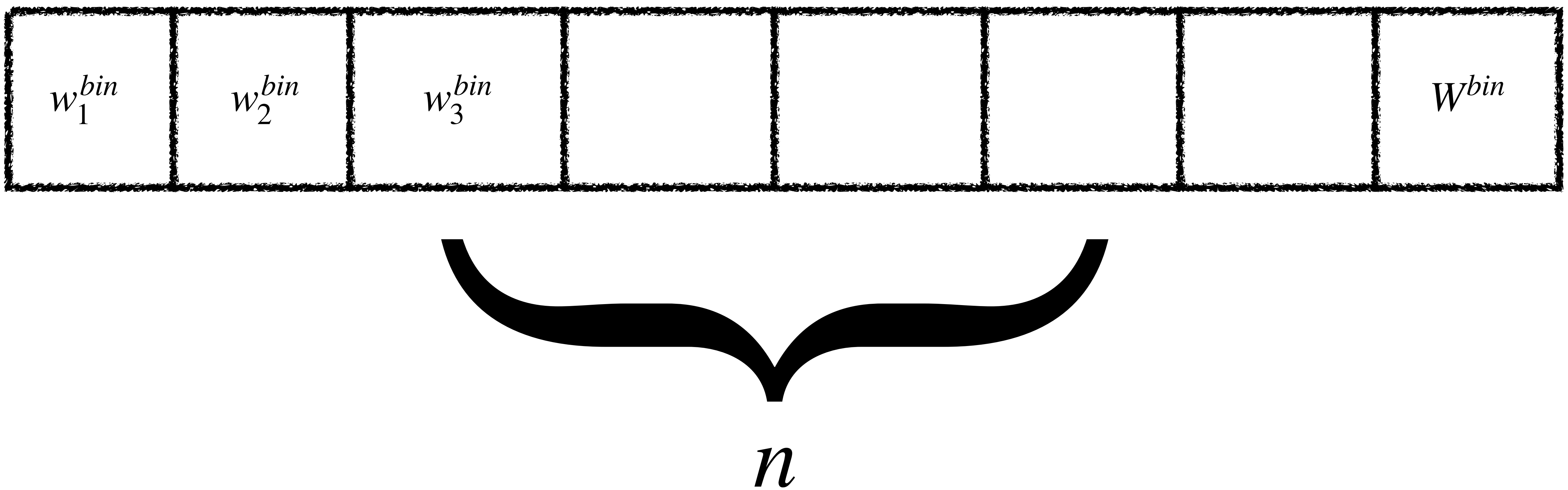
$$\sum_{i \in N} w_i \leq W \quad \text{and} \quad \sum_{i \in N} w_i \text{ is maximised.}$$

Our input



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Greedy Approaches

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  - $W = 4$, $w_1 = 3$, $w_2 = 2$, $w_3 = 2$.

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  - If yes, ?

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If n is in O

- What information do we get about the other items?
- In weighted interval scheduling, we could remove all intervals overlapping with n.
- Can we do something similar here?
  - There is no reason to a-priori exclude any remaining item, unless adding it would exceed the weight.
  - The only information that we really get is that we now have weight $W - w_n$ left.

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- To find the optimal value OPT(n), we need
  - The optimal value OPT($n-1$) if n is not in O.
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- How many subproblems do we need?

What we really need

- To find the optimal value OPT(n), we need
  - The optimal value OPT($n-1$) if n is not in O.
  - The optimal value of the solution on input $\{1, 2, \ldots, n-1\}$ and $w = W - w_n$.
- How many subproblems do we need?
  - One for each initial set $\{1, 2, \ldots, i\}$ of items and each possible value for the remaining weight w.

Subproblems

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Subproblems

- Assumptions:
  - W is an integer.
  - Every w_i is an integer.
- We will have one subproblem for each $i=0,1, \ldots ,n$ and each integer $0 \leq w \leq W$.
- Let $OPT(i,w)$ be the value of the optimal solution on subset $\{1, 2, \ldots , i\}$ and maximum allowed weight w.

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Subproblems

- Using this notation, what are we looking for?
  - $OPT(n, W)$
- Should item n be in the optimal solution O or not?
  - If no, then $OPT(n, W) = OPT(n-1, W)$.
  - If yes, then $OPT(n, W) = w_n + OPT(n-1, W-w_n)$.

Subproblems

Is j in O ?

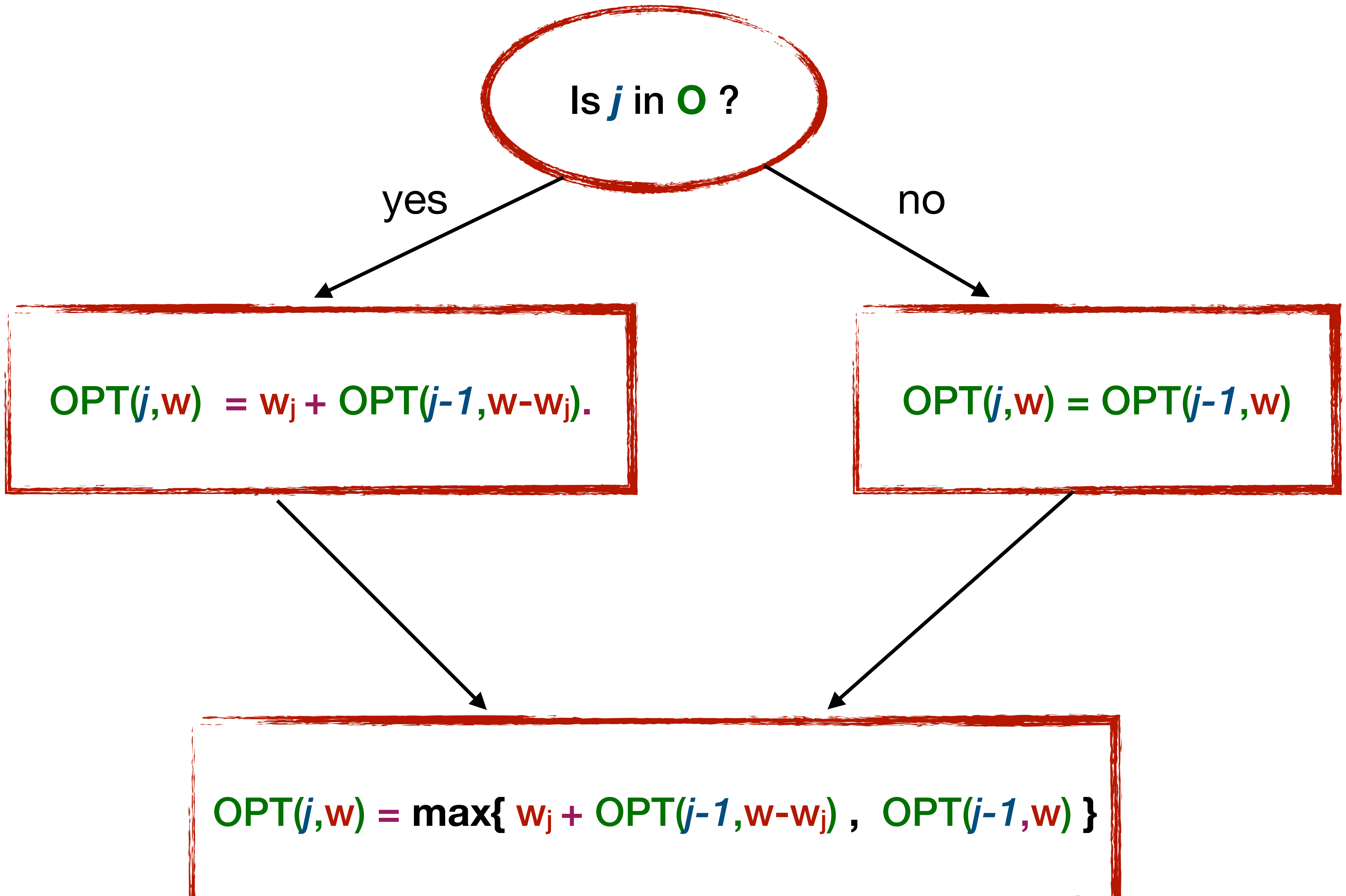
yes

$OPT(j,w) = w_j + OPT(j-1,w-w_j)$.

no

$OPT(j,w) = OPT(j-1,w)$

Subproblems



Subproblems

Is j in **O** ?

Unless $w_j > w$

yes

$$OPT(j,w) = w_j + OPT(j-1,w-w_j).$$

no

$$OPT(j,w) = OPT(j-1,w)$$

$$OPT(j,w) = \max\{ w_j + OPT(j-1,w-w_j) , \; OPT(j-1,w) \}$$

Algorithm

```
Algorithm SubsetSum(n,W)

    Array M=[0 … n, 0 … W]
    Initialise M[0, w] = 0, for each w = 0, 1 , … , W

    For i = 1, 2, … , n
        For w = 0 , … , W
          If (w_i > w)                    \* If the item does not fit *\
              M[i, w] = M[i-1, w]
          Else
              M[i, w] = max{M[i-1, w] , w_i + M[i-1, w-w_i ]}
          EndIf

     Return M[n, W]
```

Two dimensional array

n	0										
n-1	0										
…	0										
…	0										
i	0										
i-1	0										
…	0										
2	0										
1	0										
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	$w-w_i$			w				W

Two dimensional array

n	0										OPT(*n*,W)
n-1	0										
…	0										
…	0										
i	0										
i-1	0										
…	0										
2	0										
1	0										
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	$w-w_i$			w				W

Two dimensional array

OPT(n,W)

n	0										
$n-1$	0										
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0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	$w-w_i$			w				W

Two dimensional array

n	0										OPT(n,W)
$n-1$	0										OPT($n-1$,W)
…	0										
…	0										
i	0										
$i-1$	0										
…	0										
2	0										
1	0										
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	$w-w_i$			w				W

Two dimensional array

n	0										OPT(*n*,W)
n-1	0									OPT(*n-1*,W)	
…	0							OPT(*n-2*,W-w_{n-2})			
…	0										OPT(*n-2*,W)
i	0										
i-1	0										
…	0										
2	0										
1	0										
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	w-w_i			w				W

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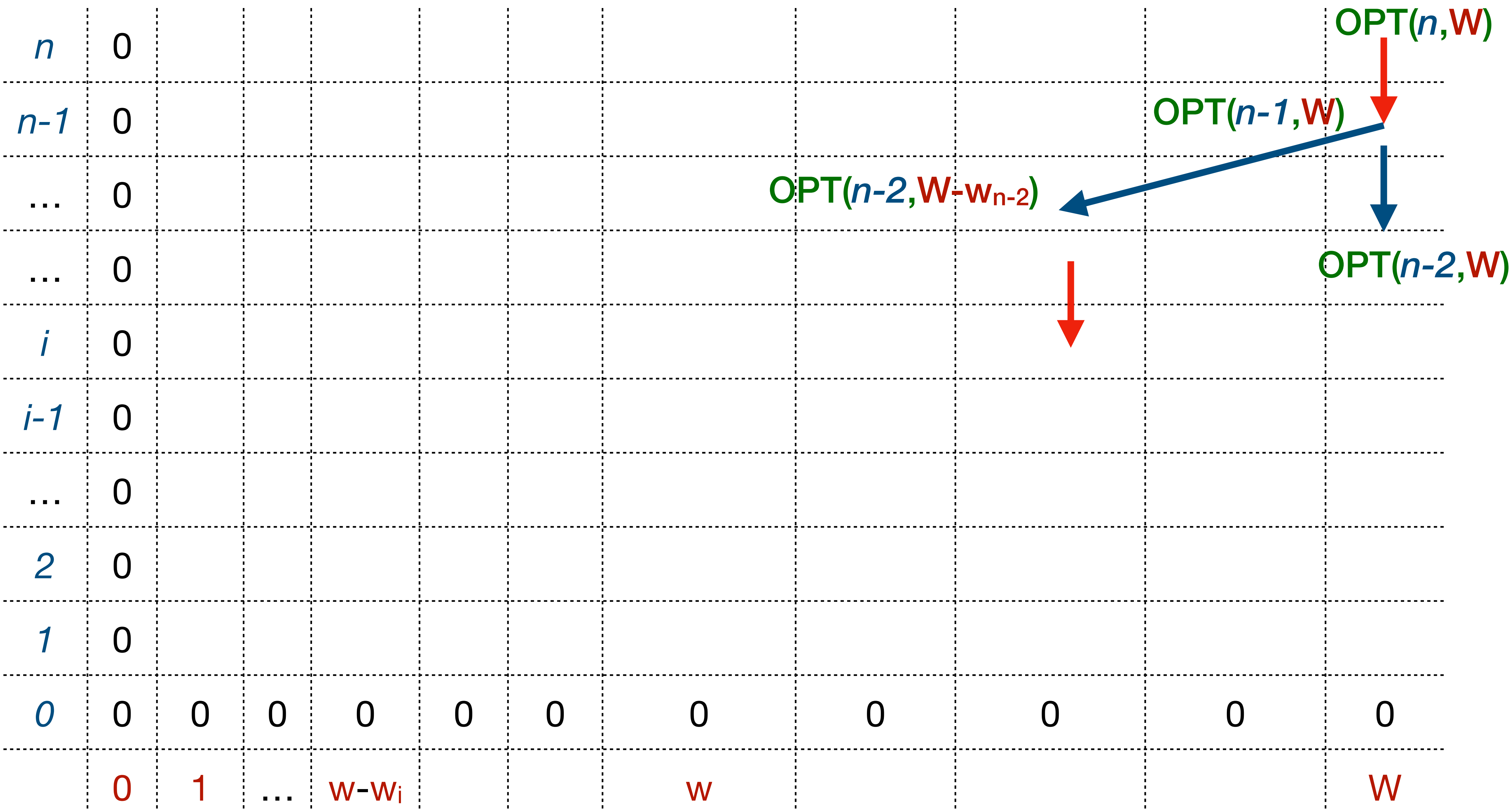

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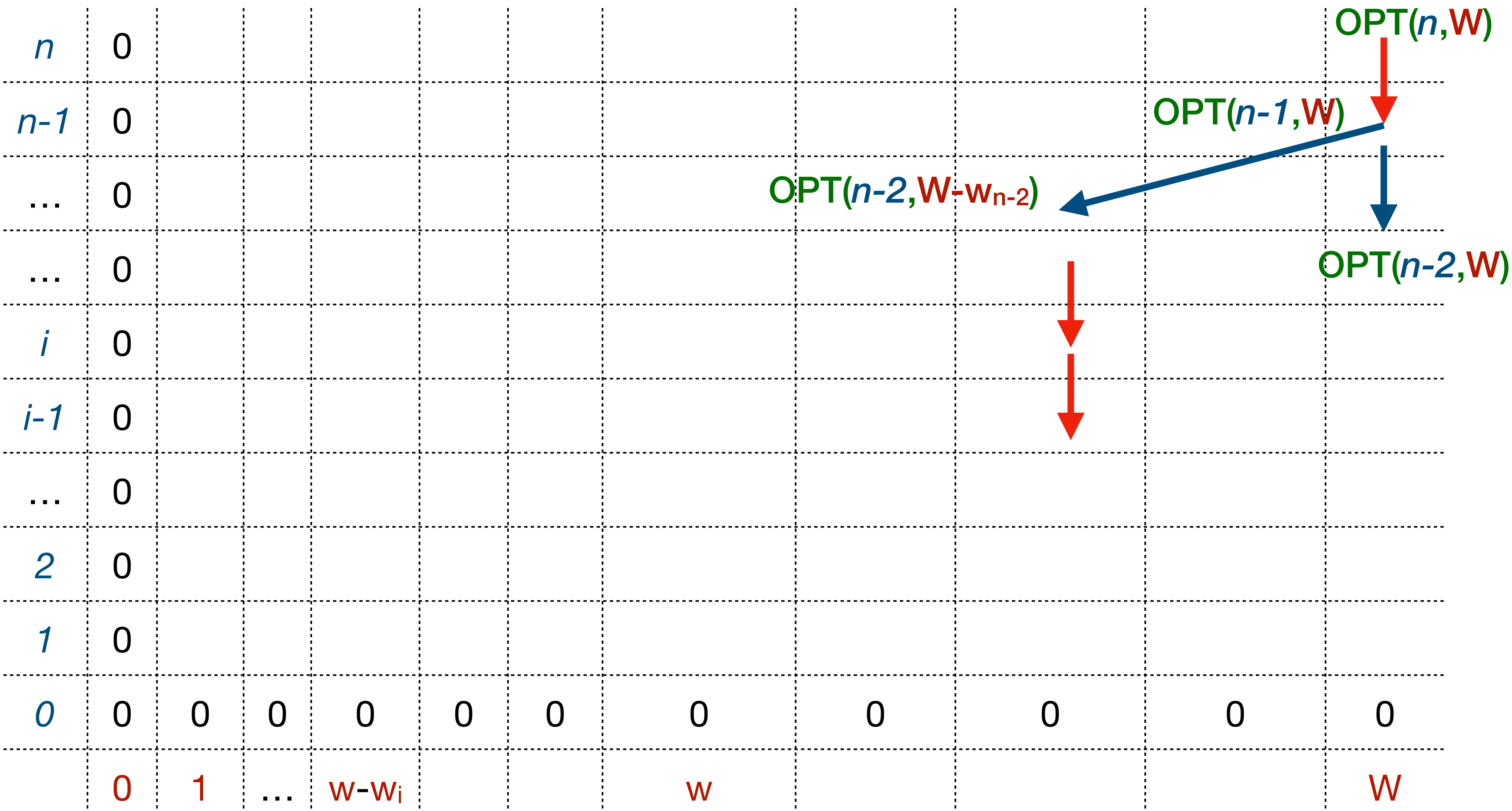

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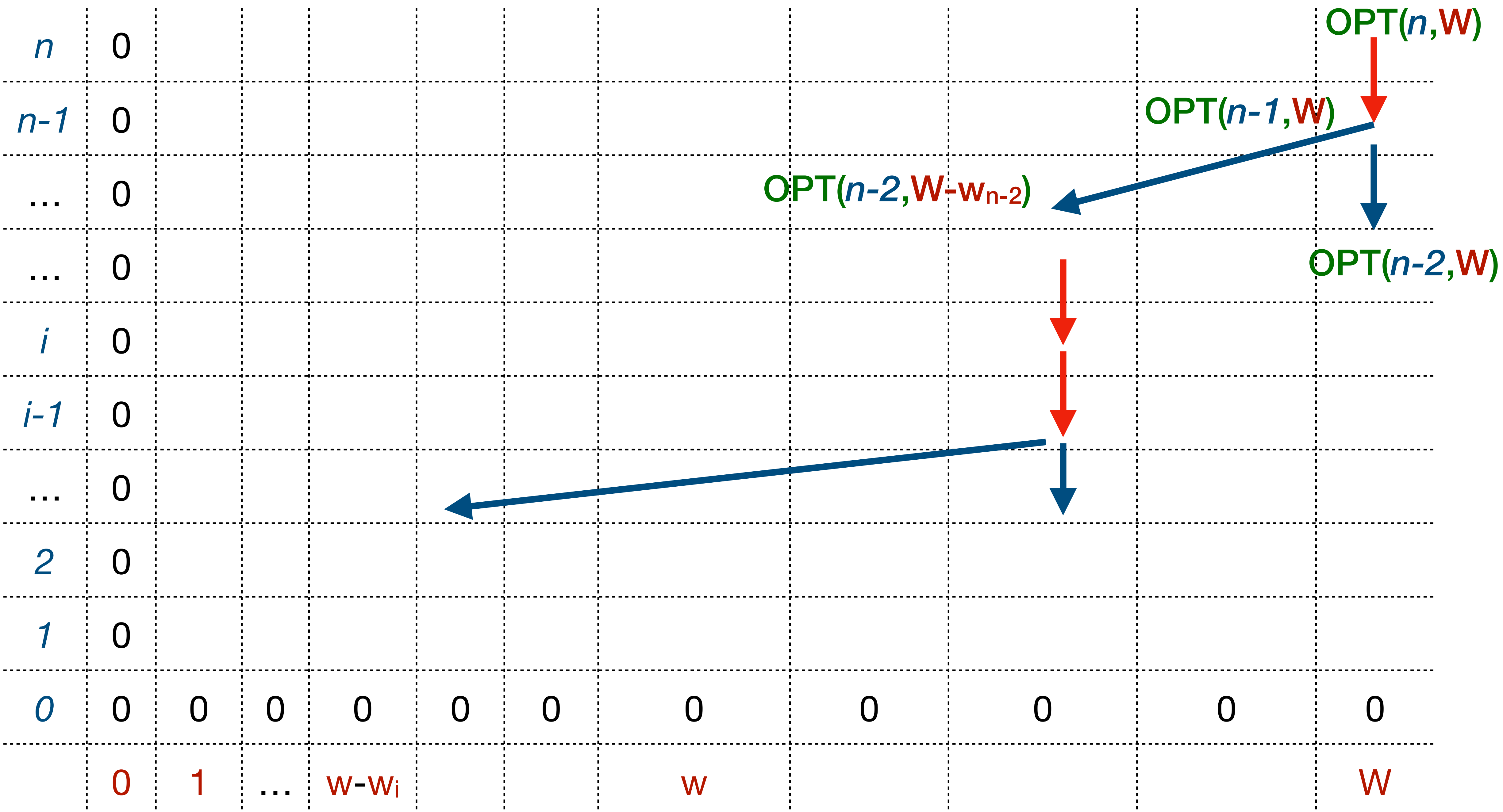

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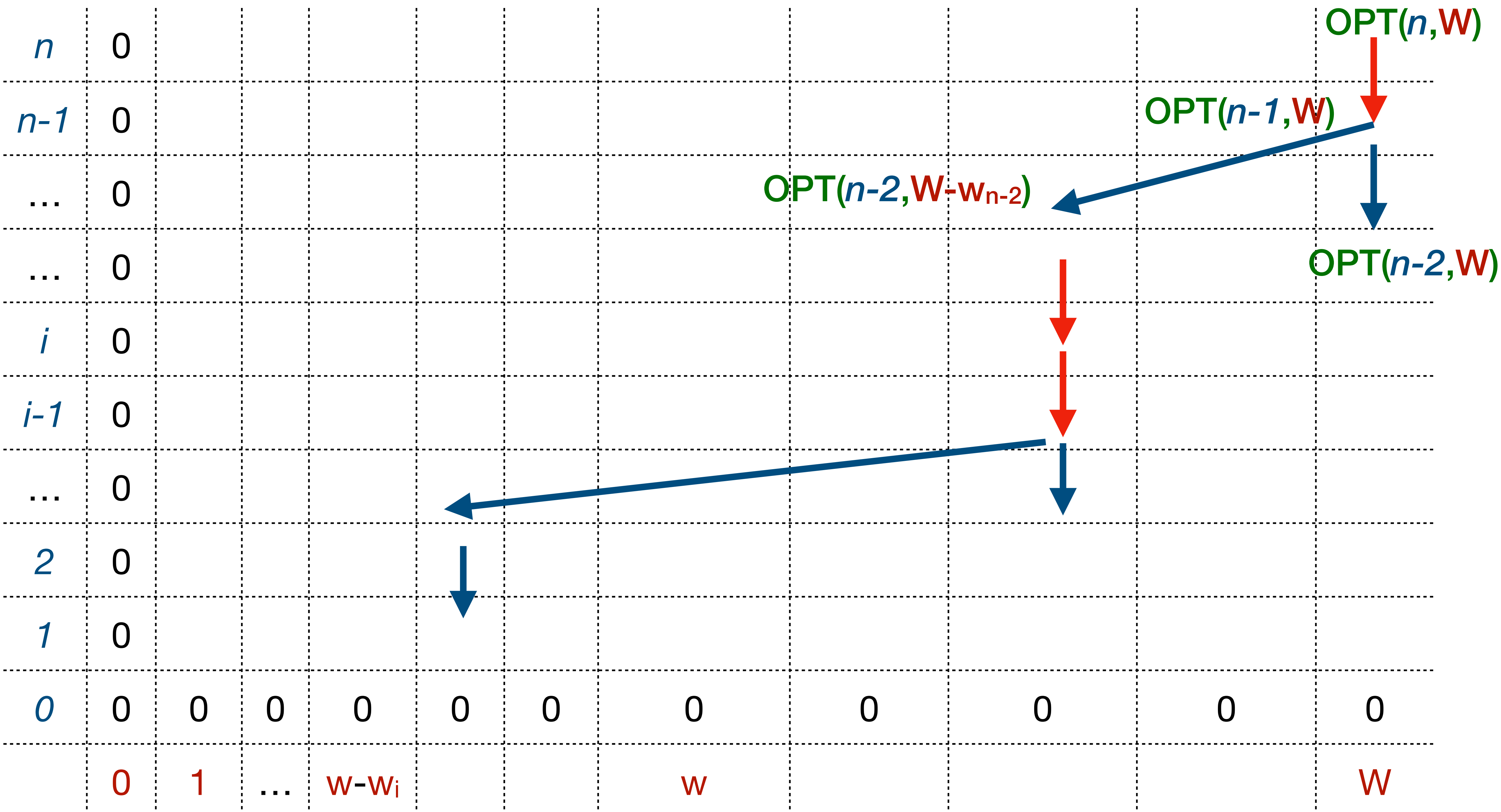

Example

- n=3, W=6, $w_1 = w_2 = 2$ and $w_3 = 3$.

3							
2							
1							
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

Array M=[$0 \ldots n$, 0 … W]
Initialise M[0, w] = 0, for each w = 0, 1 , … , W

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$\boxed{1}$							
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For w = 0 , … , W
    If (w_i > w)
        M[i, w] = M[i-1, w]
    Else
        M[i, w] = **max{**M[i-1, w] , w_i + M[i-1, w-w_i]**}**
    EndIf

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For w = 0 , … , W

If (w_i > w) **doesn't fit**

M[i, w] = M[i-1, w]

Else

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1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
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        $M[i, w] = M[i-1, w]$
    Else
        $M[i, w] = \mathbf{max}\{M[i-1, w], w_i + M[i-1, w-w_i]\}$
    EndIf

Example

- n=3, W=6, $w_1 = w_2 = 2$ and $w_3 = 3$.

$\boxed{3}$							
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

```
For w = 0 , … , W
        If (wi > w)
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        Else
          M[i, w] = max{M[i-1, w] , wi + M[i-1, w-wi ]}
        EndIf
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Example

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$\boxed{3}$	0	0	2				
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

```
For w = 0 , … , W
        If (wᵢ > w)
          M[i, w] = M[i-1, w]
        Else
          M[i, w] = max{M[i-1, w] , wᵢ + M[i-1, w-wᵢ ]}
        EndIf
```

Example

- n=3, W=6, $w_1 = w_2 = 2$ and $w_3 = 3$.

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0	0	0	0	0	0	0	0
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3	0	0	2	3	4	5	
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

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Example

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Optimal value

$\boxed{3}$	0	0	2	3	4	5	$\boxed{5}$
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

For w = 0 , … , W
    If (w_i > w)
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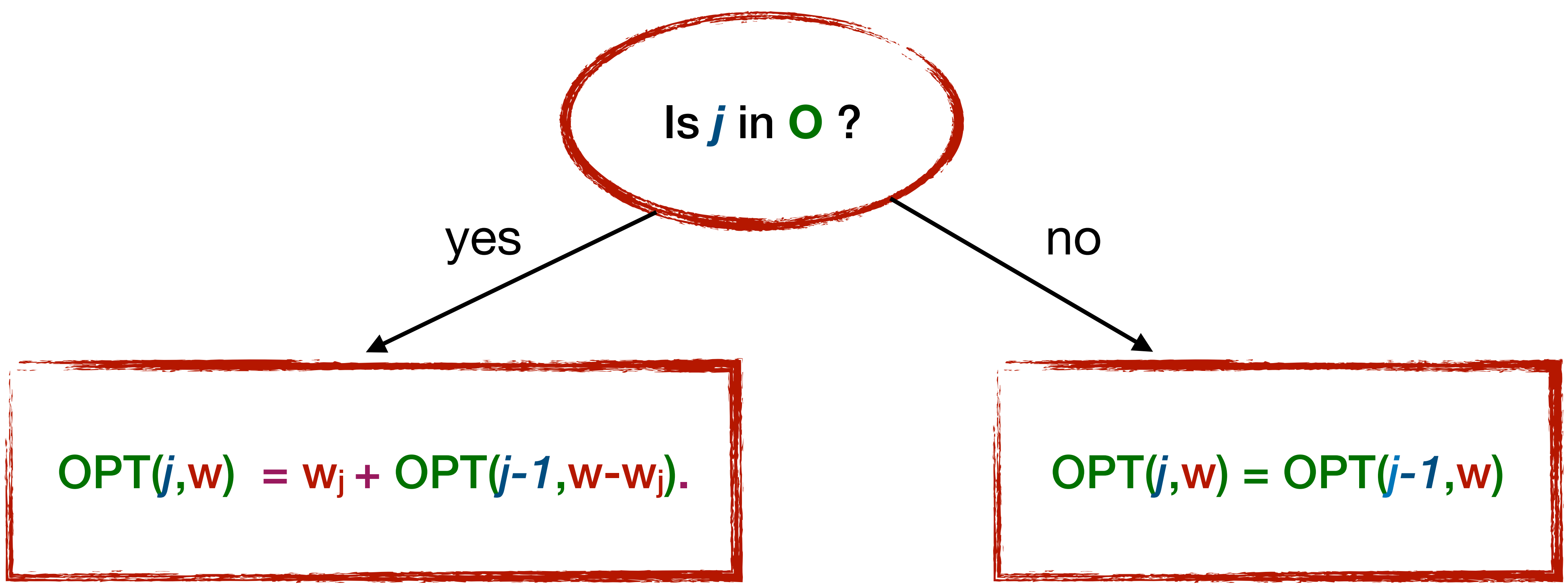
    EndIf

From values to solutions

- Very similar idea to weighted interval scheduling



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Running Time

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- We are building up a table M of solutions (instead of an array).
- We compute each value M(i, w) of the table in O(1) time using the previous values.
- What is the running time overall?
  - How many entries does the table M have?

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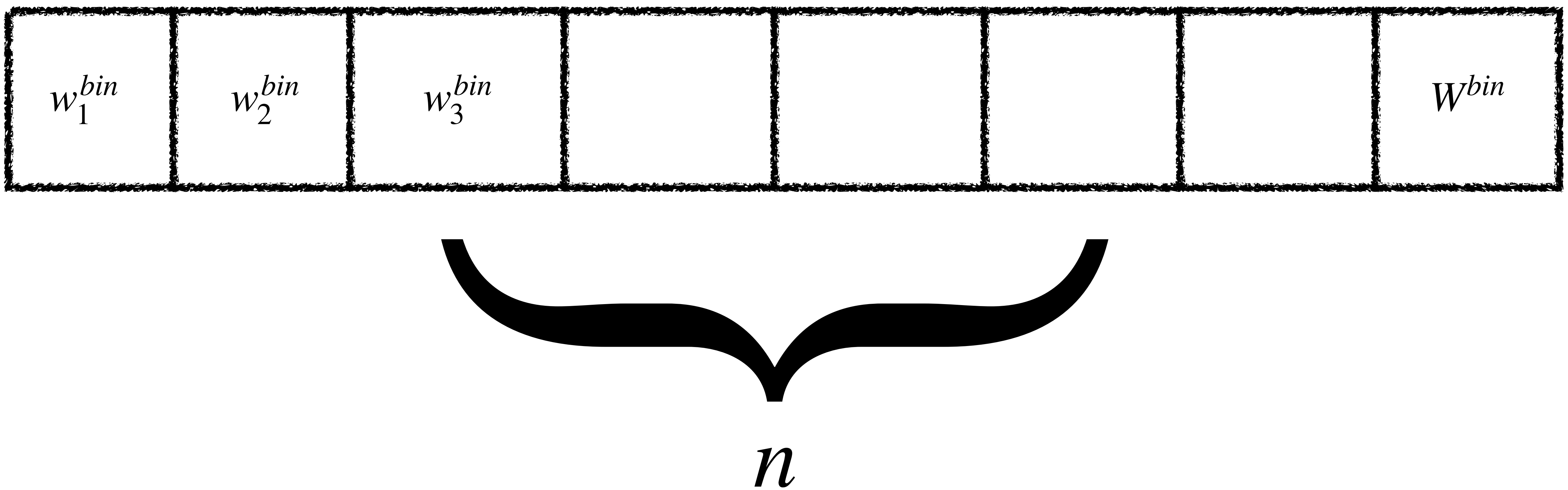
- Is this a polynomial time algorithm?

Running Time

- **SubsetSum**(n,W) runs in time O(nW).
- Is this a polynomial time algorithm?
  - Not quite, because it depends on W.

Our input



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Running Time

- **SubsetSum**(n,W) runs in time O(nW).
- Is this a polynomial time algorithm?
  - It is *pseudopolynomial*, as it runs in time polynomial in n and the unary representation of W.
  - It is fairly efficient, if in the number involved in the input are reasonably small.

Should we be happy?

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- Pseudopolynomial is good in some cases.

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- But why not polynomial?

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    - Hard enough to justify a reward of 1 million dollars!

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If you can solve this math problem you'll get a $1 million prize — and change internet security as we know it

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- Pseudopolynomial is good in some cases.
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- How hard is it to design a polynomial time algorithm for subset sum?
  - Hard enough to justify a reward of 1 million dollars!
  - Subset sum is NP-hard!

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- Pseudopolynomial is good in some cases.
- But why not polynomial?
- How hard is it to design a polynomial time algorithm for subset sum?
  - Hard enough to justify a reward of 1 million dollars!
  - Subset sum is NP-hard!
  - More about that later on in the course.

The subset sum problem

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Goal: Select a subset S of the items such that

$$\sum_{i \in N} w_i \leq W \quad \text{and} \quad \sum_{i \in N} w_i \text{ is maximised.}$$

The (0/1) knapsack problem

We are given a set of n items $\{1, 2, \ldots, n\}$.

Each item i has a non-negative weight w_i and a non-negative value v_i.

We are given a bound W.

Goal: Select a subset S of the items such that

$$\sum_{i \in N} w_i \leq W \text{ and } \sum_{i \in N} v_i \text{ is maximised.}$$

The knapsack problem

- The subset sum problem is a specific instance of the knapsack problem (why?)

3 minute exercise

Design a dynamic programming algorithm for 0/1 knapsack.

```
Algorithm SubsetSum(n,W)

    Array M=[0 … n, 0 … W]
    Initialise M[0, w] = 0, for each w = 0, 1 , … , W

    For i = 1, 2, … , n
        For w = 0 , … , W
          If (w_i > w)
             M[i, w] = M[i-1, w]
          Else
             M[i, w] = max{M[i-1, w] , w_i + M[i-1, w-w_i ]}
          EndIf

    Return M[n, W]
```

3 minute exercise

Design a dynamic programming algorithm for 0/1 knapsack.

Algorithm **SubsetSum**(n,W)

    Array M=[$0 \ldots n$, 0 … W]
    Initialise M[0, w] = 0, for each w = 0, 1 , … , W

    For i = 1, 2, … , n
        For w = 0 , … , W
            If ($w_i >$ w)
                M[i, w] = M[i-1, w]
            Else
                M[i, w] = **max{**M[i-1, w] , $\overset{v_i}{\cancel{w_i}}$ + M[i-1, w-w_i]**}**
            EndIf

    Return M[n, W]

Reading

- Kleinberg and Tardos 6.4.
- Roughgarden 16.5.