Introduction to Algorithms and Data Structures

Dynamic Programming - The Bellman-Ford Algorithm for Shortest Paths

Shortest Paths in Graphs (Lecture 17)

- **Input:** A directed graph $G = (V, E)$, and a designated node s in V. We also assume that every node u in V is reachable from s. We are also given a length $\ell_e > 0$ for every edge e in E.

- **Output:** For every node u in V, a shortest path s~u from s to u.

Shortest Paths in Graphs (today)

- **Input:** A directed graph $G = (V, E)$, and designated nodes s, t in V. We also assume that every node u in V is reachable from s. We are also given a cost $c_e \in \mathbb{R}$ for every edge e in E.

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- Motivation: e.g., Financial Networks

  positive costs (*costs* of transactions)
  negative costs (*profits* of transactions)

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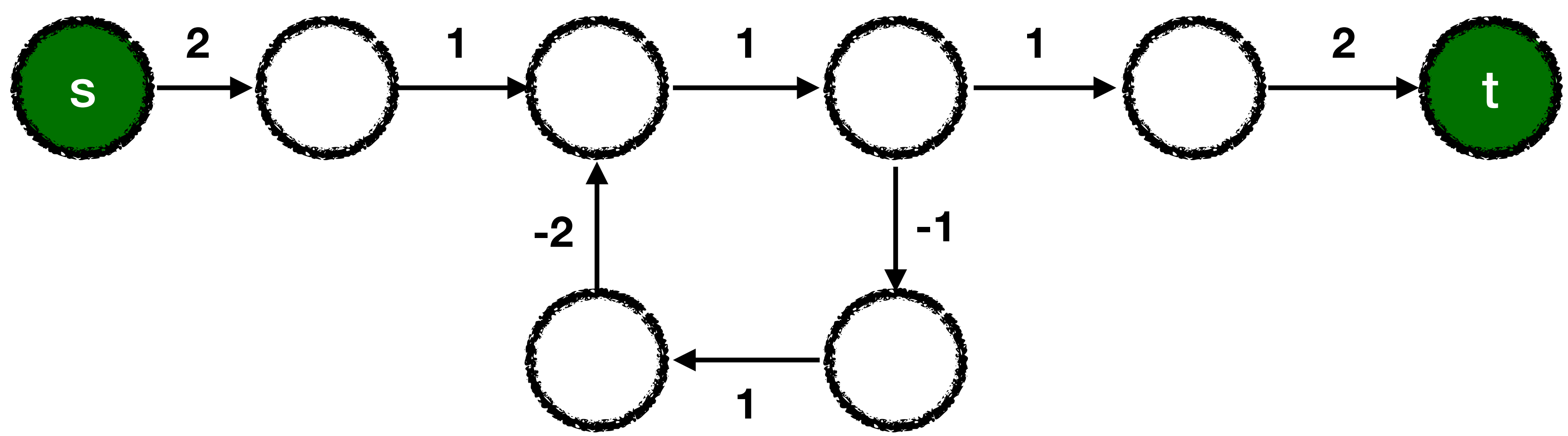
- **Input:** A directed graph $G = (V, E)$, and designated nodes s, t in V. We also assume that every node u in V is reachable from s. We are also given a cost $c_e \in \mathbb{R}$ for every edge e in E.

- **Output:** A shortest path $s \sim t$ from s to t. In other words, a path P that minimises $\sum_{(u,v) \in P} c_{uv}$

Negative Cycles

- Can we find a shortest path in the following graph?



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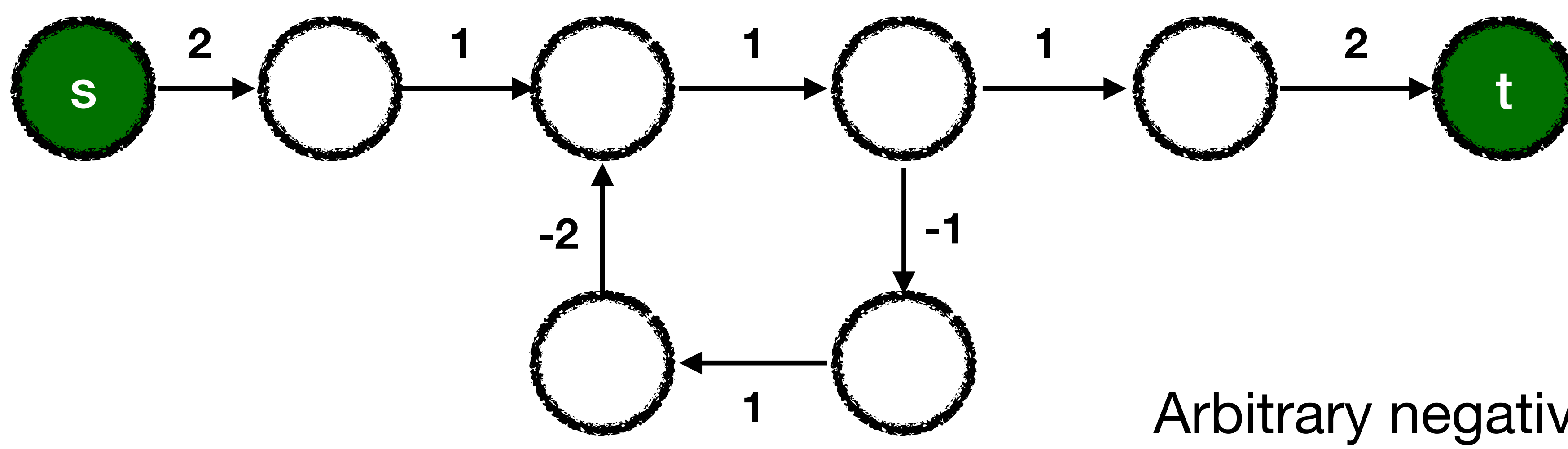


Arbitrary negative cost!

Shortest Paths in Graphs

- **Input:** A directed graph $G = (V, E)$, and designated nodes s, t in V. We also assume that every node u in V is reachable from s, and *that the graph does not have any negative cycles.* We are also given a cost $c_e \in \mathbb{R}$ for every edge e in E.

- **Output:** A shortest path $s \sim t$ from s to t. In other words, a path P that minimises $\sum_{(u,v) \in P} c_{uv}$

Why not Dijkstra?

Dijkstra's Algorithm

- For every node $v \in V - S$, we determine the shortest path that can be constructed by traveling along a path $s \sim u$ for $u \in S$, followed by (u, v).

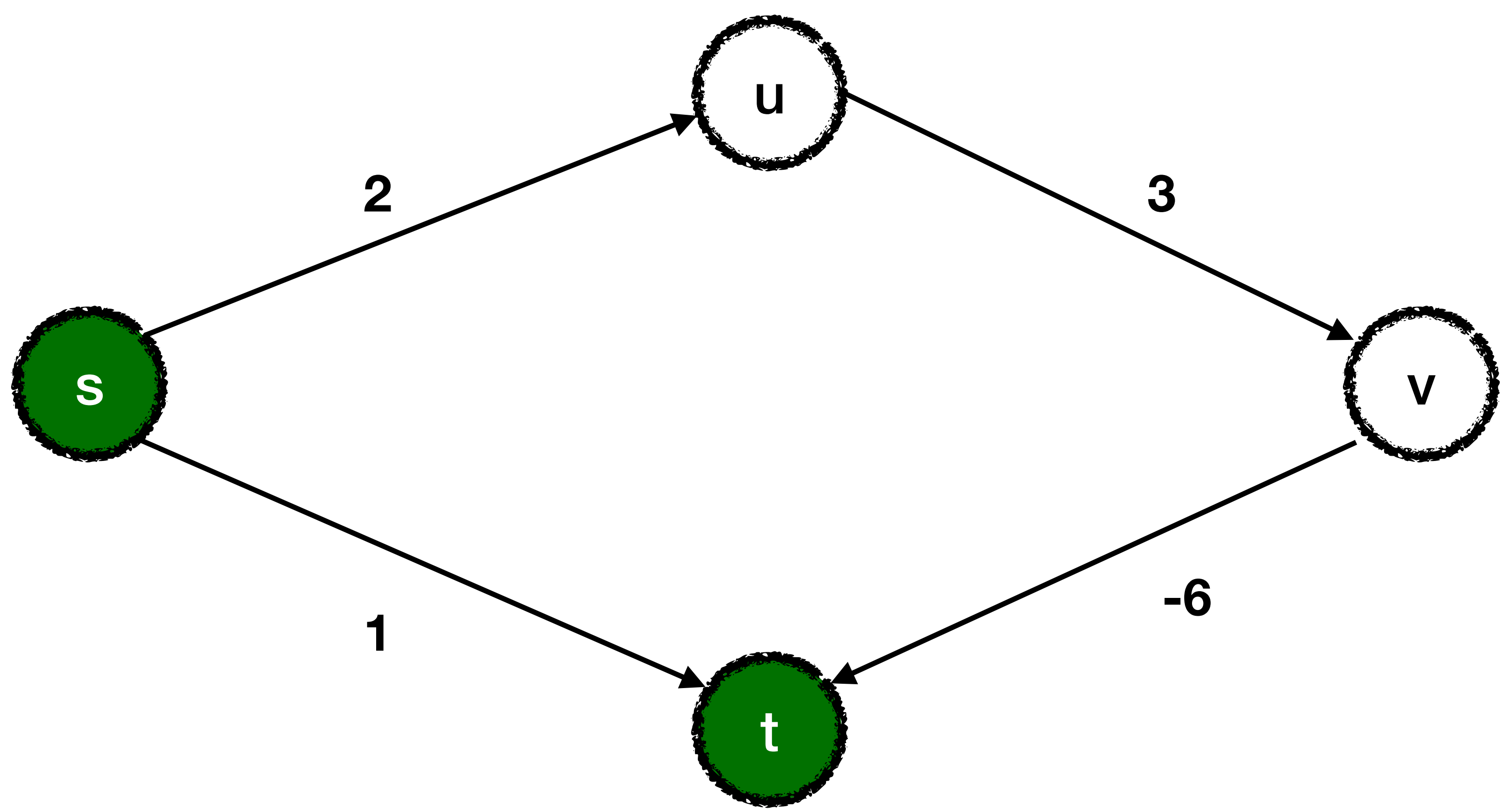
- In other words, we choose node $v \in V - S$ such that

$$d'(v) = \min_{e=(u,v):u \in S} d(u) + \ell_e$$

- Add v to S and define $d(v) = d'(v)$.

Why not Dijkstra?

- Which node would Dijkstra add in the following graph?



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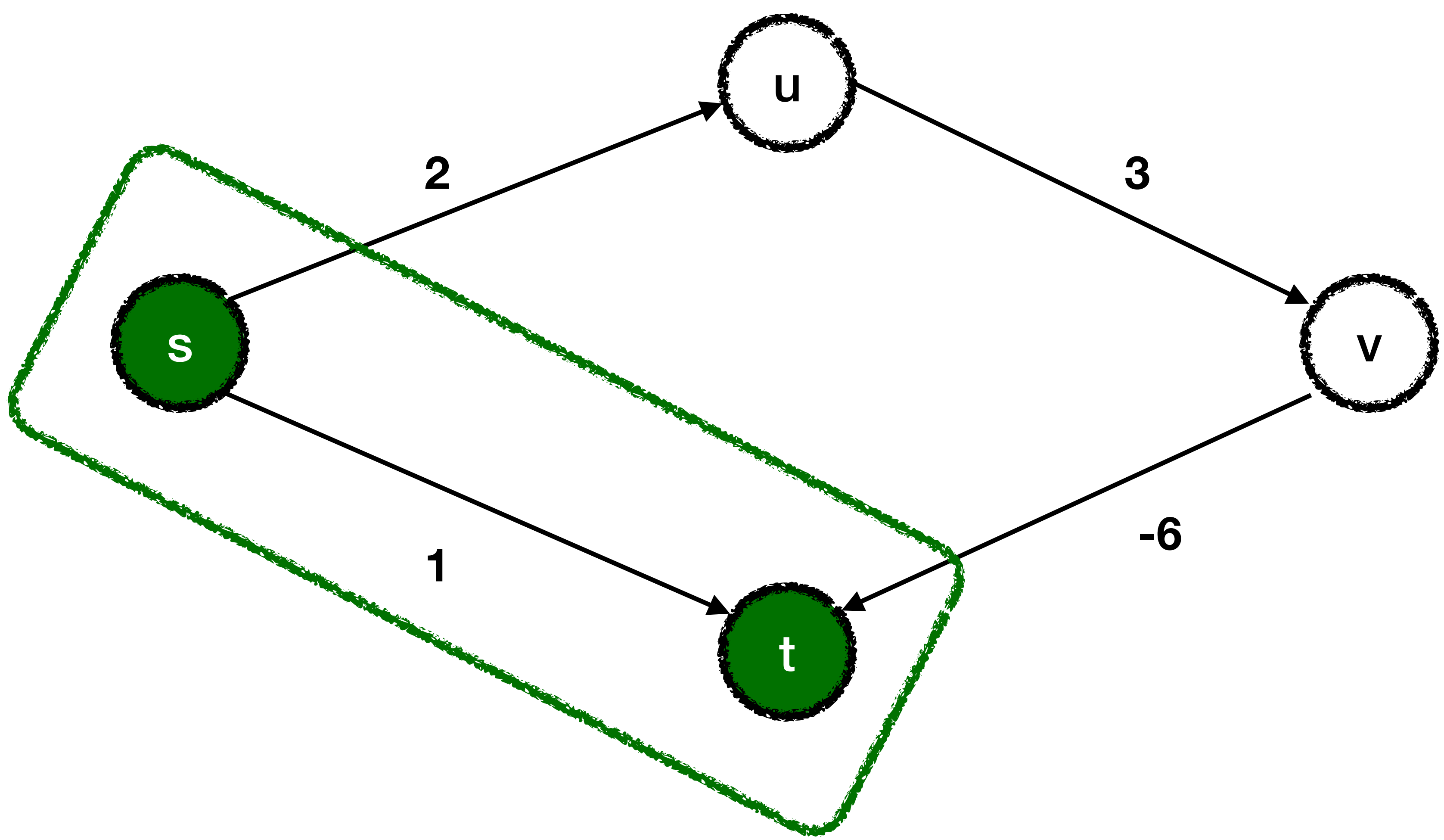

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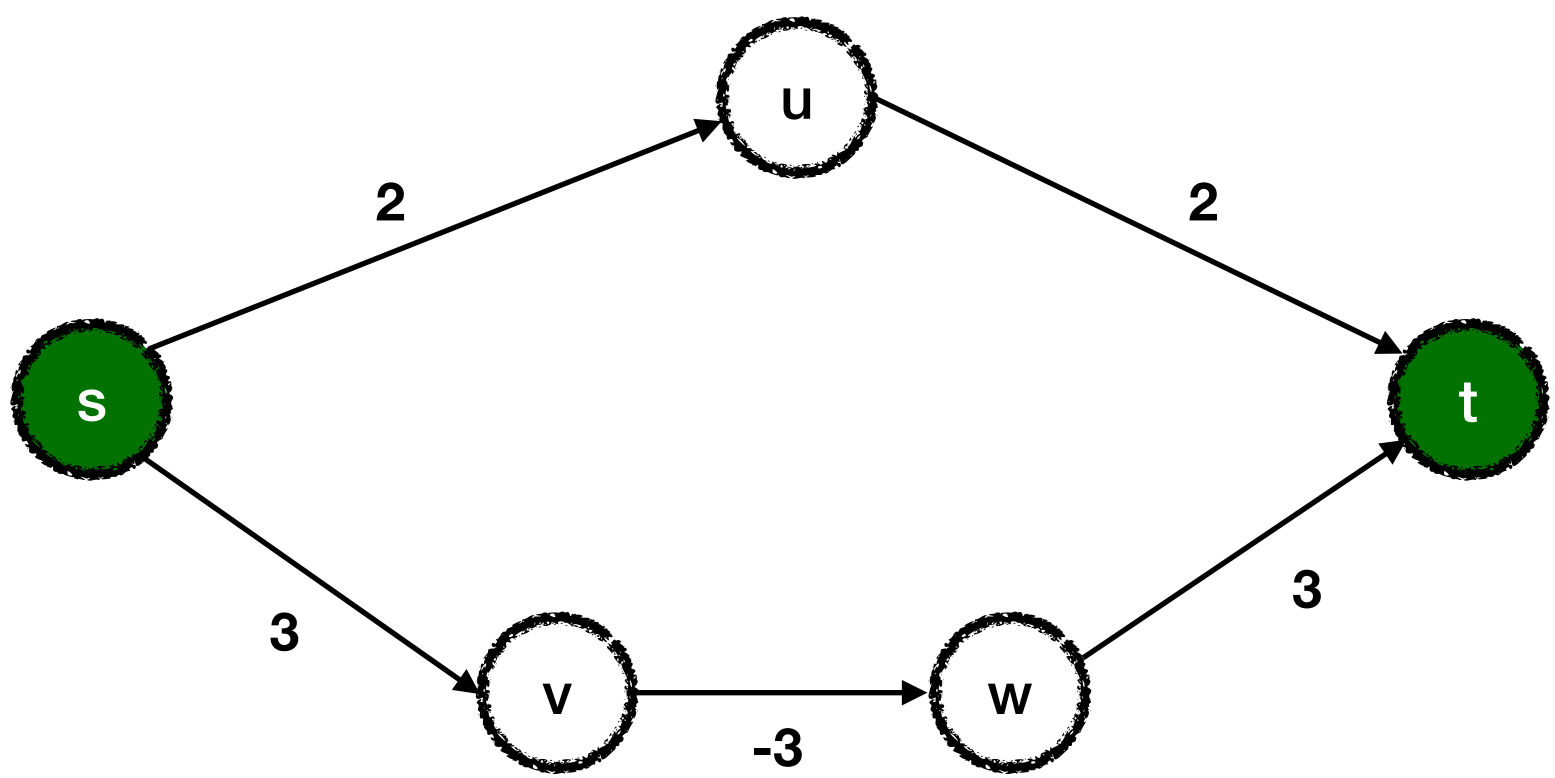

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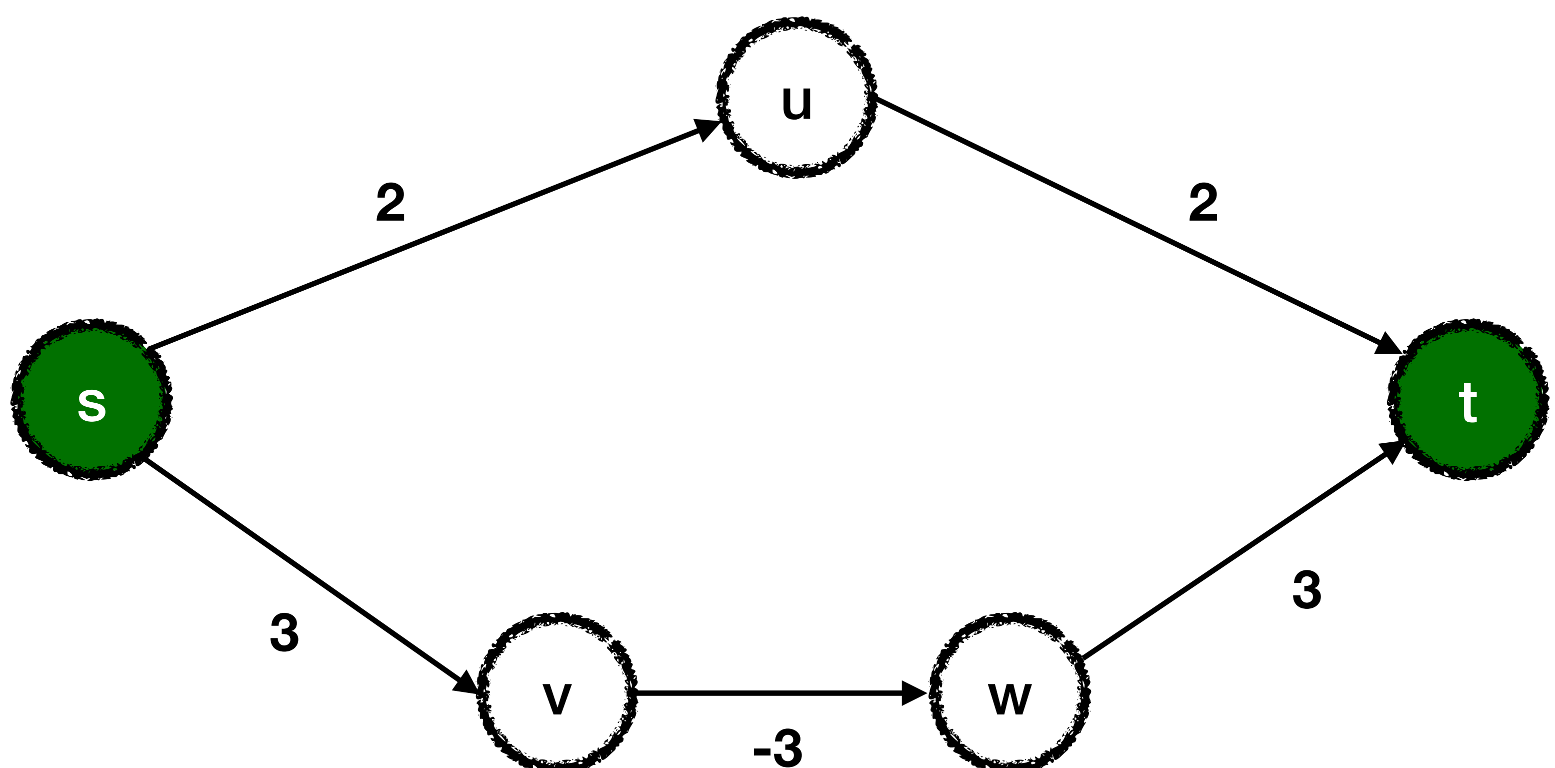

The shortest path changes!

A dynamic programming approach

- The algorithm that we will present next was developed by Bellman (1958) and Ford (1956).

- Note that Dijkstra’s algorithm was published in 1959.

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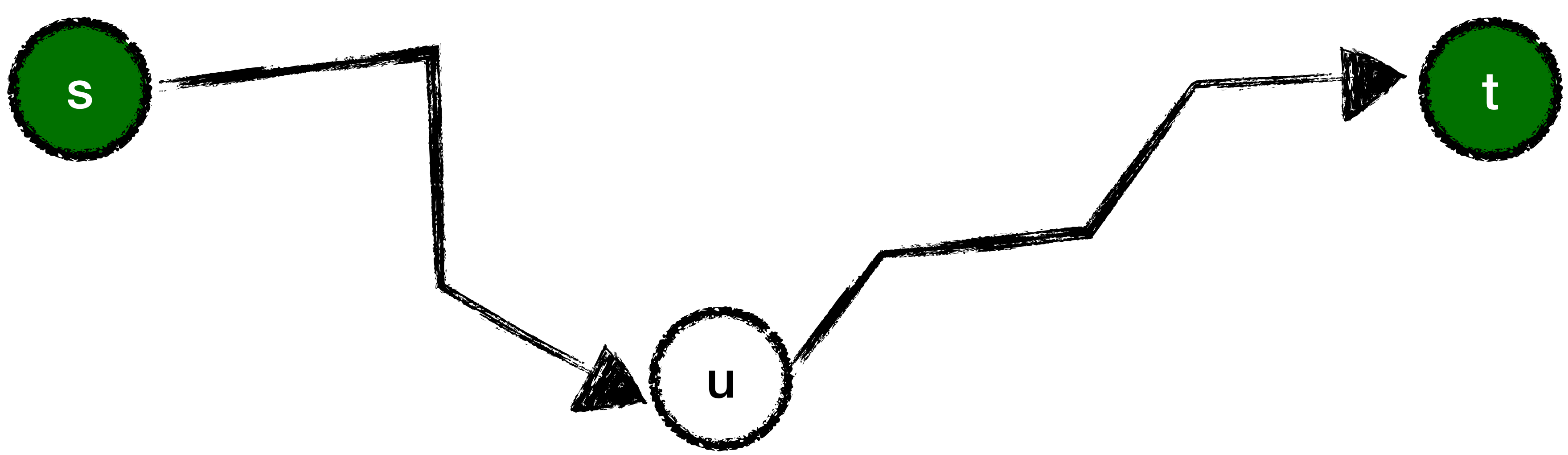

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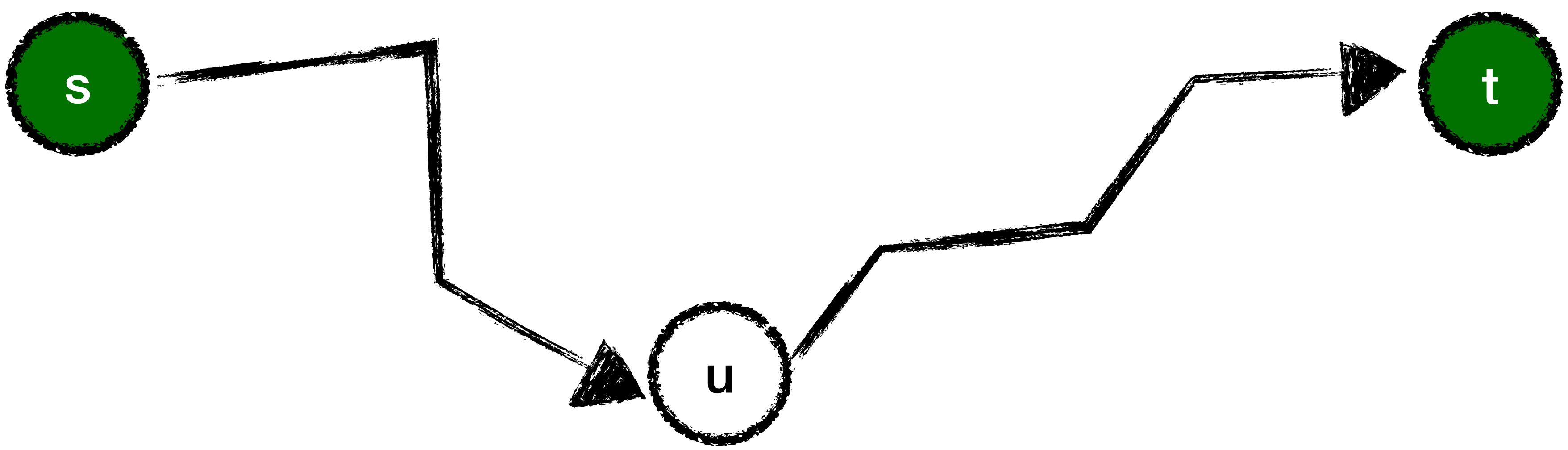

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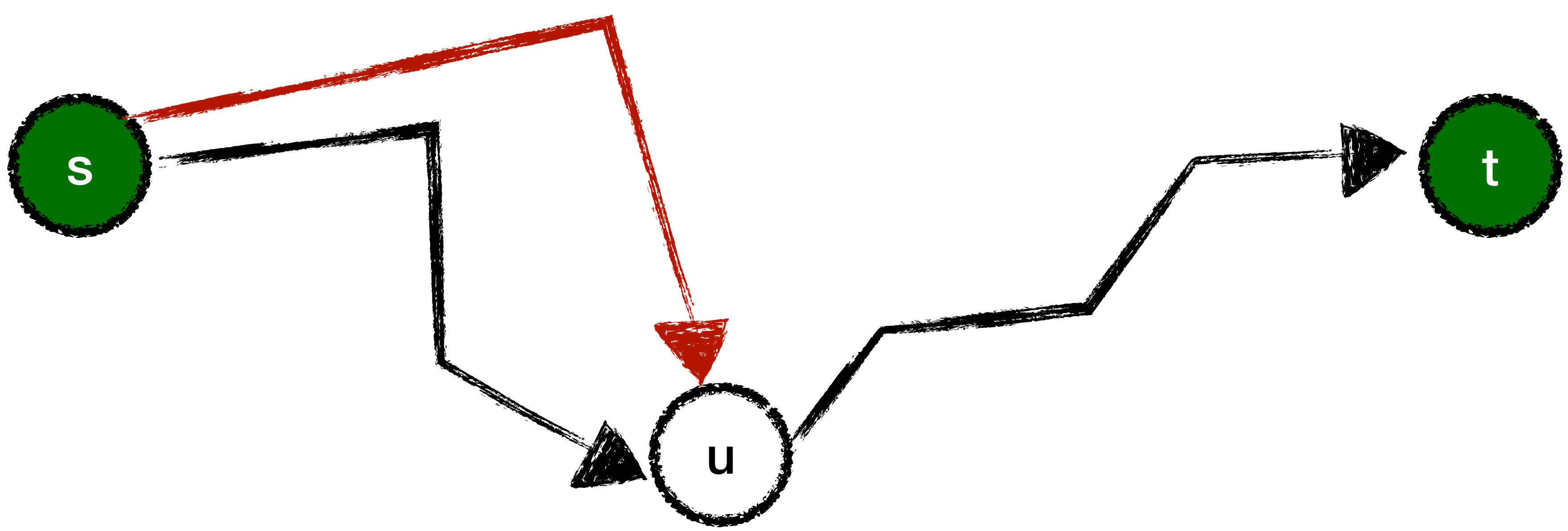

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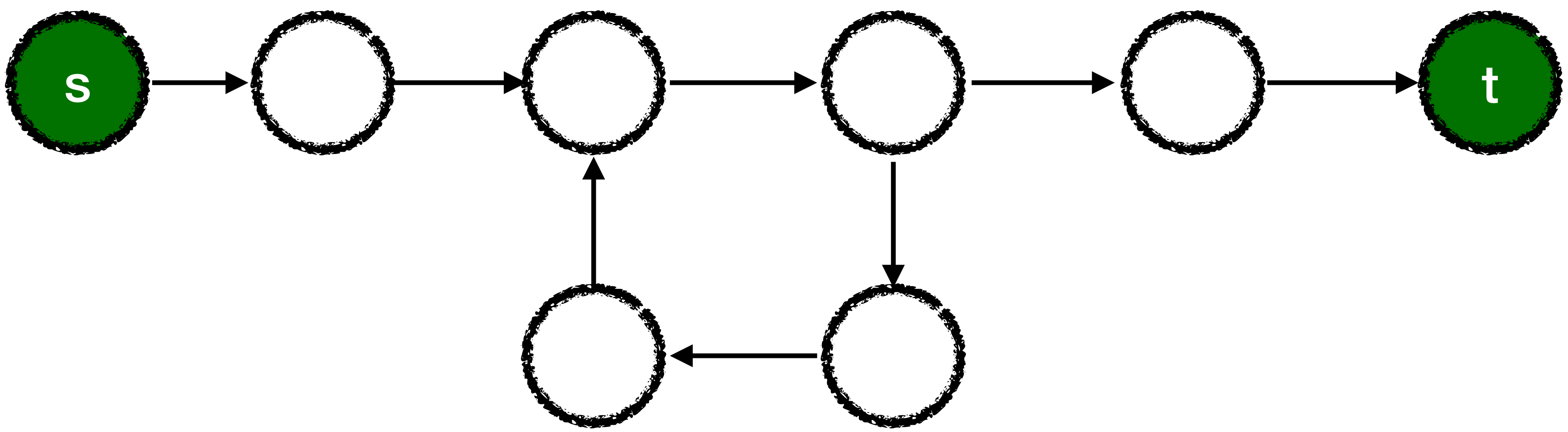

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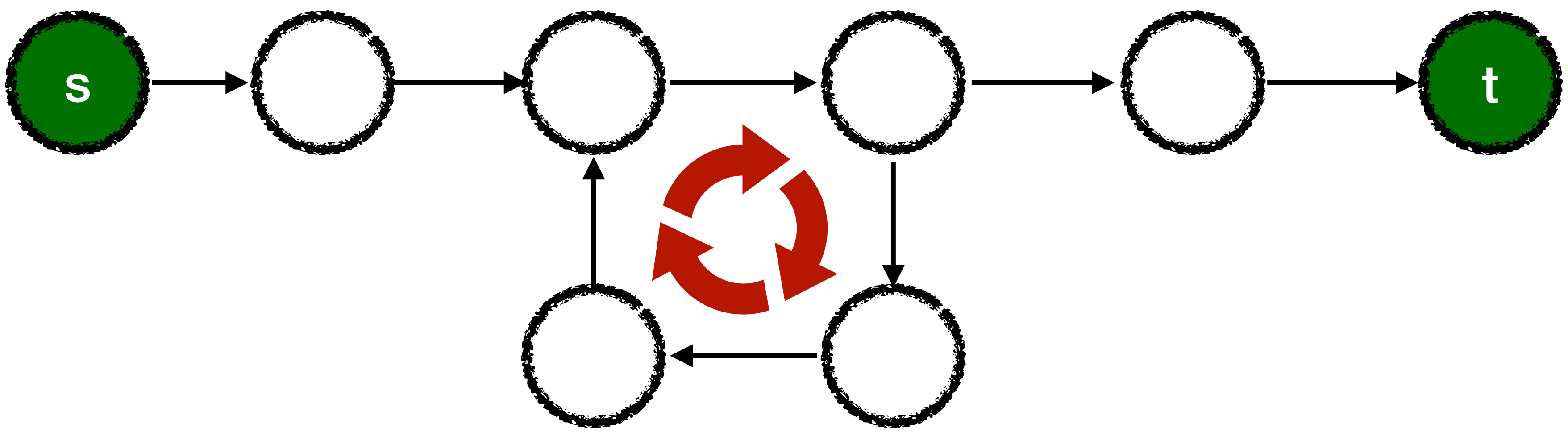

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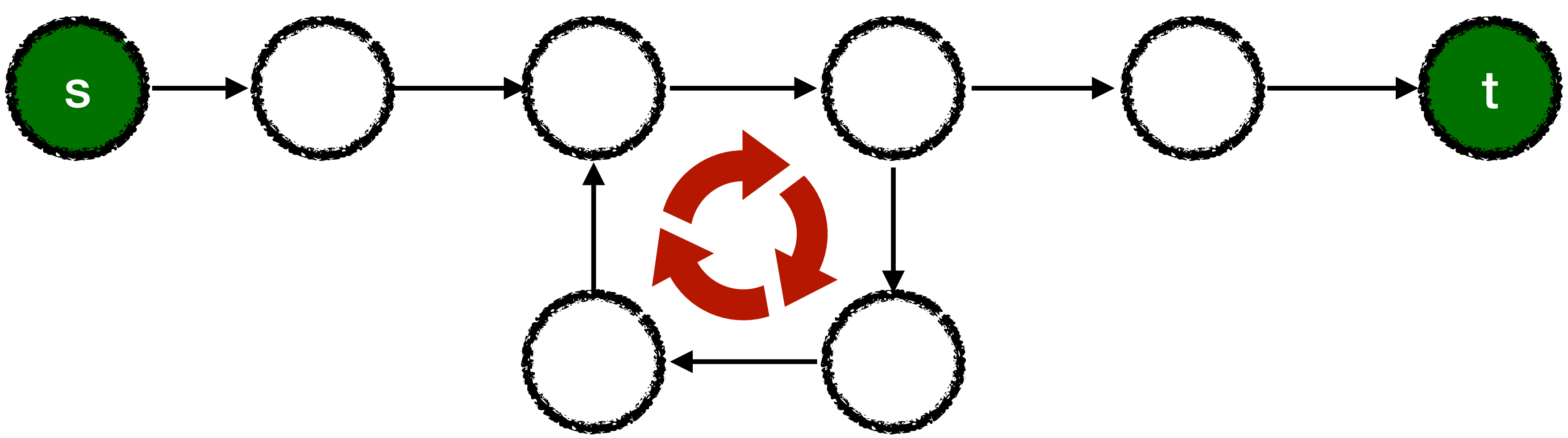

Adding the cycle cannot make the path shorter!

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- **Corollary:** The length of any shortest path s~t from s to t has at most $n - 1$ edges.

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  Subset Sum: OPT(i,w) was the value of the optimal solution on the first i items and weight w.

  Weighted Interval Scheduling: OPT(i) was the value of the optimal solution on the first i intervals.

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- We could try something similar for the “first” *i* nodes.
  - Could be made to work, but it seems complicated.
  - Instead, we will use the *number of edges*, rather than the set of nodes or edges.

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- We could also use OPT(i,v) to denote the minimum cost of a path s~v from s to node v that uses at most i edges.

  - This looks more like Dijkstra, but the former one is used in KT, because it fits better some of the other applications presented in the book.

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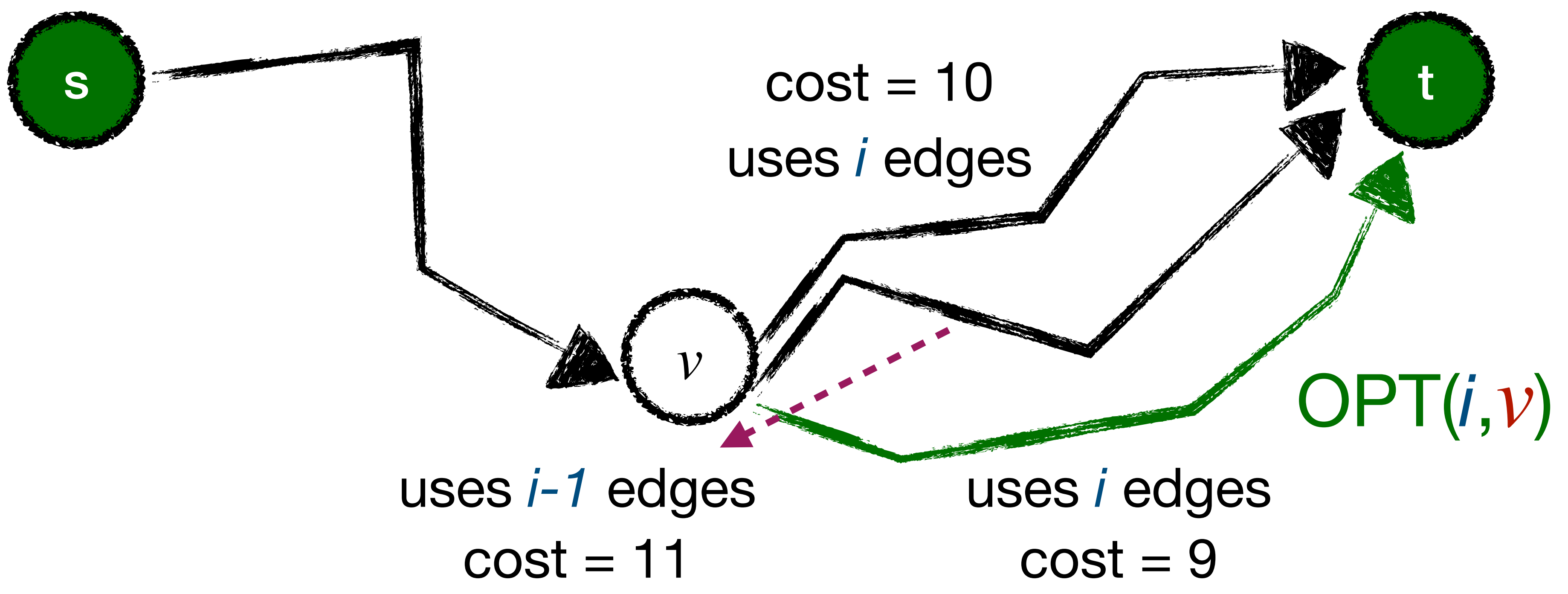

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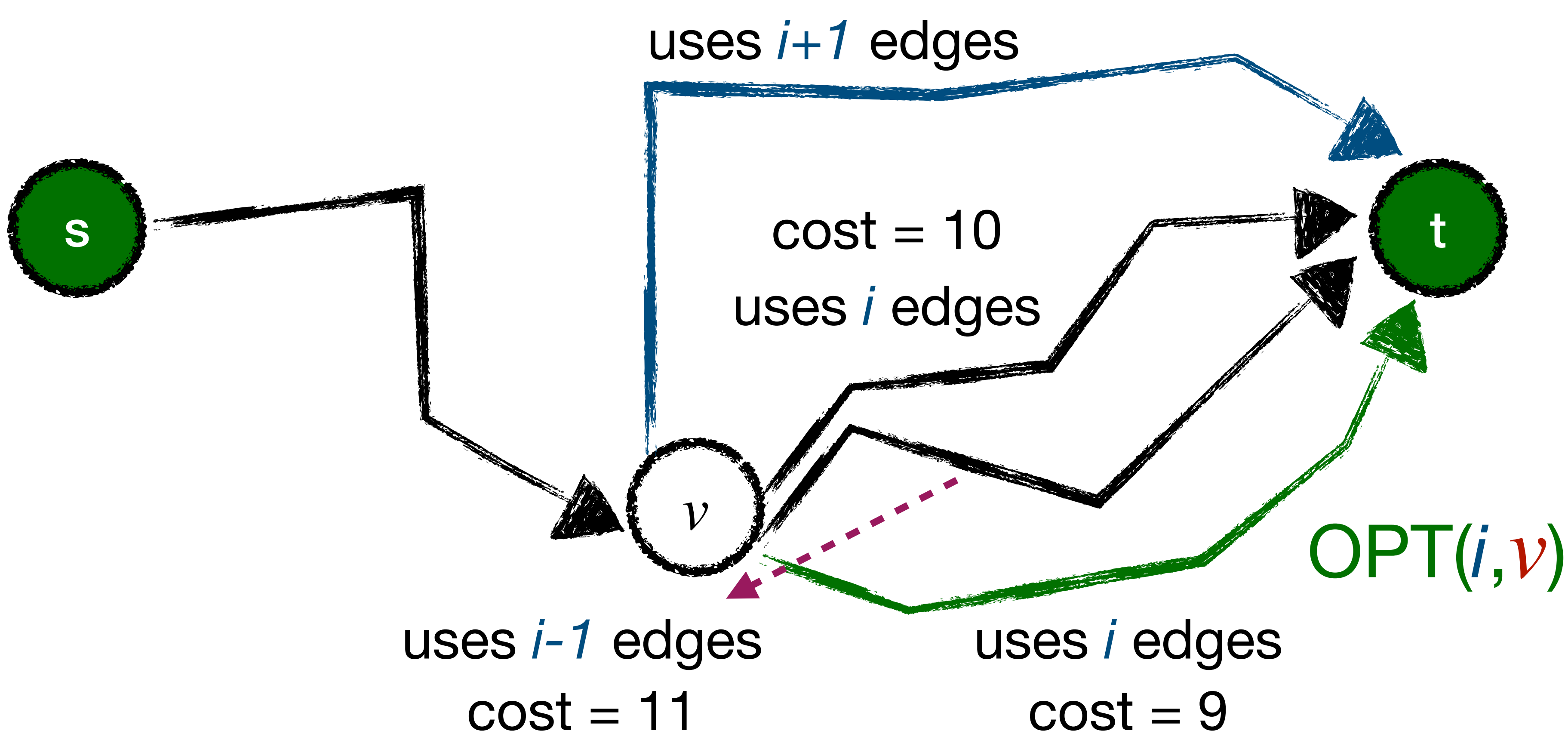

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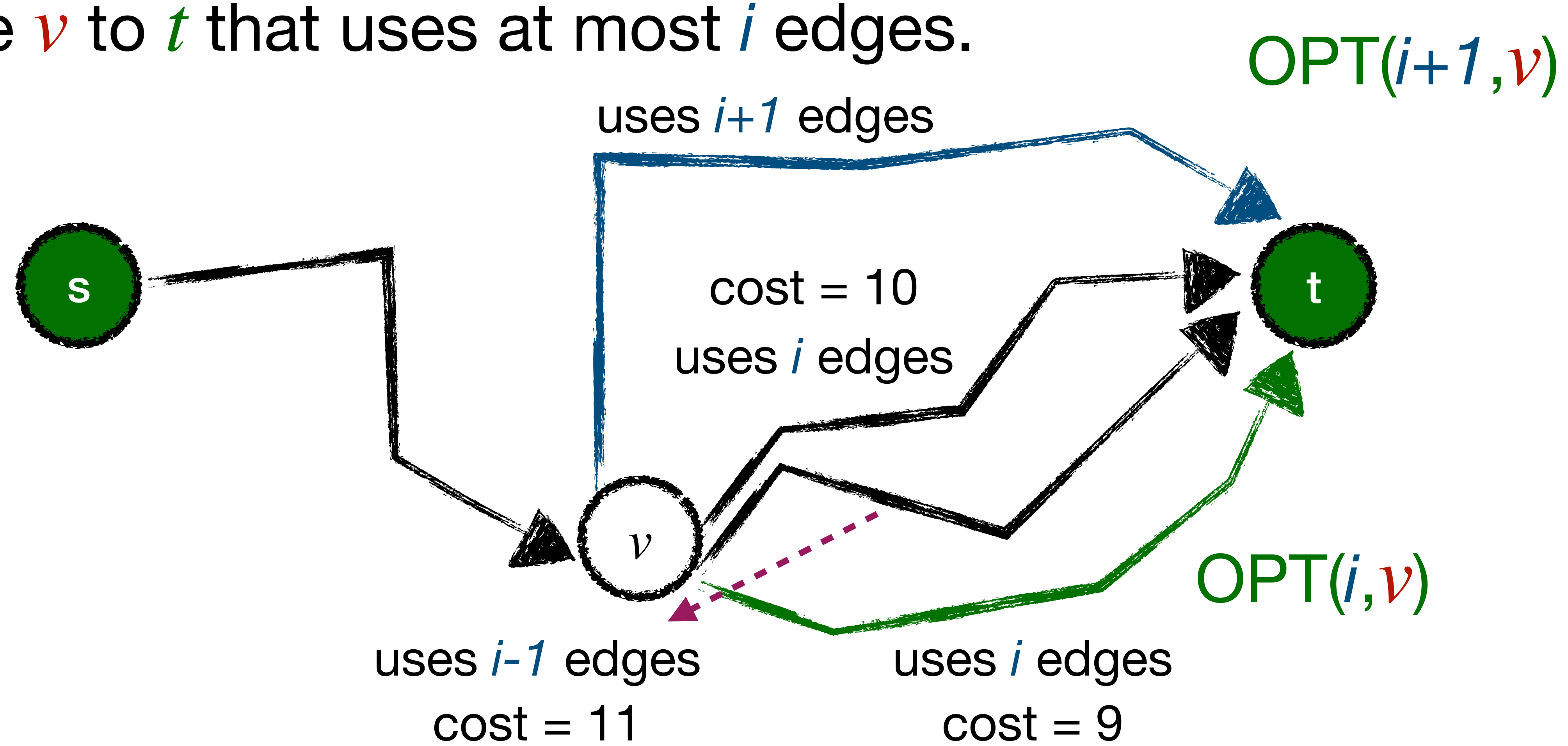

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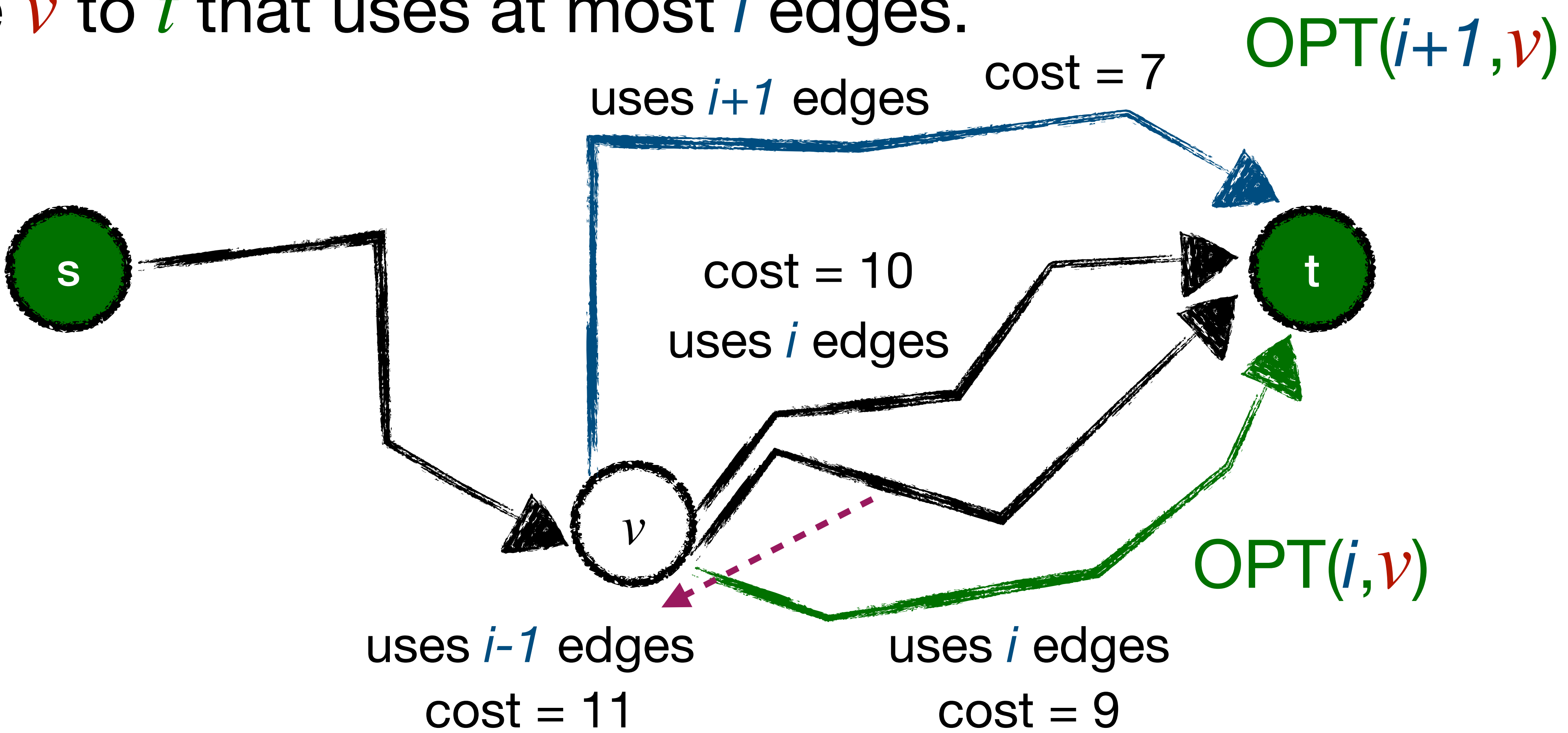

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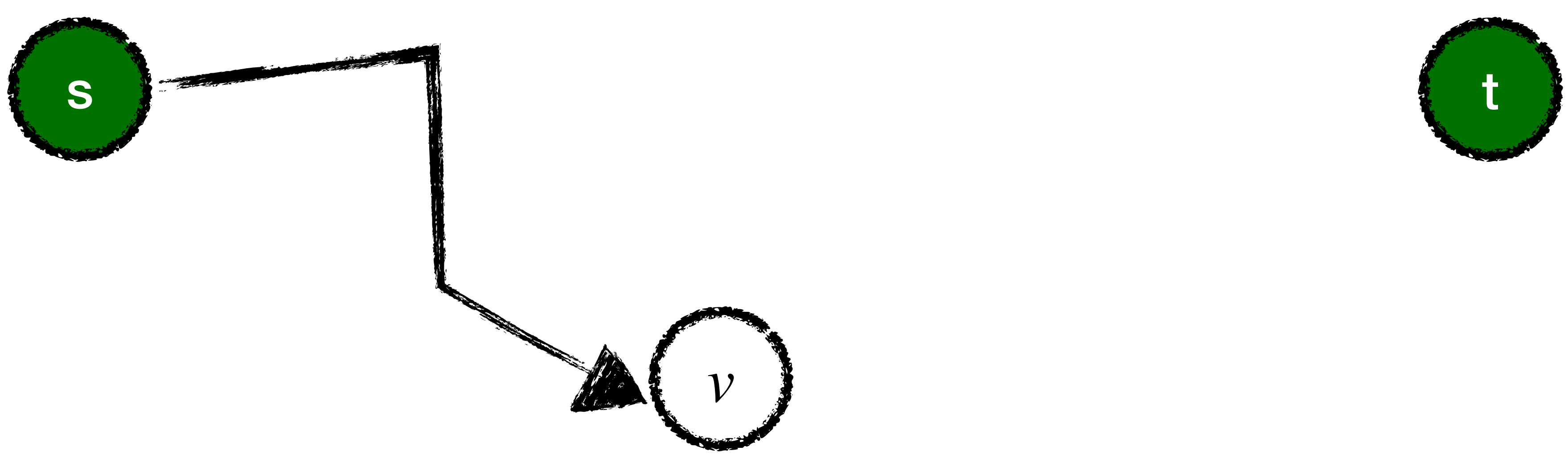

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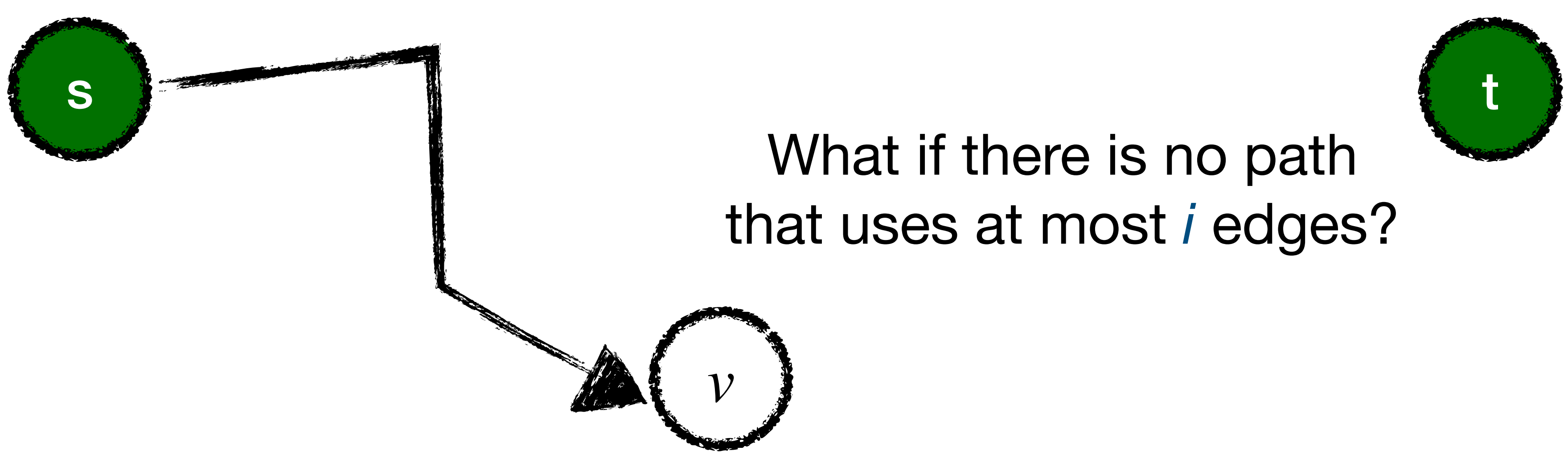

What if there is no path that uses at most i edges?

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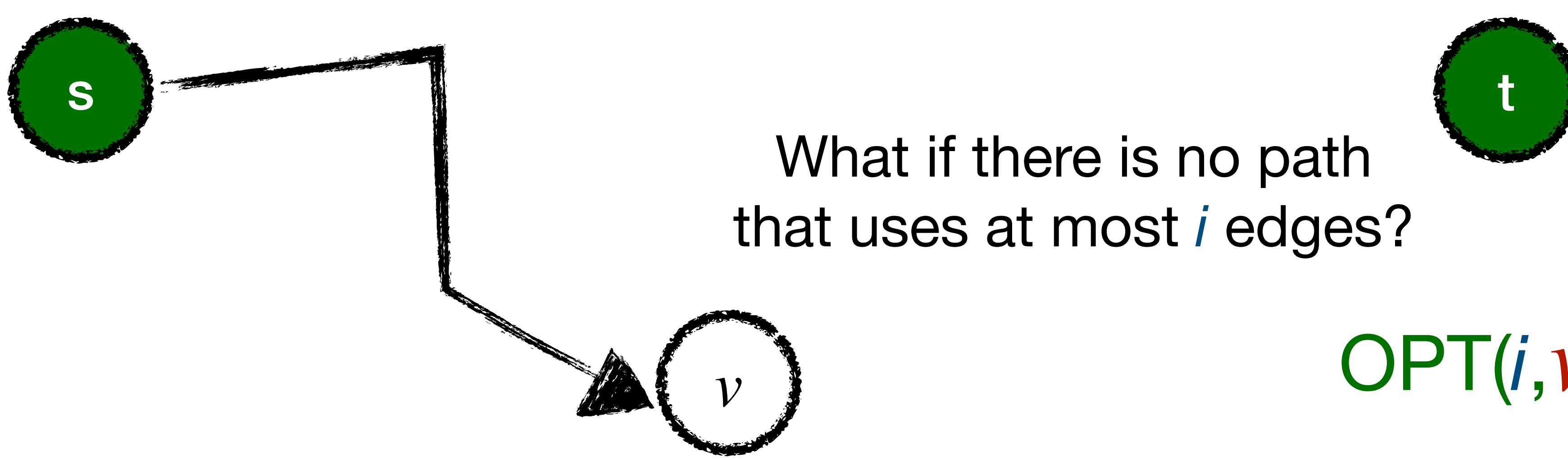

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$\text{OPT}(i,v) = \infty$

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- Let OPT(i,v) denote the minimum cost of a path $v \sim t$ from node v to t that uses at most i edges.
- What is then the (global) solution to our problem?
  - OPT(n-1,s)

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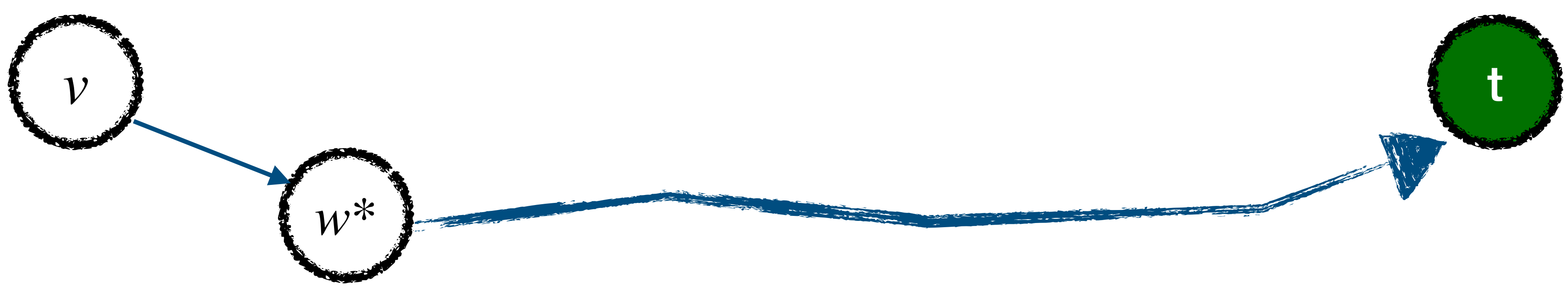

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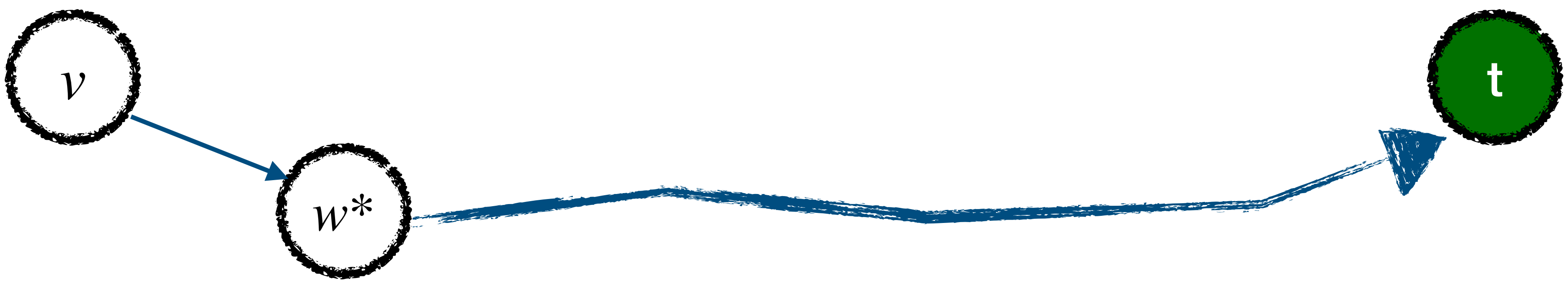

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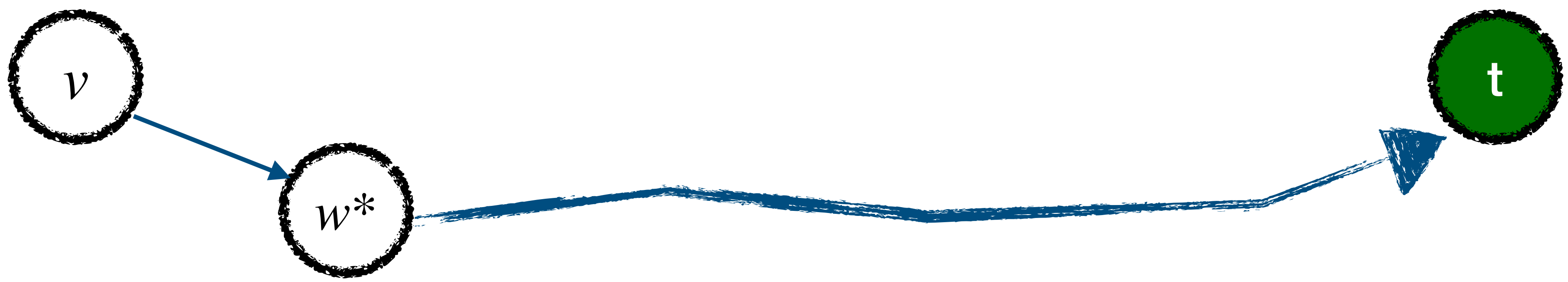

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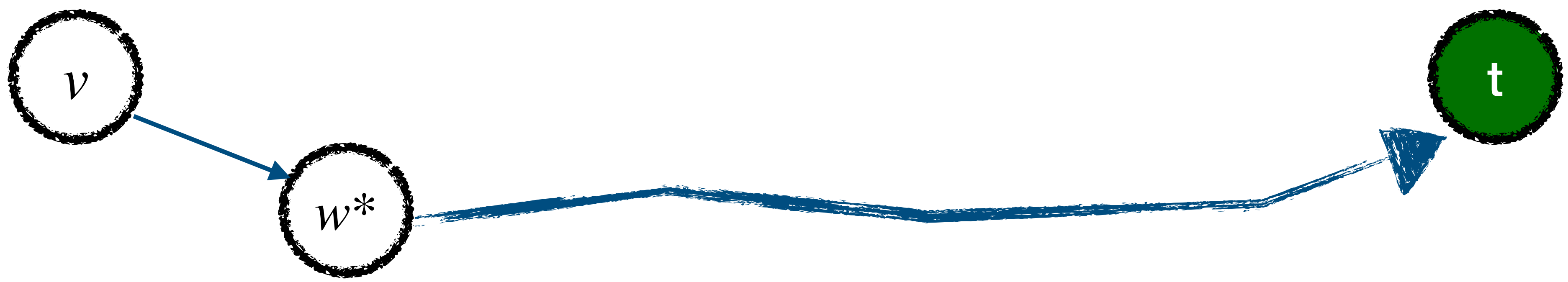

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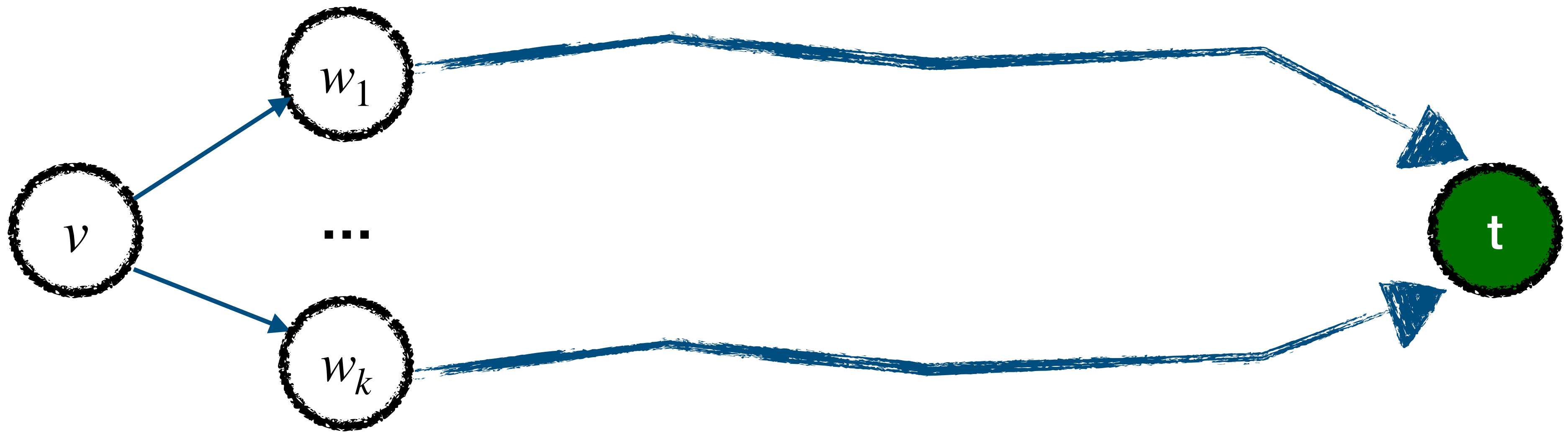

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The Bellman-Ford Algorithm

ShortestPath (G, s, t).

* Let $n = |V|$ *\

Define 2-D Array $M[0,\cdots,n-1, s, v_1, v_2, \ldots, t]$

Initialise $M[0,t] = 0$, and $M[0,v] = \infty$ for all other $v \in V$.

For $i = 1,2,\ldots,n-1$

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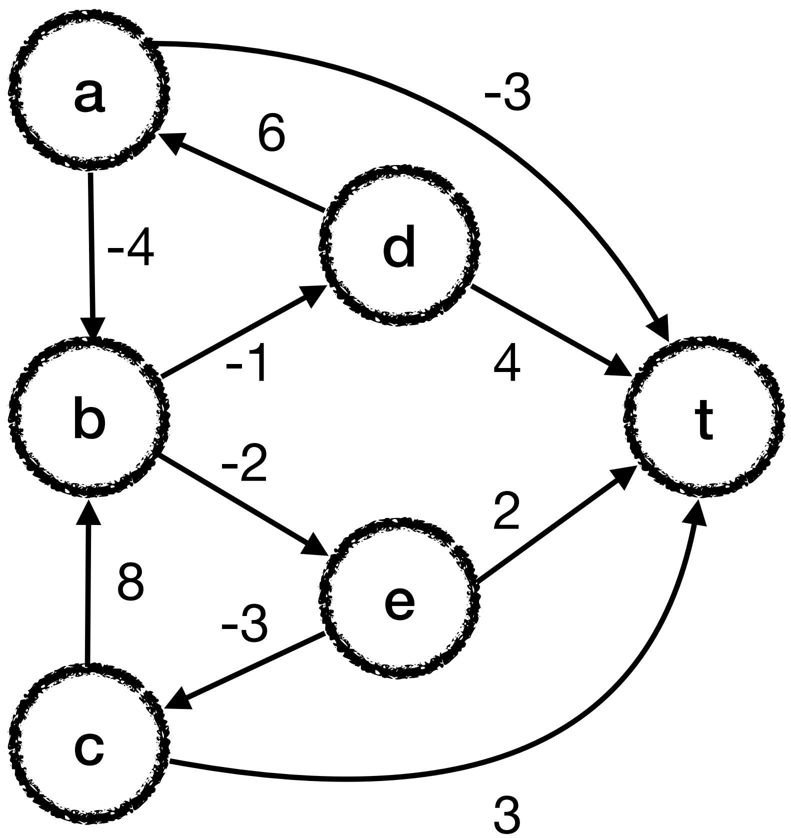
Example



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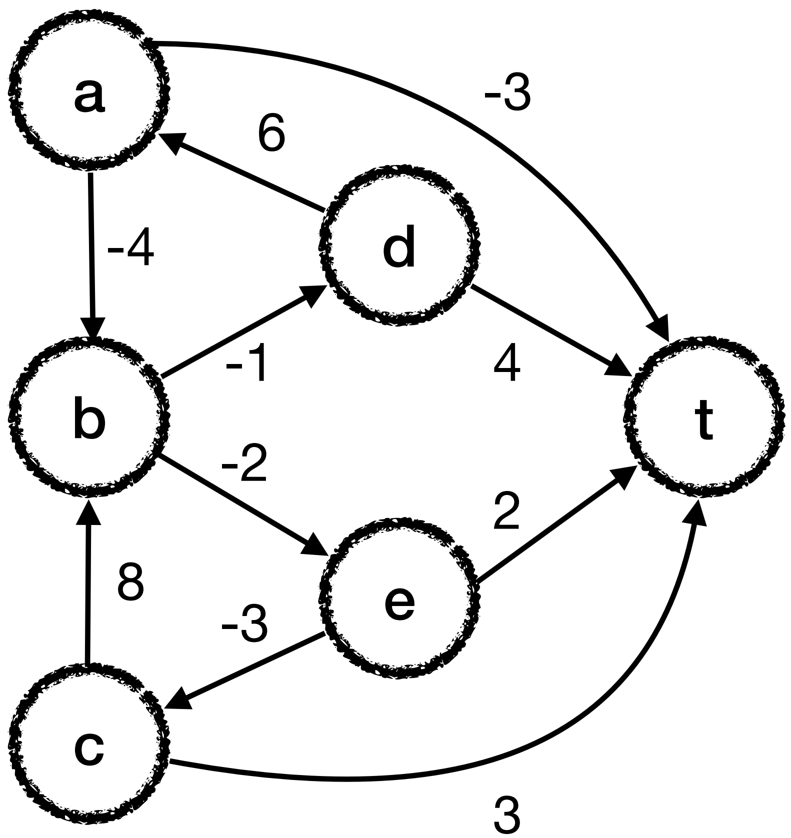

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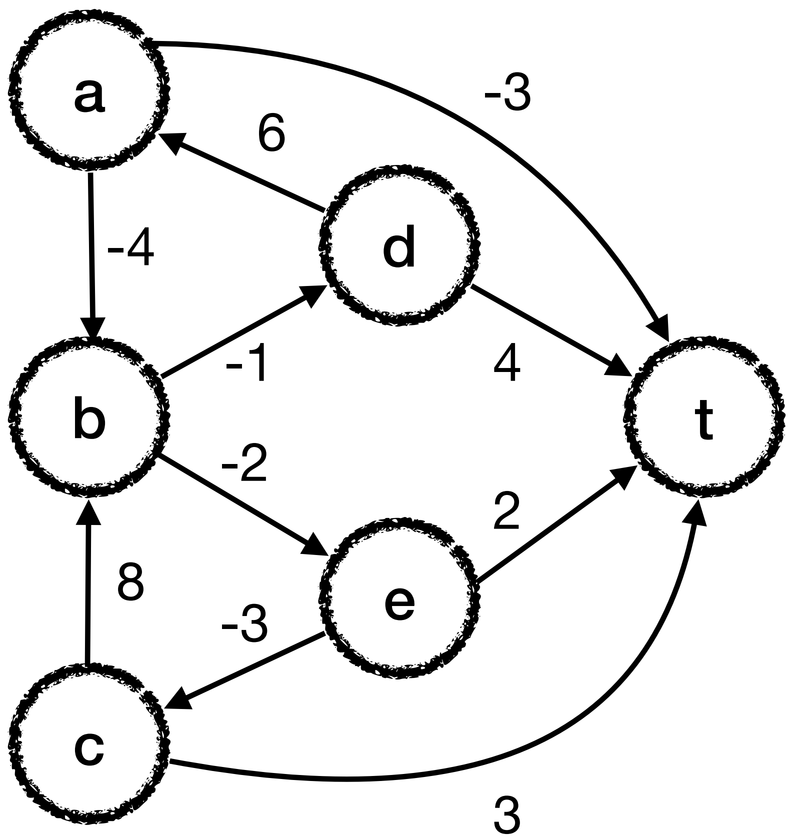

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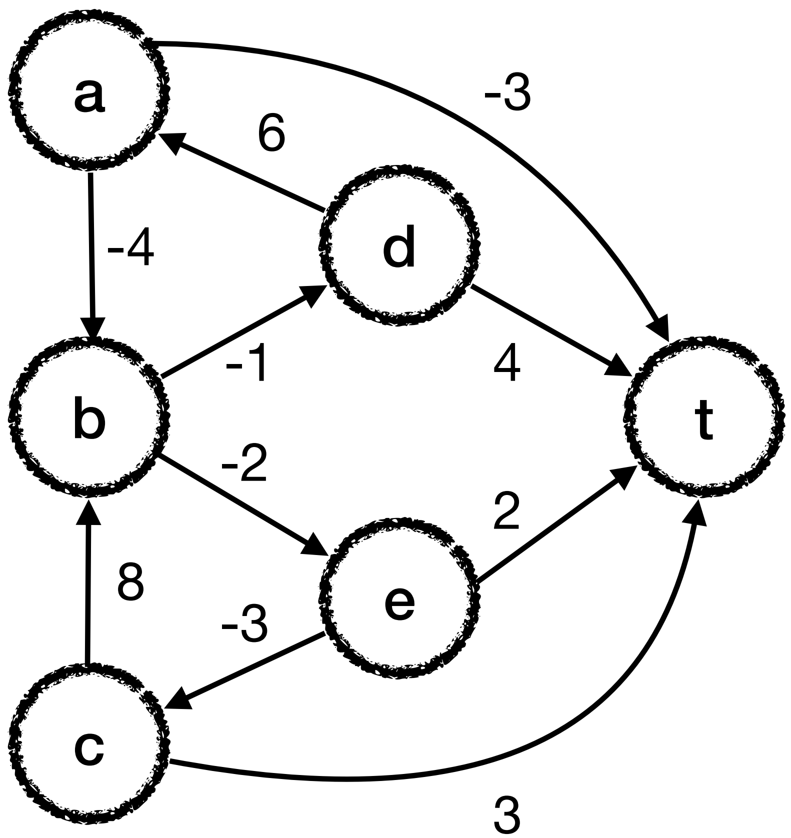

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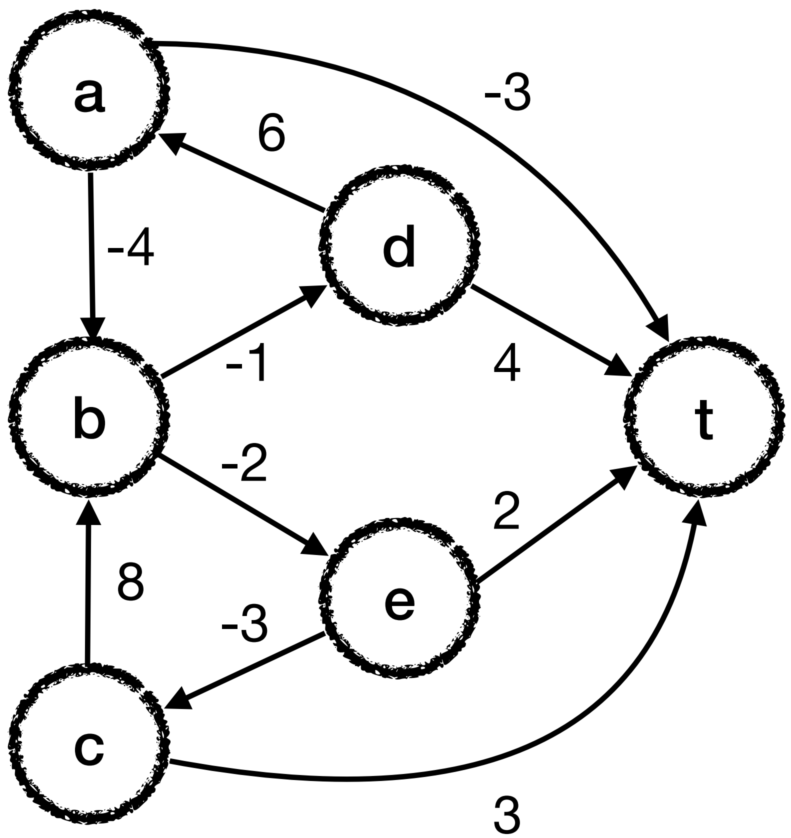

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d	∞					
e	∞					

$$M(1,t) = \min\{M(0,t)\ ,\ \min_{w \in N(t)} (0 + M(0,w))\}$$

ShortestPath (G, s, t).

* Let $n = |V|$ *\

Define 2-D Array $M[0,\cdots,n-1, s, v_1, v_2, \ldots, t]$

Initialise $M[0,t] = 0$, and $M[0,v] = \infty$ for all other $v \in V$.

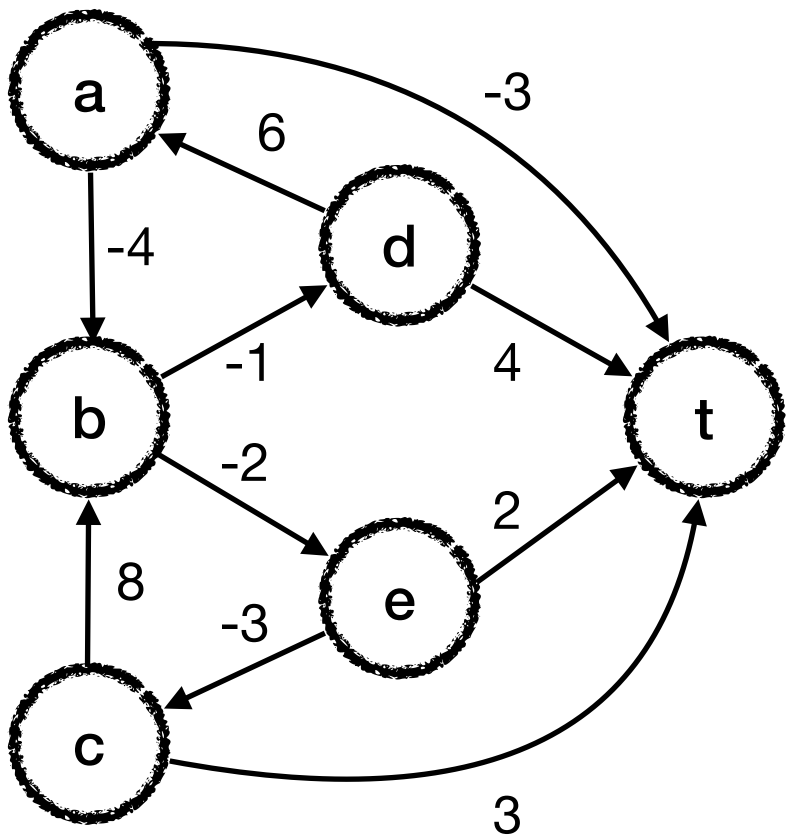
For $i = 1,2,\ldots,n-1$

For $v \in V$

$M(i,v) = \min\{M(i\text{-}1,v)\ ,\ \min_{w \in N(v)} (c_{vw} + M(i\text{-}1,w))\}$

Return $M(n\text{-}1,s)$

Example



	0	1	2	3	4	5
t	0	0				
a	∞					
b	∞					
c	∞					
d	∞					
e	∞					

ShortestPath (G, s, t).

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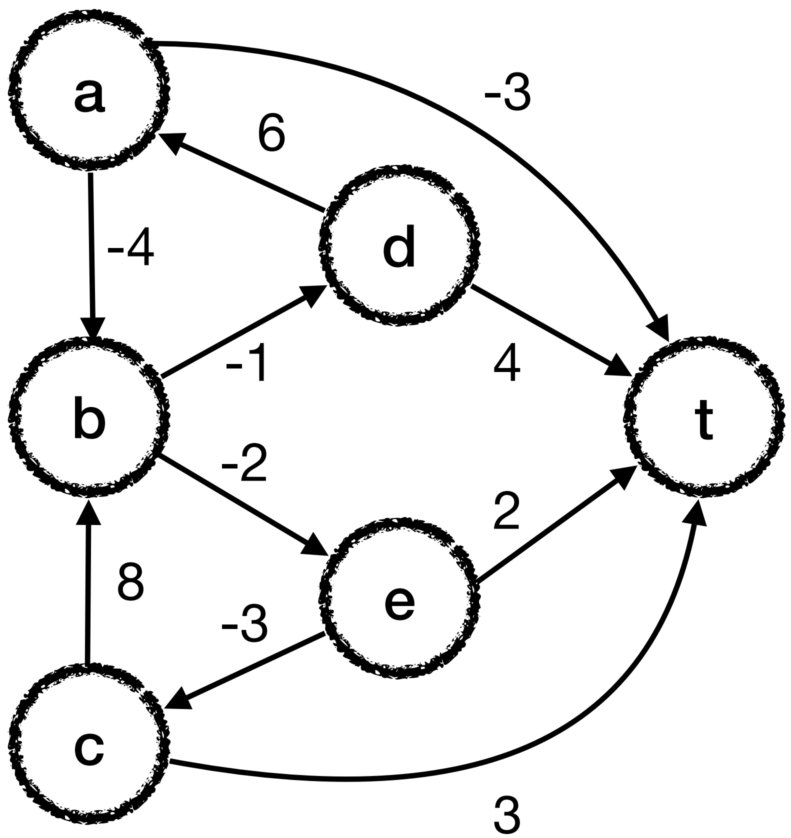

	0	1	2	3	4	5
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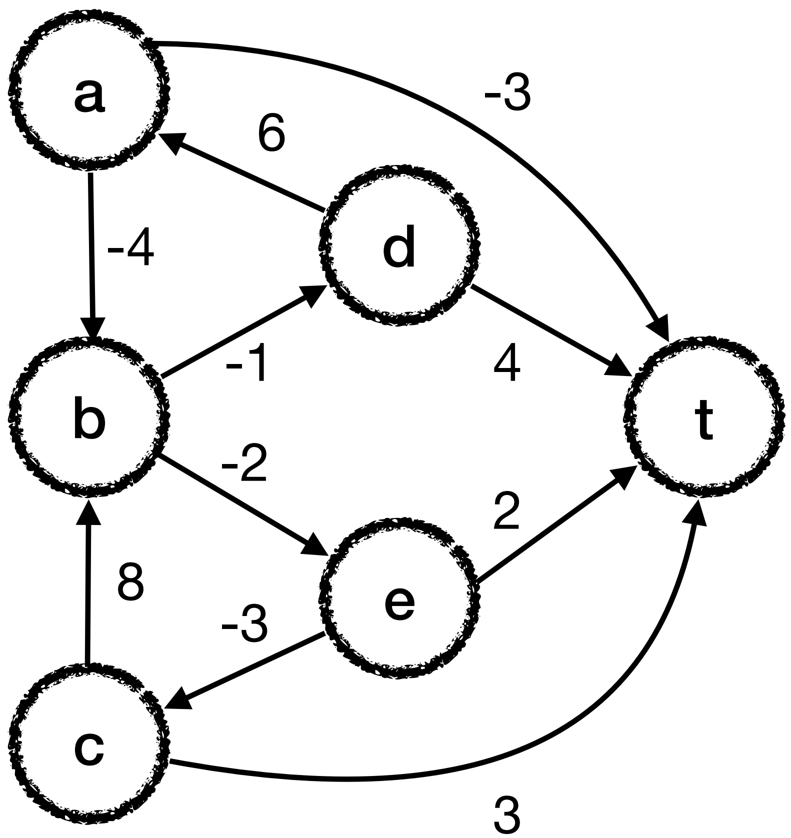
Example



	0	1	2	3	4	5
t	0	0				
a	∞					
b	∞					
c	∞					
d	∞					
e	∞					

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ShortestPath (G, s, t).

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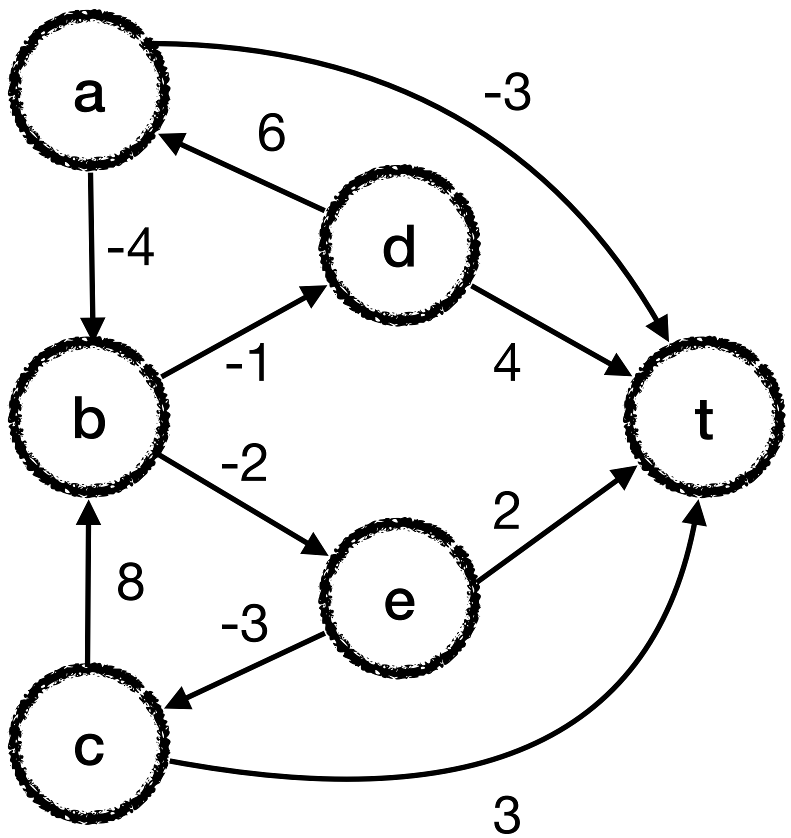
Example



	0	1	2	3	4	5
t	0	0				
a	∞					
b	∞					
c	∞					
d	∞					
e	∞					

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$$M(1,a) = c_{at} + M(0,t)$$

ShortestPath (G, s, t).

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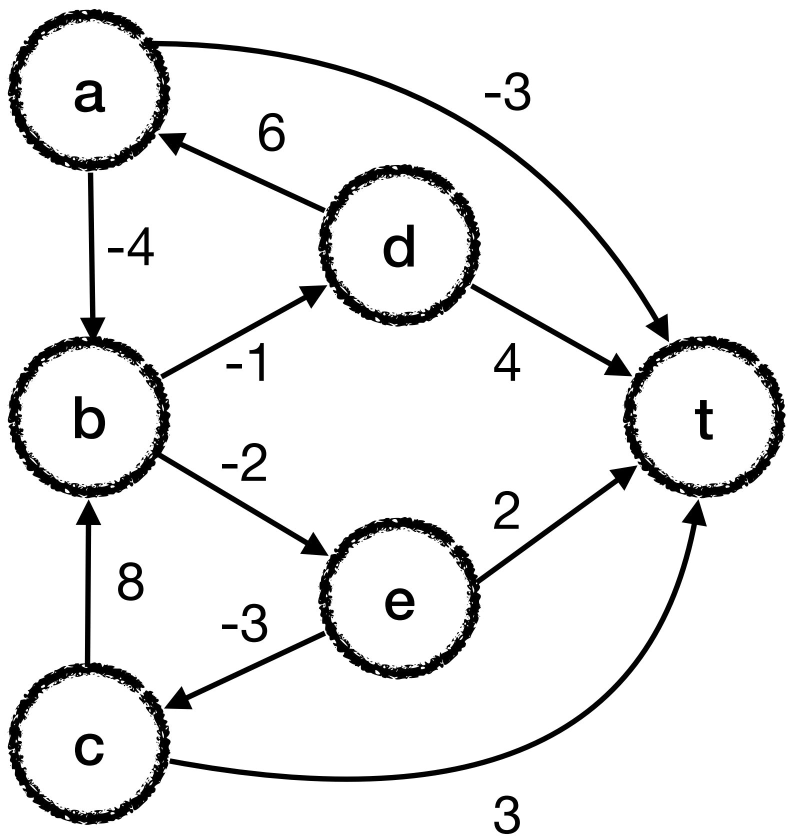
Example



	0	1	2	3	4	5
t	0	0				
a	∞	-3				
b	∞					
c	∞					
d	∞					
e	∞					

ShortestPath (G, s, t).

* Let $n = |V|$ *\

Define 2-D Array $M[0,\cdots,n-1, s, v_1, v_2, \ldots, t]$

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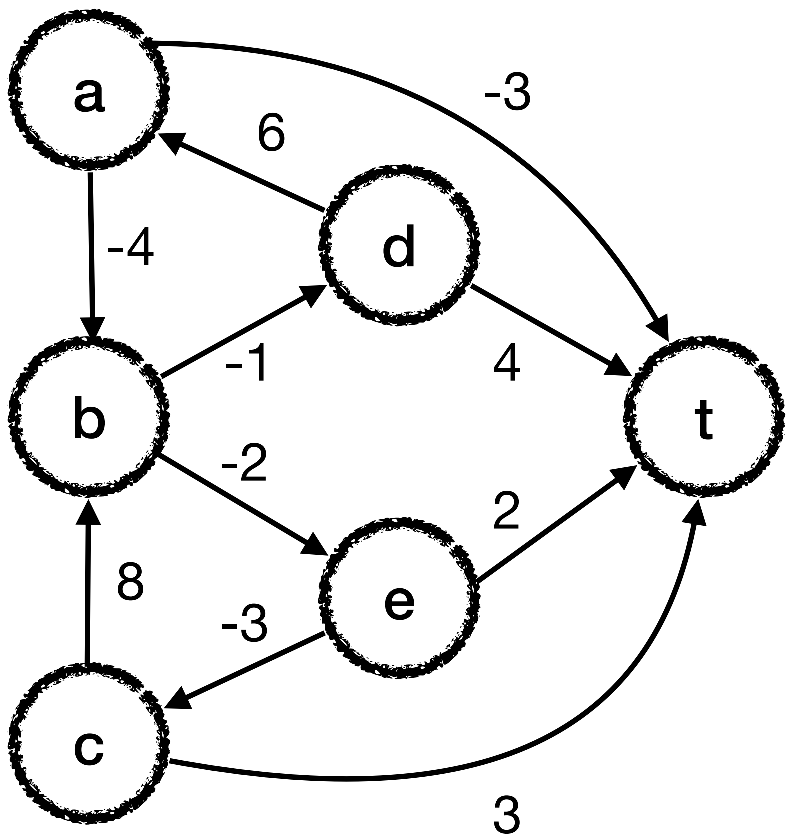
Example



	0	1	2	3	4	5
t	0	0				
a	∞	-3				
b	∞					
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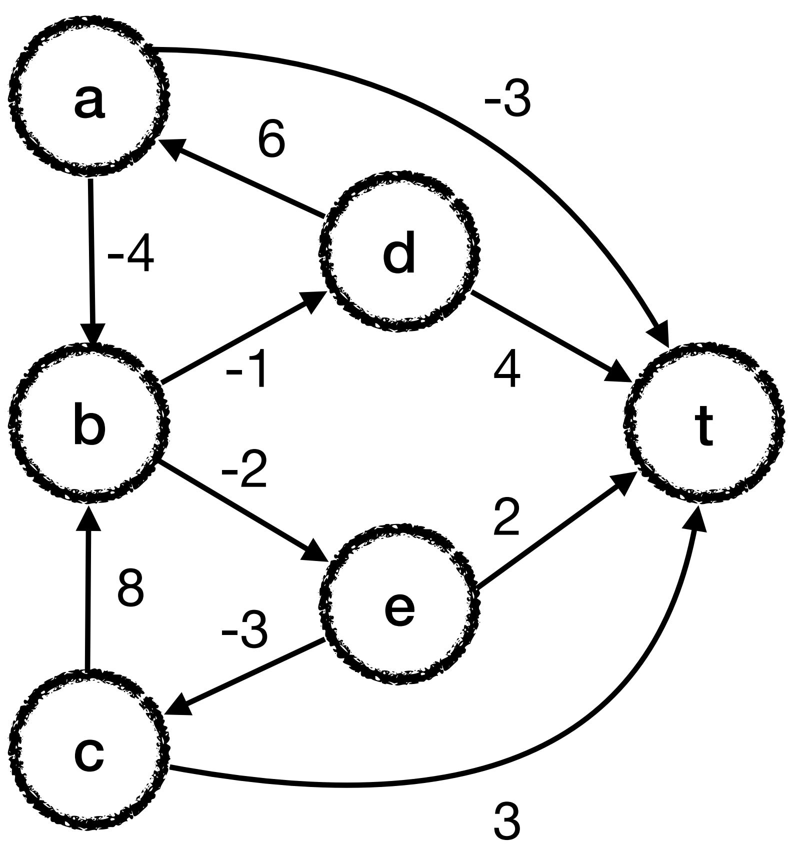

	0	1	2	3	4	5
t	0	0				
a	∞	-3				
b	∞					
c	∞					
d	∞					
e	∞					

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ShortestPath (G, s, t).

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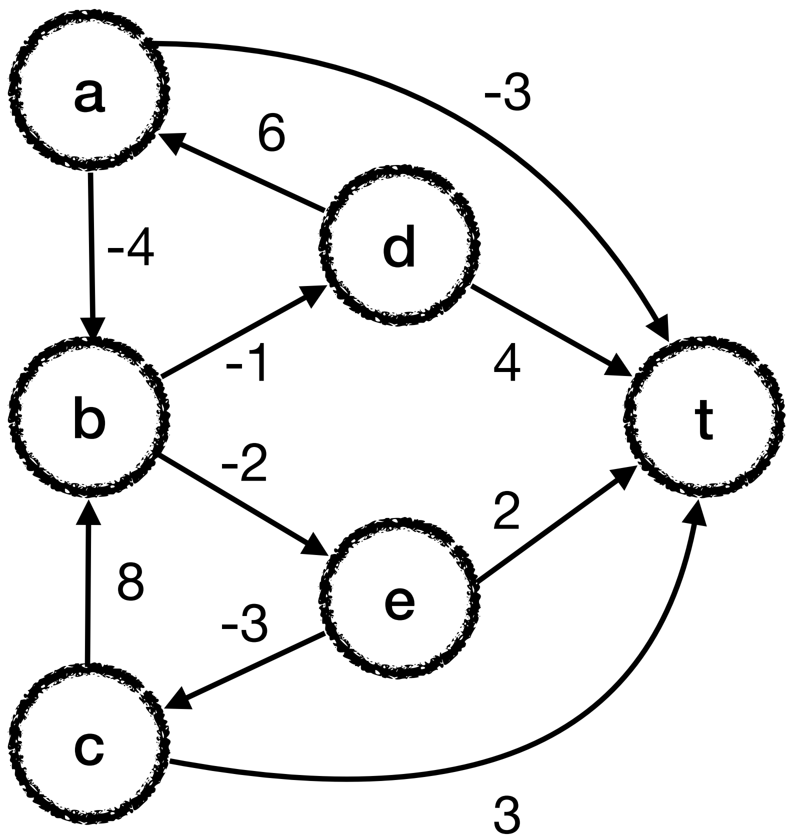
Example



	0	1	2	3	4	5
t	0	0				
a	∞	-3				
b	∞					
c	∞					
d	∞					
e	∞					

$$M(1,\mathrm{b}) = \min\{M(0,\mathrm{b}),\ \min_{w \in N(b)} (c_{bw} + M(0,w))\}$$

$$M(1,\mathrm{b}) = \infty$$

ShortestPath (G, s, t).

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Define 2-D Array $M[0,\cdots,n-1, s, v_1, v_2, \ldots, t]$

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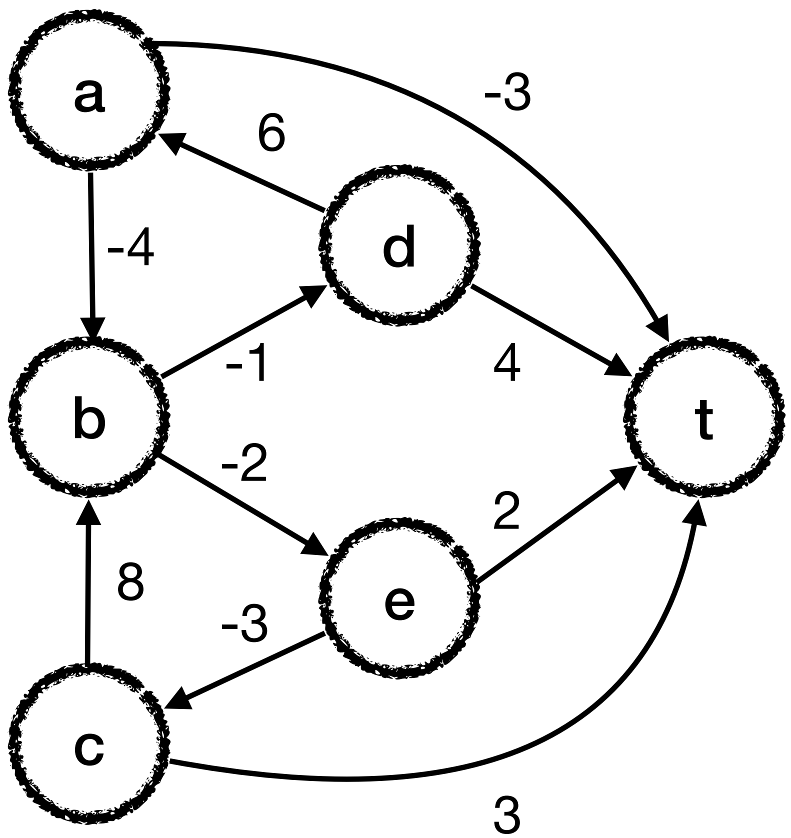
Example



	0	1	2	3	4	5
t	0	0				
a	∞	-3				
b	∞	∞				
c	∞					
d	∞					
e	∞					

$$M(1,\mathrm{b}) = \min\{M(0,\mathrm{b}),\ \min_{w \in N(b)} (c_{bw} + M(0,w))\}$$

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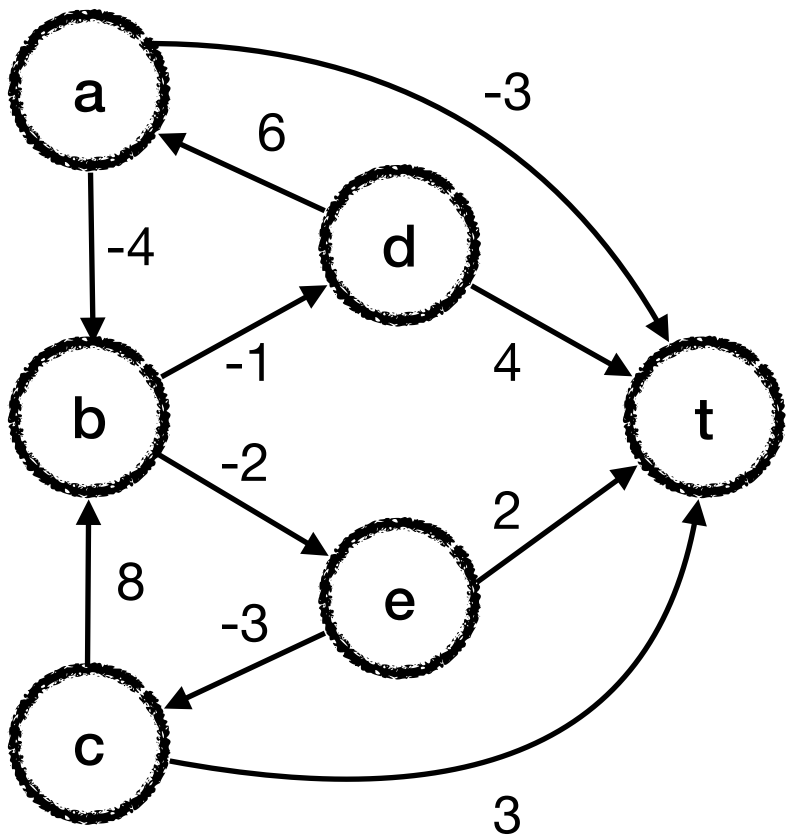
Example



	0	1	2	3	4	5
t	0	0	0	0	0	
a	∞	-3	-3	-4		
b	∞	∞	0	-2		
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

ShortestPath (G, s, t).

* Let $n = |V|$ *\

Define 2-D Array $M[0,\cdots,n-1, s, v_1, v_2, \ldots, t]$

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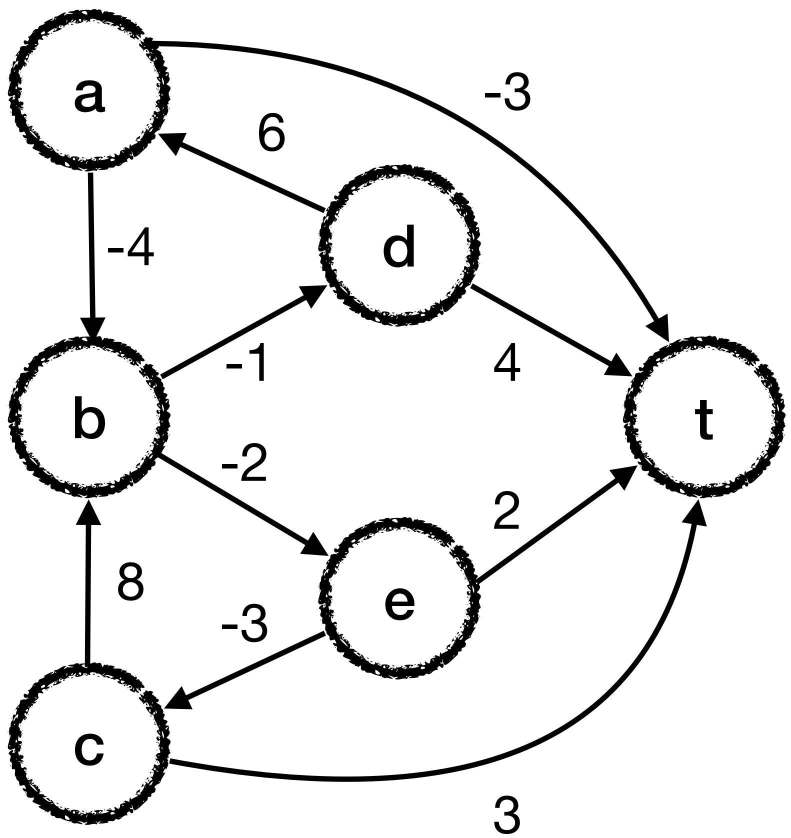
Return $M(n\text{-}1,s)$

Example



	0	1	2	3	4	5
t	0	0	0	0	0	
a	∞	-3	-3	-4		
b	∞	∞	0	-2		
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

$$M(4,\mathrm{a}) = \min\{M(3,\mathrm{a}) , \min_{w \in N(a)} (c_{aw} + M(3,\mathrm{w}))\}$$

ShortestPath (G, s, t).

* Let $n = |V|$ *\

Define 2-D Array $M[0,\cdots,n-1, s, v_1, v_2, \ldots, t]$

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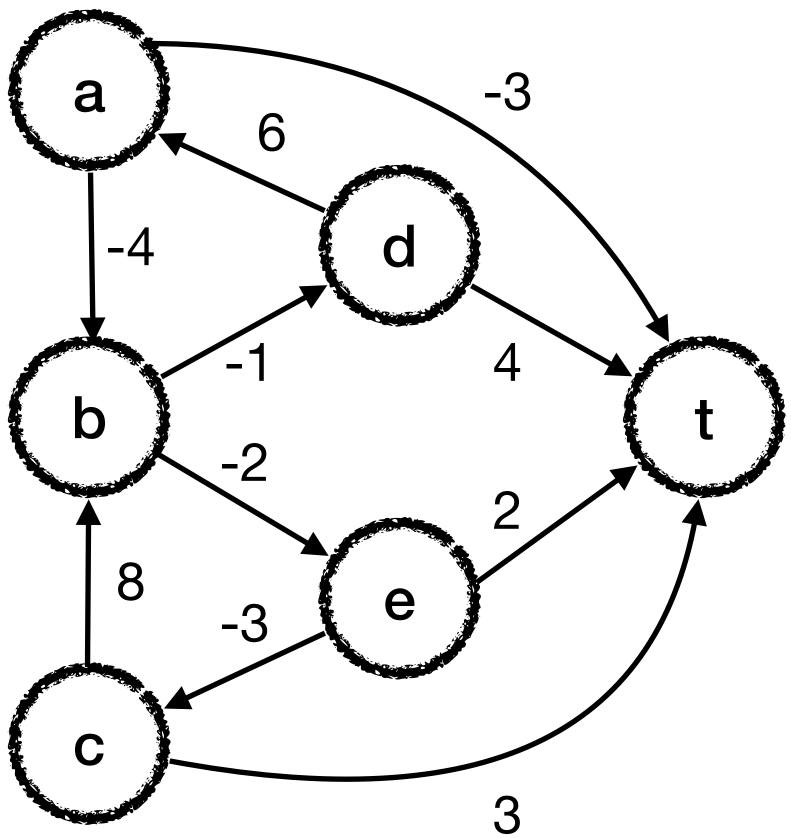
Return $M(n\text{-}1,s)$

Example



	0	1	2	3	4	5
t	0	0	0	0	0	
a	∞	-3	-3	-4		
b	∞	∞	0	-2		
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

$$M(4,\mathrm{a}) = \min\{M(3,\mathrm{a})\,,\ \min_{w\in N(a)} (c_{aw} + M(3,\mathrm{w}))\}$$

$$c_{ab} + M(3,\mathrm{b}) = \text{-}6$$

ShortestPath (G, s, t).

* Let $n = |V|$ *\

Define 2-D Array $M[0,\cdots,n-1,\ s, v_1, v_2, \ldots, t]$

Initialise $M[0,t] = 0$, and $M[0,v] = \infty$ for all other $v \in V$.

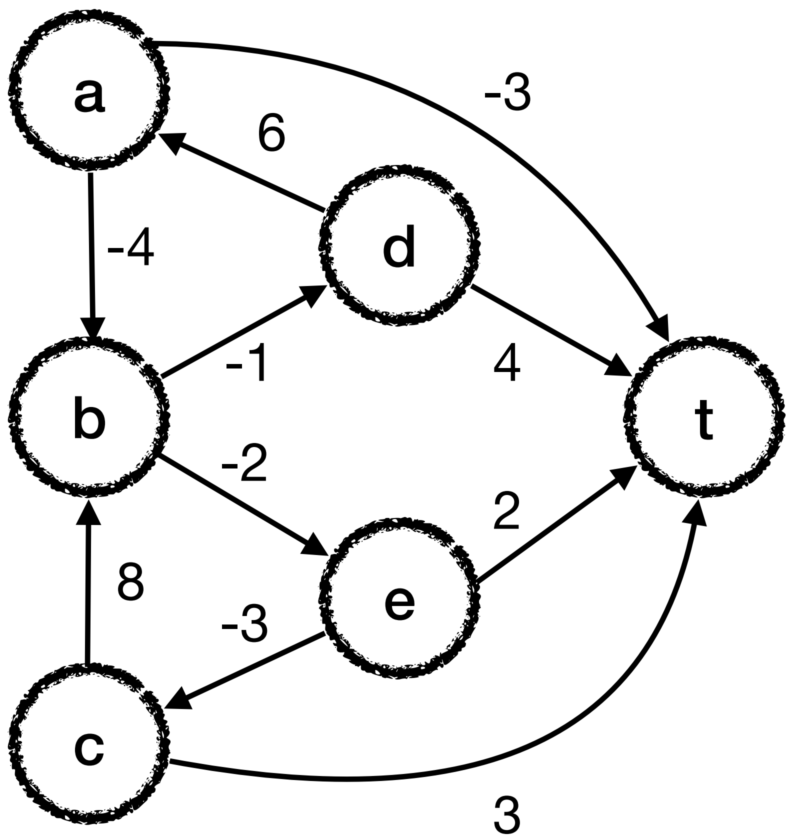
For $i = 1,2,\ldots,n-1$

For $v \in V$

$M(i,v) = \min\{M(i\text{-}1,v)\,,\ \min_{w\in N(v)} (c_{vw} + M(i\text{-}1,w))\}$

Return $M(n\text{-}1,s)$

Example



	0	1	2	3	4	5
t	0	0	0	0	0	
a	∞	-3	-3	-4		
b	∞	∞	0	-2		
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

$$M(4,a) = \min\{M(3,a)\ ,\ \min_{w \in N(a)} (c_{aw} + M(3,w))\}$$

$$c_{ab} + M(3,b) = -6$$

$$c_{at} + M(3,t) = -3$$

ShortestPath (G, s, t).

* Let $n = |V|$ *\

Define 2-D Array $M[0,\cdots,n-1,\ s, v_1, v_2, \ldots, t]$

Initialise $M[0,t] = 0$, and $M[0,v] = \infty$ for all other $v \in V$.

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For $v \in V$

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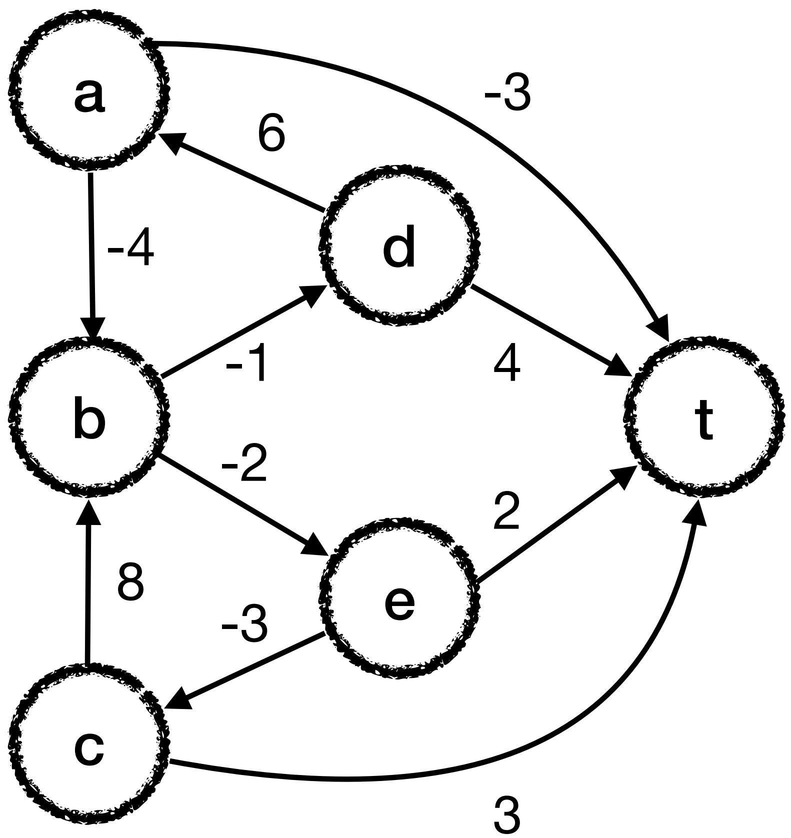
Return $M(n\text{-}1,s)$

Example



	0	1	2	3	4	5
t	0	0	0	0	0	
a	∞	-3	-3	-4	-6	
b	∞	∞	0	-2		
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

ShortestPath (G, s, t).

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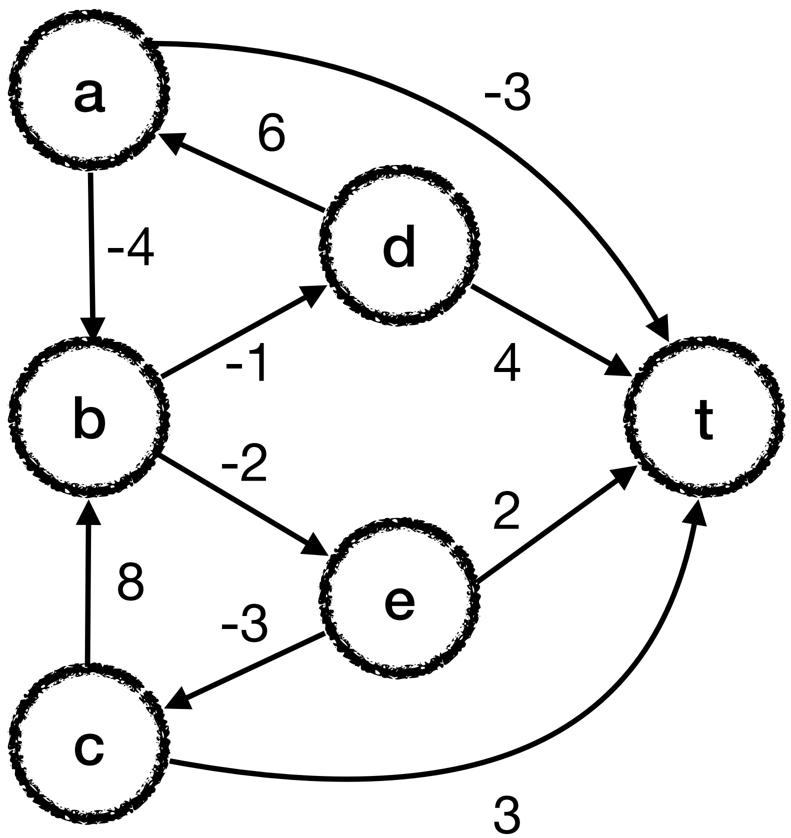
Example



	0	1	2	3	4	5
t	0	0	0	0	0	
a	∞	-3	-3	-4	-6	
b	∞	∞	0	-2		
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

ShortestPath (G, s, t).

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Define 2-D Array $M[0,\cdots,n-1, s, v_1, v_2, \ldots, t]$

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For $v \in V$

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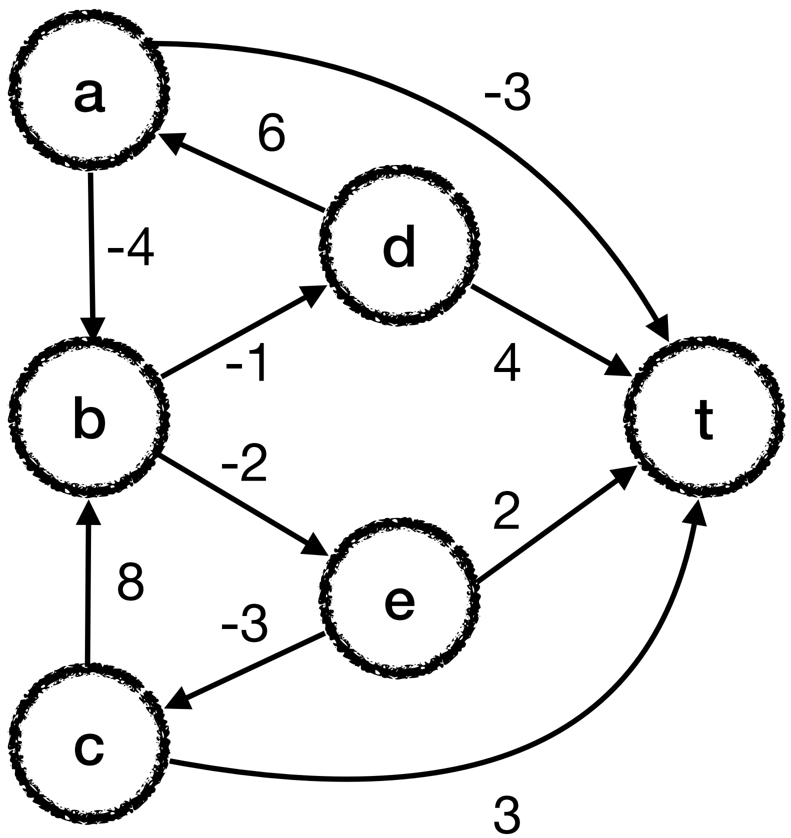
Example



	0	1	2	3	4	5
t	0	0	0	0	0	
a	∞	-3	-3	-4	-6	
b	∞	∞	0	-2		
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

$$M(4,b) = \min\{M(3,b)\ ,\ \min_{w \in N(b)} (c_{bw} + M(3,w))\}$$

ShortestPath (G, s, t).

* Let $n = |V|$ *\

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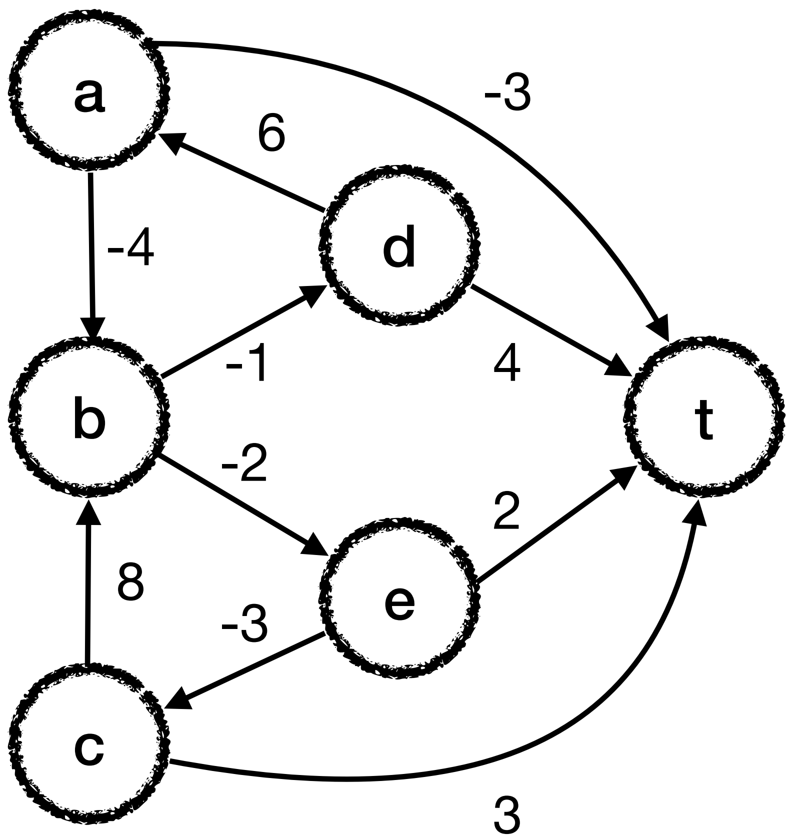
Example



	0	1	2	3	4	5
t	0	0	0	0	0	
a	∞	-3	-3	-4	-6	
b	∞	∞	0	-2		
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

$$M(4,b) = \min\{M(3,b)\ ,\ \min_{w \in N(b)} (c_{bw} + M(3,w))\}$$

$$c_{bd} + M(3,d) = -1 + 3$$

ShortestPath (G, s, t).

* Let $n = |V|$ *\

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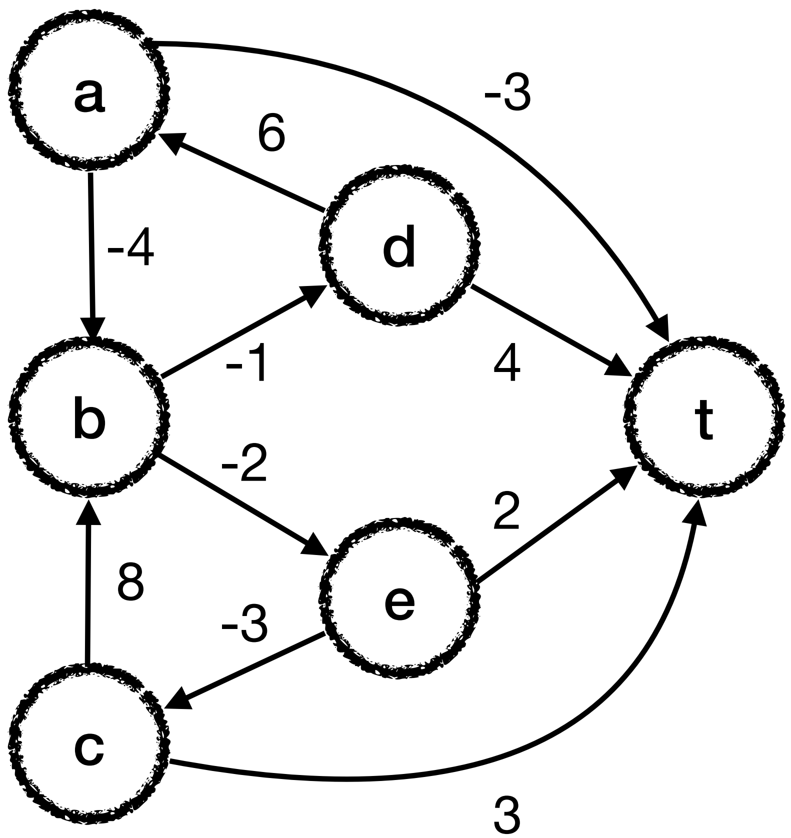
Example



	0	1	2	3	4	5
t	0	0	0	0	0	
a	∞	-3	-3	-4	-6	
b	∞	∞	0	-2		
c	∞	3	3	3		
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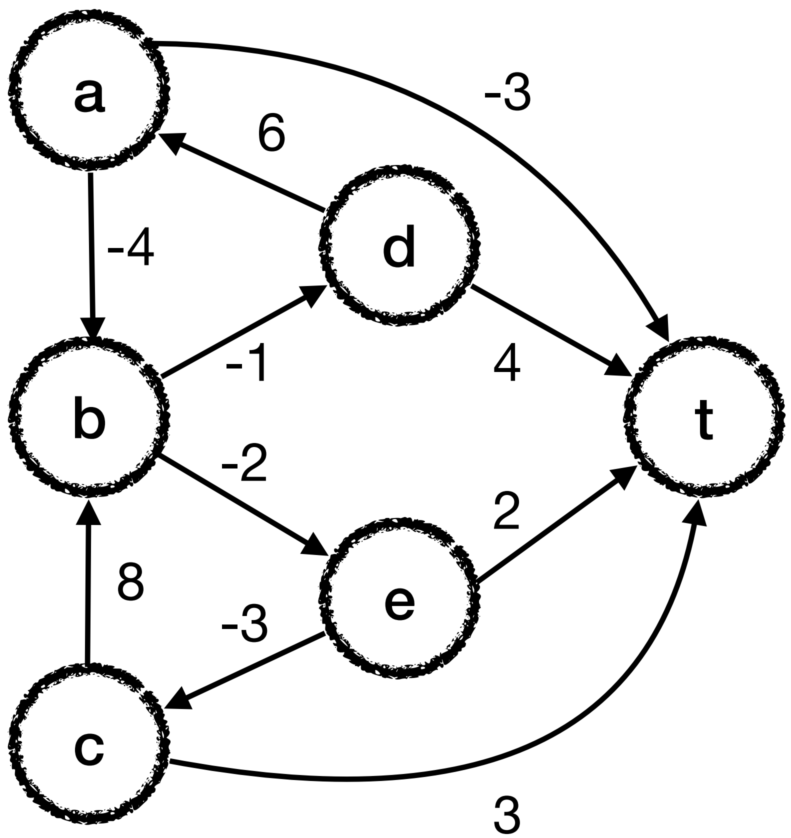
Return $M(n\text{-}1,s)$

$$c_{bd} + M(3,d) = -1 + 3$$

$$c_{be} + M(3,e) = -2 + 0$$

Example



	0	1	2	3	4	5
t	0	0	0	0	0	
a	∞	-3	-3	-4	-6	
b	∞	∞	0	-2	-2	
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

ShortestPath (G, s, t).

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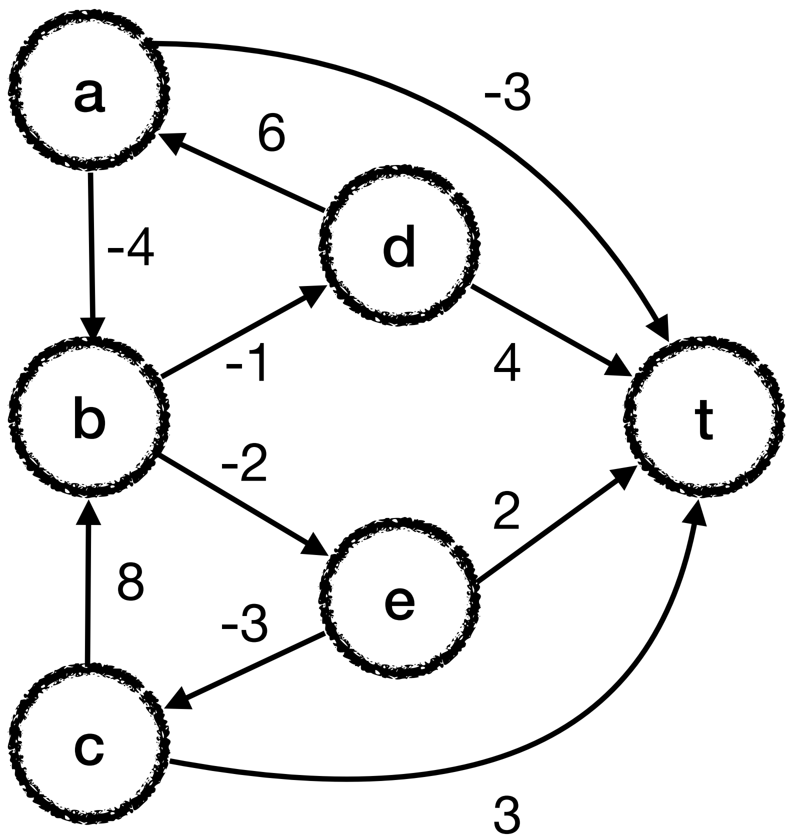

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	-6
b	∞	∞	0	-2	-2	-2
c	∞	3	3	3	3	3
d	∞	4	3	3	2	0
e	∞	2	0	0	0	0

ShortestPath (G, s, t).

* Let $n = |V|$ *\

Define 2-D Array $M[0,\cdots,n-1, s, v_1, v_2, \ldots, t]$

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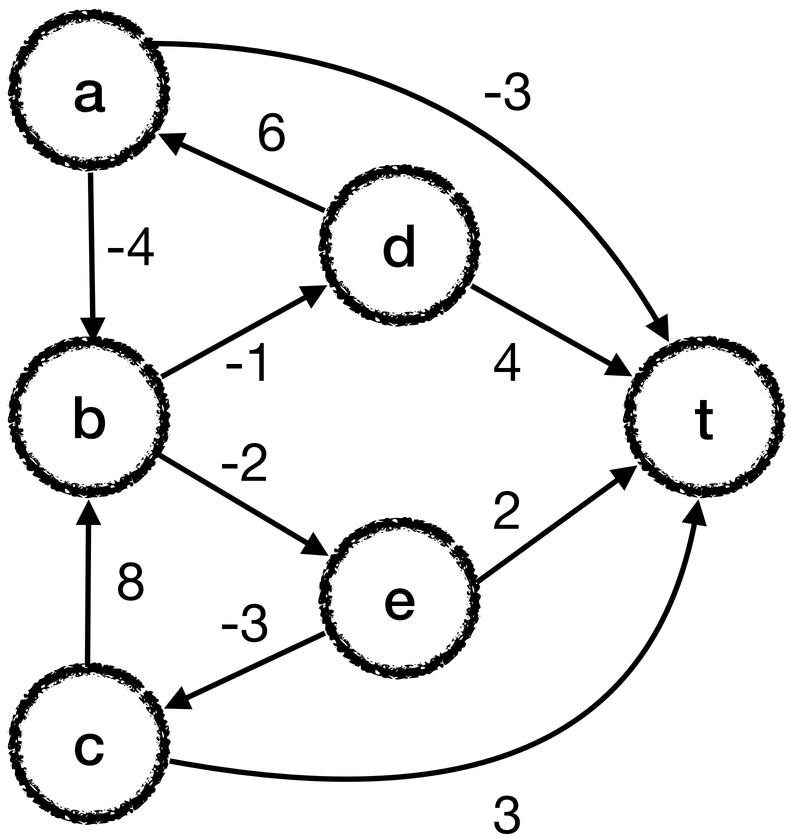
Return $M(n\text{-}1,s)$

Example



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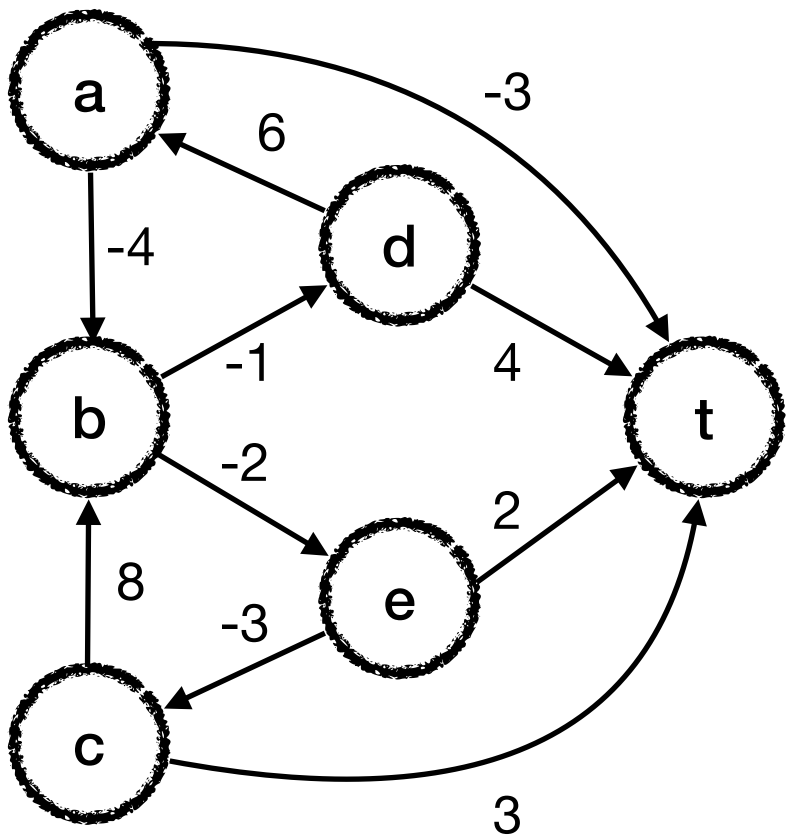

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$$M(5,d) = \min\{M(4,d)\ ,\ \min_{w \in N(d)} (c_{dw} + M(4,w))\}$$

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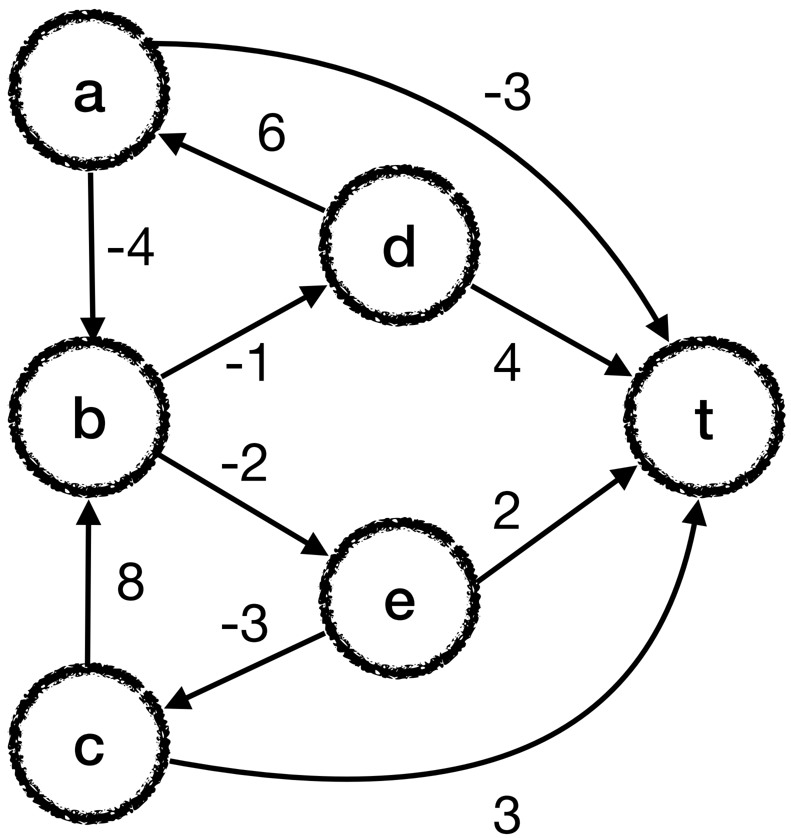

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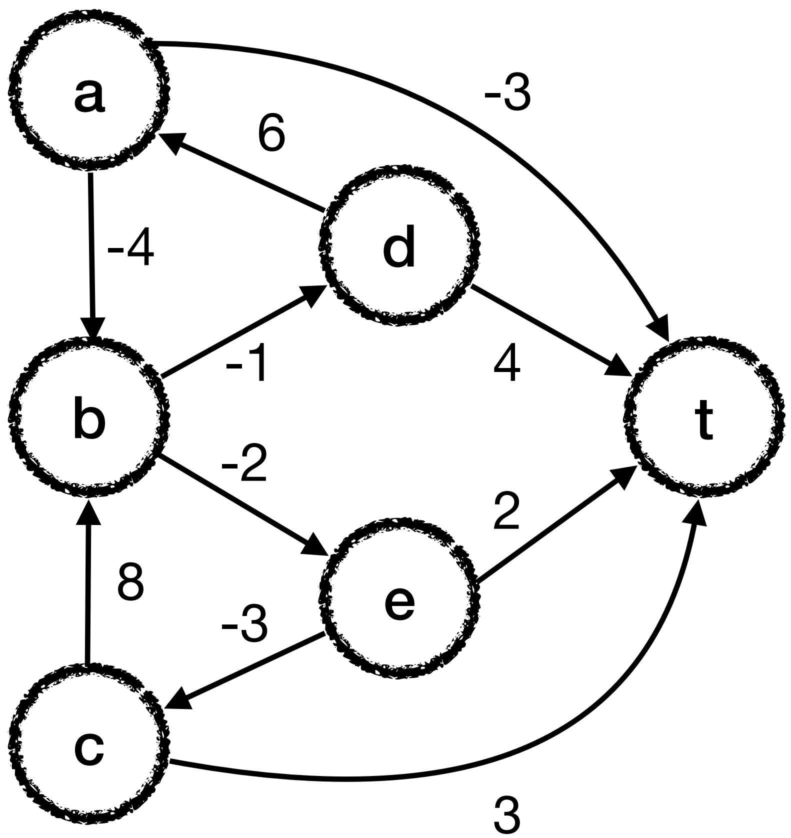
So the first edge must be (d, a).

Example



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Next we consider $M(4, a)$, etc.

Running Time

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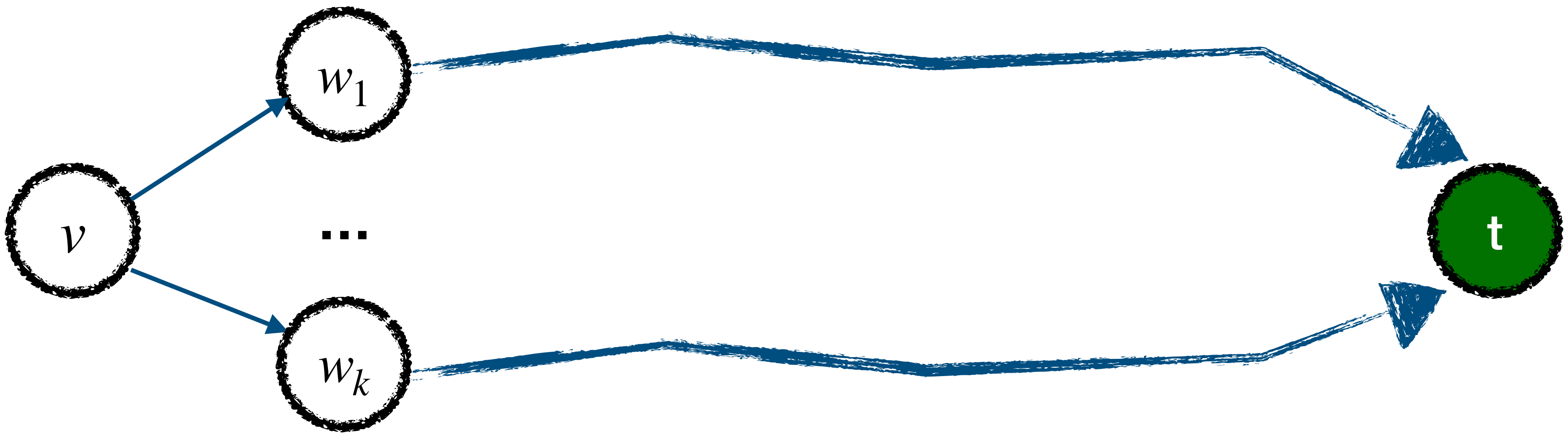
Return $M(n\text{-}1,s)$

Overall: $O\left(n^3\right)$

Improved Analysis



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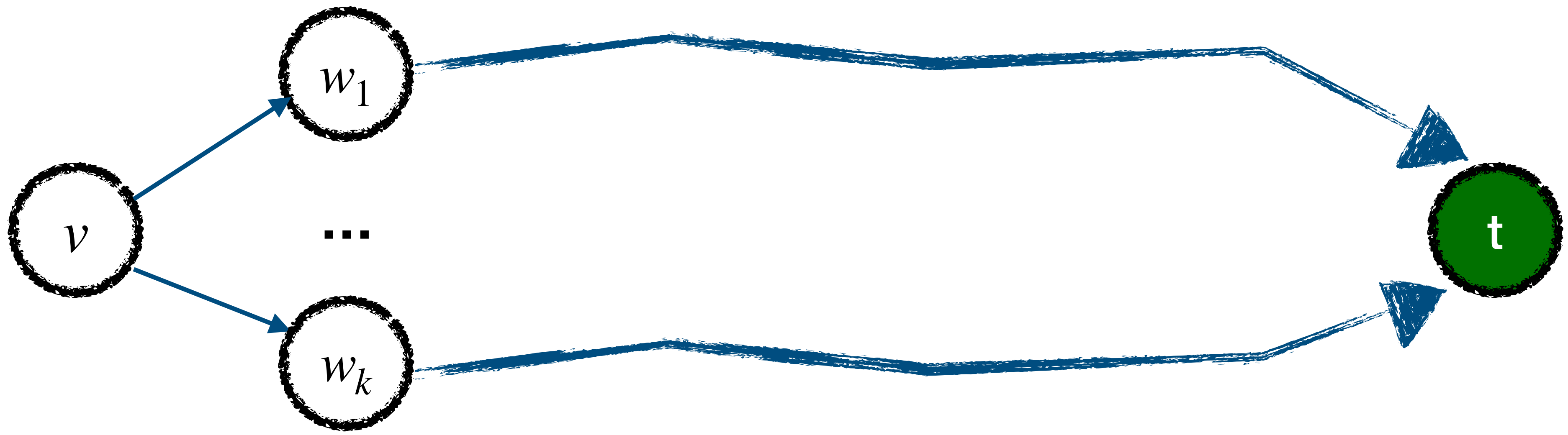

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Suffices to only check w such that $(v, w) \in E$.

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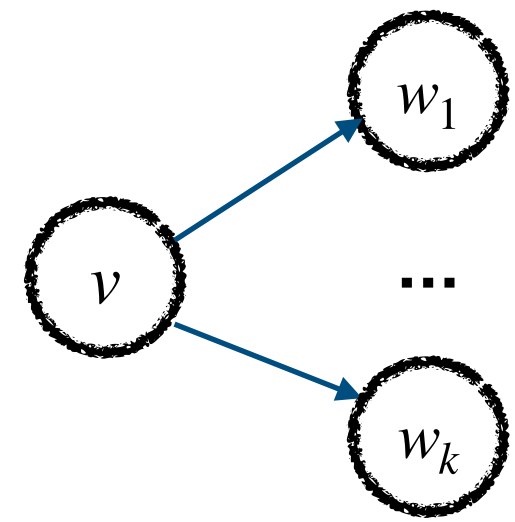
We have to compute an entry for every $v \in V$ and every index $i \in [0,n-1]$, so in total we need time $O\left(n \sum_{v \in V} n_v\right)$

Improved Analysis

How large is $\sum_{v \in V} n_v$?



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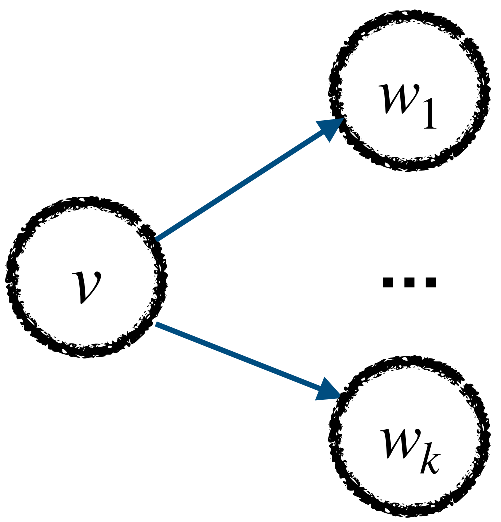

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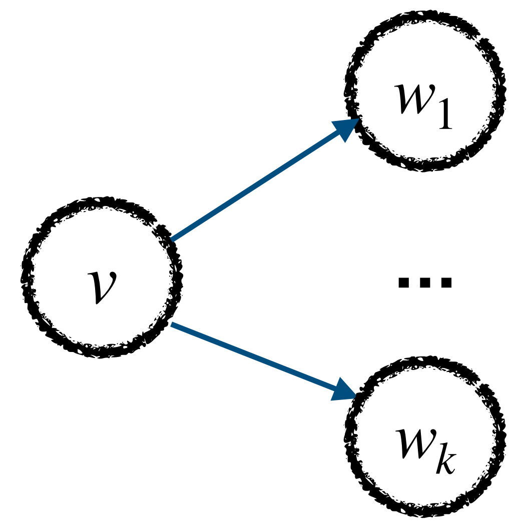

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Overall: $O(nm)$

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- That works, however more work is needed to recover the shortest paths!

Shortest Paths in Graphs

- **Input:** A directed graph $G = (V, E)$, and designated nodes s, t in V. We also assume that every node u in V is reachable from s, and *that the graph does not have any negative cycles.* We are also given a cost $c_e \in \mathbb{R}$ for every edge e in E.

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What if the graph has negative cycles? Can we at least detect that?

Detecting Negative Cycles

- Can be done in time $O(mn)$.
- It is in fact usually included as a part of the Bellman-Ford algorithm.
- We will not cover this here, see KT 6.10 for the details if you are interested.

Reading and References

- Kleinberg and Tardos 6.8.
- CLRS 22.1.
- Roughgarden 18.1, 18.2.
- The Bellman-Ford visualiser: https://algorithms.discrete.ma.tum.de/graph-algorithms/spp-bellman-ford/index_en.html