Introduction to Algorithms and Data Structures

Dynamic Programming - The Bellman-Ford Algorithm for Shortest Paths
Shortest Paths in Graphs (Lecture 17)

- **Input:** A directed graph $G = (V, E)$, and a designated node $s$ in $V$. We also assume that every node $u$ in $V$ is reachable from $s$. We are also given a length $\ell_e > 0$ for every edge $e$ in $E$.

- **Output:** For every node $u$ in $V$, a shortest path $s \sim u$ from $s$ to $u$. 
Shortest Paths in Graphs (today)

• **Input:** A directed graph $G = (V, E)$, and designated nodes $s, t$ in $V$. We also assume that every node $u$ in $V$ is reachable from $s$. We are also given a cost $c_e \in \mathbb{R}$ for every edge $e$ in $E$.

• **Output:** A shortest path $s \sim t$ from $s$ to $t$. 
Shortest Paths in Graphs (today)
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- The difference is that the edge “lengths” can be positive or negative. In this context they are better interpreted as costs, and denoted by $c_e$ or $c_{uv}$. 
Shortest Paths in Graphs (today)

• The difference is that the edge “lengths” can be positive or negative. In this context they are better interpreted as costs, and denoted by $c_e$ or $c_{uv}$.

• Motivation: e.g., Financial Networks

  positive costs (costs of transactions)
  negative costs (profits of transactions)
Shortest Paths in Graphs (today)

- **Input:** A directed graph \( G = (V, E) \), and designated nodes \( s, t \) in \( V \). We also assume that every node \( u \) in \( V \) is reachable from \( s \). We are also given a cost \( c_e \in \mathbb{R} \) for every edge \( e \) in \( E \).

- **Output:** A shortest path \( s \sim t \) from \( s \) to \( t \). In other words, a path \( P \) that minimises \( \sum_{(u,v) \in P} c_{uv} \)
Negative Cycles

- Can we find a shortest path in the following graph?
Negative Cycles

• Can we find a shortest path in the following graph?

Arbitrary negative cost!
Shortest Paths in Graphs

• **Input:** A directed graph $G = (V, E)$, and designated nodes $s, t$ in $V$. We also assume that every node $u$ in $V$ is reachable from $s$, and *that the graph does not have any negative cycles*. We are also given a cost $c_e \in \mathbb{R}$ for every edge $e$ in $E$.

• **Output:** A shortest path $s \sim t$ from $s$ to $t$. In other words, a path $P$ that minimises $\sum_{(u,v) \in P} c_{uv}$.
Why not Dijkstra?
Dijkstra’s Algorithm

• For every node \( v \in V-S \), we determine the shortest path that can be constructed by traveling along a path \( s \sim u \) for \( u \in S \), followed by \( (u, v) \).

• In other words, we choose node \( v \in V-S \) such that

\[
d'(v) = \min_{e=(u,v): u \in S} d(u) + \ell_e
\]

• Add \( v \) to \( S \) and define \( d(v) = d'(v) \).
Why not Dijkstra?

• Which node would Dijkstra add in the following graph?
• Which node would Dijkstra add in the following graph?
Maybe modified Dijkstra?

- **Idea:** “Get rid” of the negative costs by adding a large number $M$ to all the edge costs.
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The shortest path changes!
A dynamic programming approach

- The algorithm that we will present next was developed by Bellman (1958) and Ford (1956).

- Note that Dijkstra’s algorithm was published in 1959.
Why dynamic programming?

Let’s look at a shortest path \( s \sim t \) from \( s \) to \( t \).
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This consists of a shortest path $s \sim u$ from $s$ to $t$, and a shortest path $u \sim t$ from $u$ to $t$ (why?).
Why dynamic programming?

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Why dynamic programming?

Let’s look at a shortest path $s \sim t$ from $s$ to $t$.

This consists of a shortest path $s \sim u$ from $s$ to $t$, and a shortest path $u \sim t$ from $u$ to $t$ (why?).
Simple Observation

- **Observation**: If a graph does not have any negative cycles, then there is a shortest path $s \leadsto t$ from $s$ to $t$ that is simple, i.e., it does not repeat any nodes.
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Adding the cycle cannot make the path shorter!
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Simple Observation

• **Observation:** If a graph *does not have any negative cycles*, then there is a shortest path \( s \sim t \) from \( s \) to \( t \) that is simple, i.e., it does not repeat any nodes.

• **Corollary:** The length of any shortest path \( s \sim t \) from \( s \) to \( t \) has at most \( n - 1 \) edges.
Setting up our subproblems
Setting up our subproblems

• Previously:

  Subset Sum: \( \text{OPT}(i, w) \) was the value of the optimal solution on the first
  \( i \) items and weight \( w \).

  Weighted Interval Scheduling: \( \text{OPT}(i) \) was the value of the optimal
  solution on the first \( i \) intervals.
Setting up our subproblems

• Previously:

  Subset Sum: $OPT(i, w)$ was the value of the optimal solution on the first $i$ items and weight $w$.

  Weighted Interval Scheduling: $OPT(i)$ was the value of the optimal solution on the first $i$ intervals.

• We could try something similar for the “first” $i$ nodes.
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  • Could be made to work, but it seems complicated.
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• We could try something similar for the “first” \( i \) nodes.

  • Could be made to work, but it seems complicated.

  • Instead, we will use the *number of edges*, rather than the set of nodes or edges.
Setting up our subproblems
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• Let $\text{OPT}(i,v)$ denote the minimum cost of a path $v\sim t$ from node $v$ to $t$ that uses at most $i$ edges.
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• We could also use $\text{OPT}(i,v)$ to denote the minimum cost of a path $s \sim v$ from $s$ to node $v$ that uses at most $i$ edges.
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• We could also use $\text{OPT}(i,v)$ to denote the minimum cost of a path $s \sim v$ from $s$ to node $v$ that uses at most $i$ edges.

• This looks more like Dijkstra, but the former one is used in KT, because it fits better some of the other applications presented in the book.
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\[
\begin{align*}
\text{OPT}(i,v) & \quad \text{uses } i+1 \text{ edges} \\
& \quad \text{cost } = 7 \\
\text{OPT}(i,v) & \quad \text{uses } i \text{ edges} \\
& \quad \text{cost } = 10 \\
v & \quad \text{uses } i-1 \text{ edges} \\
& \quad \text{cost } = 11 \\
& \quad \text{uses } i \text{ edges} \\
& \quad \text{cost } = 9 \\
s & \quad \text{uses } i \text{ edges} \\
& \quad \text{cost } = 9 \\
& \quad \text{uses } i+1 \text{ edges} \\
& \quad \text{cost } = 7 \\
t & \quad \text{uses } i \text{ edges} \\
& \quad \text{cost } = 10 \\
& \quad \text{uses } i+1 \text{ edges} \\
& \quad \text{cost } = 7
\end{align*}
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Setting up our subproblems

Let \( \text{OPT}(i,v) \) denote the minimum cost of a path \( v \sim t \) from node \( v \) to \( t \) that uses at most \( i \) edges.
Setting up our subproblems

Let $\text{OPT}(i,v)$ denote the minimum cost of a path $v \sim t$ from node $v$ to $t$ that uses at most $i$ edges.

What if there is no path that uses at most $i$ edges?
Setting up our subproblems

Let $\text{OPT}(i,v)$ denote the minimum cost of a path $v \sim t$ from node $v$ to $t$ that uses at most $i$ edges.

What if there is no path that uses at most $i$ edges?

$\text{OPT}(i,v) = \infty$
Setting up our subproblems

• Let $\text{OPT}(i,v)$ denote the minimum cost of a path $v \sim t$ from node $v$ to $t$ that uses at most $i$ edges.
Setting up our subproblems

- Let $OPT(i,v)$ denote the minimum cost of a path $v \sim t$ from node $v$ to $t$ that uses at most $i$ edges.

- What is then the (global) solution to our problem?
Setting up our subproblems

- Let \( \text{OPT}(i,v) \) denote the minimum cost of a path \( v \sim t \) from node \( v \) to \( t \) that uses at most \( i \) edges.

- What is then the (global) solution to our problem?
  - \( \text{OPT}(n-1,s) \)
Simple Observation

• **Observation:** If a graph *does not have any negative cycles*, then there is a shortest path $s \sim t$ from $s$ to $t$ that is simple, i.e., it does not repeat any nodes.

• **Corollary:** The length of any shortest path $s \sim t$ from $s$ to $t$ has at most $n - 1$ edges.
The recurrence relation
The recurrence relation

Let $P$ a minimum-cost path using at most $i$ edges from $v$ to $t$ with cost $\text{OPT}(i,v)$. 
The recurrence relation

Let \( P \) a minimum-cost path using at most \( i \) edges from \( v \) to \( t \) with cost \( \text{OPT}(i,v) \).

Case 1: \( P \) uses at most \( i-1 \) edges. Then \( \text{OPT}(i,v) = \text{OPT}(i-1,v) \).
The recurrence relation

Let $P$ a minimum-cost path using at most $i$ edges from $v$ to $t$ with cost $\text{OPT}(i,v)$.

**Case 1:** $P$ uses at most $i-1$ edges. Then $\text{OPT}(i,v) = \text{OPT}(i-1,v)$.

**Case 2:** $P$ uses exactly $i$ edges. Let $(v, w^*)$ be the first edge of $P$. Then $\text{OPT}(i,v) = c_{vw^*} + \text{OPT}(i-1,w^*)$. 
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We don’t know $w^*$. 
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We don't know $w^*$.

Take $\min_{w \in N(v)} (c_{vw} + \text{OPT}(i-1, w))$.
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Recurrence: $\text{OPT}(i,v) = \min\{ \text{OPT}(i-1,v) , \min_{w \in N(v)} (c_{vw} + \text{OPT}(i-1,w)) \}$
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The Bellman-Ford Algorithm

**ShortestPath** \(( G, s, t )\).

\[
\text{\textbackslash* Let } n = |V| \text{ \textbackslash*}\n\]
Define 2-D Array \( M[0, \ldots, n - 1, s, v_1, v_2, \ldots, t] \)
Initialise \( M[0, t] = 0 \), and \( M[0, v] = \infty \) for all other \( v \in V \).

For \( i = 1, 2, \ldots, n - 1 \)
For \( v \in V \)
\[
M(i, v) = \min\{ M(i-1, v) , \min_{w \in N(v)} (c_{vw} + M(i-1, w))\}
\]
Return \( M(n-1, s) \)
ShortestPath \((G, s, t)\).
\[\text{\# Let } n = |V| \text{ \#}\]
Define 2-D Array \(M[0, \cdots, n - 1, s, v_1, v_2, \ldots, t]\)
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For \(i = 1, 2, \ldots, n - 1\)
For \(v \in V\)
\[M(i, v) = \min\{M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w))\}\]
Return \(M(n-1, s)\)
ShortestPath \((G, s, t)\).

\* Let \(n = |V|\) \* 

Define 2-D Array \(M[0, \cdots, n-1, s, v_1, v_2, \ldots, t]\)

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For \(i = 1, 2, \ldots, n-1\)

For \(v \in V\)

\[M(i,v) = \min\{M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w))\}\]

Return \(M(n-1,s)\)
ShortestPath \(( G, s, t )\).

\[
\text{\"Let } n = \lvert V \rvert \text{\"}
\]

Define 2-D Array \( M[0, \ldots, n - 1, s, v_1, v_2, \ldots, t] \)

Initialise \( M[0, t] = 0 \), and \( M[0, v] = \infty \) for all other \( v \in V \).

For \( i = 1, 2, \ldots, n - 1 \)

For \( v \in V \)

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M(i, v) = \min\{ M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w)) \}
\]

Return \( M(n-1, s) \)
\textbf{Example}

\textbf{ShortestPath}( G, s, t ).

\texttt{/* Let } n = |V| \texttt{ /*
Define 2-D Array } M[0,\ldots, n-1, s, v_1, v_2, \ldots, t] 
Initialise } M[0, t] = 0, \text{ and } M[0, v] = \infty \text{ for all other } v \in V.

For } i = 1, 2, \ldots, n-1 
For } v \in V
\quad M(i, v) = \min \{ M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w)) \} 

Return } M(n-1, s)
Example

\[ M(i,t) = \min \{ M(0,t), \min_{w \in N(t)} (0 + M(0,w)) \} \]

ShortestPath \( (G, s, t) \).
\[
\text{\"Let } n = |V| \text{\"}
\]
Define 2-D Array \( M[0,\ldots,n-1,s,v_1,v_2,\ldots,t] \)
Initialise \( M[0,t] = 0 \), and \( M[0,v] = \infty \) for all other \( v \in V \).

For \( i = 1,2,\ldots,n-1 \)
For \( v \in V \)
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Return \( M(n-1,s) \)
ShortestPath \((G, s, t)\).

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For \(i = 1,2,\ldots, n - 1\)

For \(v \in V\)

\[
M(i, v) = \min\{M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w))\}
\]

Return \(M(n-1, s)\)
**Example**

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

\[ \begin{array}{ccccccc}
\text{t} & 0 & 0 & \text{a} & \infty & \text{b} & \infty & \text{c} & \infty \\
\text{d} & \infty & \text{e} & \infty & \text{f} & \infty \\
\end{array} \]

**ShortestPath** \((G, s, t)\).

\[ \text{Let } n = |V| \]

Define 2-D Array \(M[0,\ldots,n-1, s, v_1, v_2, \ldots, t]\)

Initialise \(M[0,t] = 0\), and \(M[0,v] = \infty\) for all other \(v \in V\).

For \(i = 1,2,\ldots,n-1\)

For \(v \in V\)

\[ M(i,v) = \min\{M(i-1,v) \cdot \min_{w \in N(v)} (c_{vw} + M(i-1,w))\} \]

Return \(M(n-1,s)\)
Example

Let $n = |V|$.

Define 2-D Array $M[0,\ldots,n-1,s,v_1,v_2,\ldots,t]$

Initialise $M[0,t] = 0$, and $M[0,v] = \infty$ for all other $v \in V$.

For $i = 1,2,\ldots,n-1$

For $v \in V$

$$M(i,v) = \min\{M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w))\}$$

Return $M(n-1,s)$

$$M(1,a) = \min\{M(0,a), \min_{w \in N(a)} (c_{aw} + M(0,w))\}$$
**Example**

ShortestPath \((G, s, t)\).

* Let \(n = |V|\) *

Define 2-D Array \(M[0,..., n-1, s, v_1, v_2, ..., t]\)

Initialise \(M[0, t] = 0\), and \(M[0, v] = \infty\) for all other \(v \in V\).

For \(i = 1, 2, ..., n - 1\)

For \(v \in V\)

\[ M(i,v) = \min\{M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w))\} \]

Return \(M(n-1,s)\)

\[
\begin{array}{cccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline 
 t & 0 & 0 & & & & \\
 a & \infty & & & & & \\
 b & \infty & & & & & \\
 c & \infty & & & & & \\
 d & \infty & & & & & \\
 e & \infty & & & & & \\
\end{array}
\]

\[
M(1, a) = \min\{M(0, a), \min_{w \in N(a)} (c_{aw} + M(0, w))\}
\]

\[
M(1, a) = c_{at} + M(0, t)
\]
ShortestPath (G, s, t).

\* Let n = |V| \* \\
Define 2-D Array $M[0, \ldots, n-1, s, v_1, v_2, \ldots, t]$ \\
Initialise $M[0, t] = 0$, and $M[0, v] = \infty$ for all other $v \in V$.

For $i = 1, 2, \ldots, n-1$
\quad For $v \in V$
\quad \quad $M(i, v) = \min\{M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w))\}$

Return $M(n-1, s)$
\textbf{ShortestPath} \((G, s, t)\).

\texttt{\textbackslash * Let }\ n = |V|\ 	exttt{\textbackslash *}

Define 2-D Array \(M[0, \ldots, n - 1, s, v_1, v_2, \ldots, t]\)

Initialise \(M[0, t] = 0\), and \(M[0, v] = \infty\) for all other \(v \in V\).

For \(i = 1, 2, \ldots, n - 1\)

For \(v \in V\)

\[M(i, v) = \min\{M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w))\}\]

Return \(M(n-1, s)\)
ShortestPath \((G, s, t)\).

\[*\]
\[
\text{Let } n = |V| \quad *\]

Define 2-D Array \(M[0,\ldots, n-1, s, v_1, v_2, \ldots, t]\)

Initialise \(M[0,t] = 0\), and \(M[0,v] = \infty\) for all other \(v \in V\).

For \(i = 1,2,\ldots, n-1\)

For \(v \in V\)

\[
M(i,v) = \min \{ M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w)) \}
\]

Return \(M(n-1,s)\)

\[
\begin{array}{ccccccc}
  & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
  t & 0 & 0 &  &  &  &  \\
  a & \infty & -3 &  &  &  &  \\
  b & \infty &  &  &  &  &  \\
  c & \infty &  &  &  &  &  \\
  d & \infty &  &  &  &  &  \\
  e & \infty &  &  &  &  &  \\
\end{array}
\]

\[
M(1,b) = \min \{ M(0,b), \min_{w \in N(b)} (c_{bw} + M(0,w)) \}
\]
Example

ShortestPath \((G, s, t)\).

\* Let \(n = |V|\) \* \\
Define 2-D Array \(M[0, \ldots, n-1, s, v_1, v_2, \ldots, t]\) \\
Initialise \(M[0, t] = 0\), and \(M[0, v] = \infty\) for all other \(v \in V\).

For \(i = 1, 2, \ldots, n-1\) \\
For \(v \in V\) \\
\(M(i, v) = \min\{M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w))\}\)

Return \(M(n-1, s)\)

\[
\begin{array}{ccccccc}
\text{t} & 0 & 1 & 2 & 3 & 4 & 5 \\
\text{a} & \infty & 0 & \cdot & \cdot & \cdot & \cdot \\
\text{b} & \infty & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{c} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{d} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{e} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\[
M(1, b) = \min\{M(0, b), \min_{w \in N(b)} (c_{bw} + M(0, w))\}
\]

\[
M(1, b) = \infty
\]
**Example**

ShortestPath \((G, s, t)\).

```
\* Let \(n = |V|\) \*

Define 2-D Array \(M[0,\ldots,n-1, s, v_1, v_2, \ldots, t]\)
Initialise \(M[0,t] = 0\), and \(M[0,v] = \infty\) for all other \(v \in V\).

For \(i = 1,2,\ldots,n-1\)
For \(v \in V\)

\(M(i,v) = \min\{M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w))\}\)

Return \(M(n-1,s)\)
```

\[
M(1,b) = \min \{ M(0,b) , \min_{w \in N(b)} (c_{bw} + M(0,w)) \} = \infty
\]
ShortestPath \((G, s, t)\).

```
\* Let \(n = |V| \* \\
Define 2-D Array \(M[0, \ldots, n-1, s, v_1, v_2, \ldots, t]\) \\
Initialise \(M[0,t] = 0\), and \(M[0,v] = \infty\) for all other \(v \in V\).
```

For \(i = 1,2,\ldots, n-1\)

For \(v \in V\)

\(M(i,v) = \min\{M(i-1,v), \min\limits_{w \in N(v)} (c_{vw} + M(i-1,w))\}\)

Return \(M(n-1,s)\)
\textbf{Example}

\textbf{ShortestPath} \((G, s, t)\).

\begin{verbatim}
\* Let \( n = |V| \) *\
Define 2-D Array \( M[0,\ldots, n-1, s, v_1, v_2, \ldots, t] \)
Initialise \( M[0,t] = 0 \), and \( M[0,v] = \infty \) for all other \( v \in V \).

For \( i = 1,2,\ldots, n-1 \)
For \( v \in V \)
    \( M(i,v) = \min \{ M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w)) \} \)

Return \( M(n-1,s) \)
\end{verbatim}

\[
M(4,a) = \min \{ M(3,a) , \min_{w \in N(a)} (c_{aw} + M(3,w)) \}
\]
**Example**

\[
\begin{align*}
M(4,a) &= \min \{ M(3,a) , \\
& \quad \min_{w \in N(a)} (c_{aw} + M(3,w)) \}\} \\
\min_{w \in N(a)} (c_{ab} + M(3,b)) &= -6
\end{align*}
\]
Example

\[ \text{ShortestPath}(G, s, t) \]

\[
\text{\texttt{\textbackslash \textbackslash}} \text{ Let } n = |V| \text{ \texttt{\textbackslash \textbackslash}} \\
\text{Define 2-D Array } M[0, \ldots, n-1, s, v_1, v_2, \ldots, t] \\
\text{ Initialise } M[0,t] = 0, \text{ and } M[0,v] = \infty \text{ for all other } v \in V. \\
\]

For \( i = 1, 2, \ldots, n-1 \)

For \( v \in V \)

\[ M(i,v) = \min\{M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w))\} \]

Return \( M(n-1,s) \)

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<tr>
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<td>0</td>
<td>0</td>
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</tbody>
</table>

\[ M(4,a) = \min\{M(3,a), \min_{w \in N(a)} (c_{aw} + M(3,w))\} \]

\[ c_{ab} + M(3,b) = -6 \]

\[ c_{at} + M(3,t) = -3 \]
Example

ShortestPath \(( G, s, t)\).

\* Let \( n = |V| \) \*

*Define 2-D Array \( M[0, \ldots, n - 1, s, v_1, v_2, \ldots, t] \)*

*Initialise \( M[0,t] = 0 \), and \( M[0,v] = \infty \) for all other \( v \in V \).*

*For \( i = 1,2,\ldots, n - 1 \)*

*For \( v \in V \)*

\[ M(i,v) = \min\{ M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w)) \} \]

*Return \( M(n-1,s) \)*
ShortestPath \( (G, s, t) \).

\[ \text{Let } n = |V| \]

Define 2-D Array \( M[0, \ldots, n - 1, s, v_1, v_2, \ldots, t] \)

Initialise \( M[0,t] = 0 \), and \( M[0,v] = \infty \) for all other \( v \in V \).

For \( i = 1,2,\ldots, n - 1 \)
For \( v \in V \)
\[
M(i,v) = \min \{ M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w)) \}
\]

Return \( M(n-1,s) \)
Example

\[ M(4, b) = \min \{ M(3, b), \min_{w \in N(b)} (c_{bw} + M(3, w)) \} \]

ShortestPath \((G, s, t)\).

\* Let \(n = |V|\) \*\n
Define 2-D Array \(M[0, \ldots, n-1, s, v_1, v_2, \ldots, t]\)

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For \(v \in V\)

\[ M(i, v) = \min \{ M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w)) \} \]

Return \(M(n-1, s)\)
Example

\[ M(4, b) = \min \{ M(3, b), \min_{w \in N(b)} (c_{bw} + M(3, w)) \} \]

\[ c_{bd} + M(3, d) = -1 + 3 \]
ShortestPath \((G, s, t)\).

\[
\text{\texttt{\textbackslash a Let } n = |V| \texttt{ \textbackslash a}} \\
\text{Define 2-D Array } M[0,\ldots,n-1, s, v_1, v_2, \ldots, t] \\
\text{Initialise } M[0,t] = 0, \text{ and } M[0,v] = \infty \text{ for all other } v \in V.
\]

For \(i = 1,2,\ldots,n-1\)

For \(v \in V\)

\[M(i,v) = \min\{M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w))\}\]

Return \(M(n-1,s)\)

\[
\begin{array}{c|c|c|c|c|c|c|c}
\hline
& 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\hline
t & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
a & \infty & -3 & -3 & -4 & -6 & \infty \\
\hline
b & \infty & \infty & 0 & -2 & \infty & \infty \\
\hline
c & \infty & 3 & 3 & 3 & \infty & \infty \\
\hline
d & \infty & 4 & 3 & 3 & \infty & \infty \\
\hline
e & \infty & 2 & 0 & 0 & \infty & \infty \\
\hline
\end{array}
\]

\[
M(4,b) = \min\{M(3,b), \min_{w \in N(b)} (c_{bw} + M(3,w))\}
\]

\[
c_{bd} + M(3,d) = -1 + 3
\]

\[
c_{be} + M(3,e) = -2 + 0
\]
**ShortestPath** \((G, s, t)\).

\* Let \(n = |V|\) \* 

Define 2-D Array \(M[0,\ldots, n-1, s, v_1, v_2, \ldots, t]\)

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For \(i = 1, 2, \ldots, n-1\)

For \(v \in V\)

\[M(i, v) = \min\{M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w))\}\]

Return \(M(n-1, s)\)
**Example**

**ShortestPath** \((G, s, t)\).

```plaintext
\* Let \(n = |V|\) \*

Define 2-D Array \(M[0, \ldots, n-1, s, v_1, v_2, \ldots, t]\)

Initialise \(M[0, t] = 0\), and \(M[0, v] = \infty\) for all other \(v \in V\).

For \(i = 1, 2, \ldots, n - 1\)

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\[ M(i, v) = \min\{M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w))\} \]

Return \(M(n-1, s)\)
```

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</tbody>
</table>
Example

ShortestPath \(( G, s, t )\).

```
\* Let \( n = |V| \) \*

Define 2-D Array \( M[0, \ldots, n-1, s, v_1, v_2, \ldots, t] \)
Initialise \( M[0,t] = 0, \) and \( M[0,v] = \infty \) for all other \( v \in V. \)

For \( i = 1, 2, \ldots, n-1 \)
For \( v \in V \)
\[
M(i,v) = \min\{ M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w)) \} 
\]

Return \( M(n-1,s) \)
```

We can find the actual paths via tracing backwards:
**Example**

\[
\begin{array}{ccccccc}
  t & 0 & 0 & 0 & 0 & 0 & 0 \\
  a & \infty & -3 & -3 & -4 & -6 & -6 \\
  b & \infty & \infty & 0 & -2 & -2 & -2 \\
  c & \infty & 3 & 3 & 3 & 3 & 3 \\
  d & \infty & 4 & 3 & 3 & 2 & 0 \\
  e & \infty & 2 & 0 & 0 & 0 & 0 \\
\end{array}
\]

We can find the actual paths via tracing backwards:

\[
M(5,d) = \min \{ M(4,d) , \min_{w \in N(d)} (c_{d,w} + M(4,w)) \}
\]

**ShortestPath** \((G,s,t)\).

\[
\text{\textbackslash \textbackslash \ Let } n = |V| \text{ \textbackslash \textbackslash} \\
\text{Define 2-D Array } M[0,\cdots,n-1,s,v_1,v_2,\ldots,t] \\
\text{Initialise } M[0,t] = 0, \text{ and } M[0,v] = \infty \text{ for all other } v \in V. \\
\]

For \(i = 1,2,\ldots,n-1\)

\[
\text{For } v \in V \\
M(i,v) = \min \{ M(i-1,v) , \min_{w \in N(v)} (c_{v,w} + M(i-1,w)) \}
\]

Return \(M(n-1,s)\)
**Example**

```
\text{ShortestPath} (G, s, t).

\text{\textbackslash{}* Let } n = |V| \textbackslash{}*
\text{Define 2-D Array } M[0, \ldots, n-1, s, v_1, v_2, \ldots, t]
\text{Initialise } M[0,t] = 0, \text{ and } M[0,v] = \infty \text{ for all other } v \in V.

\text{For } i = 1, 2, \ldots, n-1
\text{For } v \in V
\quad M(i,v) = \min\{M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w))\}
\text{Return } M(n-1,s)
```

We can find the actual paths via tracing backwards:

\[ M(5,d) = \min \{ M(4,d) , \]
\[ \min_{w \in N(d)} (c_{dw} + M(4,w)) \} \]

So the first edge must be \((d, a)\).
Example

ShortestPath \((G, s, t)\).

\[
\text{\texttt{\textbackslash{}x2211} Let } n = |V| \text{ \texttt{\textbackslash{}x2211} }
\]
Define 2-D Array \(M[0, \ldots, n - 1, s, v_1, v_2, \ldots, t]\)
Initialise \(M[0, t] = 0\), and \(M[0, v] = \infty\) for all other \(v \in V\).

For \(i = 1, 2, \ldots, n - 1\)
For \(v \in V\)
\[
M(i, v) = \min\{M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w))\}
\]
Return \(M(n-1, s)\)

We can find the actual paths via tracing backwards:

\[
M(5, d) = \min\{M(4, d), \min_{w \in N(d)} (c_{dw} + M(4, w))\}
\]

So the first edge must be \((d, a)\).
Next we consider \(M(4, a)\), etc.
Running Time

**ShortestPath** \((G, s, t)\).

\* Let \(n = |V|\) *

Define 2-D Array \(M[0, \ldots, n - 1, s, v_1, v_2, \ldots, t]\)
Initialise \(M[0, t] = 0\), and \(M[0, v] = \infty\) for all other \(v \in V\).

For \(i = 1, 2, \ldots, n - 1\)

For \(v \in V\)

\[M(i, v) = \min\{M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w))\}\]

Return \(M(n-1, s)\)
Running Time

\textbf{ShortestPath} \(( G, s, t ) \).

\[ \text{\texttt{\textbackslash* Let } n = |V| \texttt{\textbackslash*}} \]

Define 2-D Array \( M[0,\ldots,n-1, s, v_1, v_2, \ldots, t] \)

Initialise \( M[0,t] = 0 \), and \( M[0,v] = \infty \) for all other \( v \in V \).

\[ O(1) \]

For \( i = 1, 2, \ldots, n - 1 \)

For \( v \in V \)

\[ M(i,v) = \min \{ M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w)) \} \]

Return \( M(n-1,s) \)
Running Time

**ShortestPath** \((G, s, t)\).

\* Let \(n = |V| \)*\n
Define 2-D Array \(M[0,\ldots,n-1, s, v_1, v_2, \ldots, t]\)

Initialise \(M[0,t] = 0\), and \(M[0,v] = \infty\) for all other \(v \in V\).

\(\mathcal{O}(1)\) \(\mathcal{O}(n)\)

For \(i = 1,2,\ldots,n-1\)

For \(v \in V\)

\(M(i,v) = \min\{M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w))\}\)

Return \(M(n-1,s)\)
Running Time

\textbf{ShortestPath}\ (G, s, t).

\texttt{\/* Let }n = \mid V\mid *\texttt{*

Define 2-D Array }\textbf{M}[0, \cdots, n - 1, s, v_1, v_2, \cdots, t]\texttt{.

Initialise }\textbf{M}[0, t] = 0,\text{ and }\textbf{M}[0, v] = \infty\text{ for all other }v \in V.\texttt{\}

\texttt{O(1)}\texttt{ \quad O(n)}

\texttt{n - 1}\texttt{For }i = 1, 2, \ldots, n - 1

\texttt{For }v \in V

\textbf{M}(i, v) = \min\{\textbf{M}(i - 1, v), \ \min_{w \in N(v)} (c_{vw} + \textbf{M}(i - 1, w))\}\texttt{\}

\texttt{Return }\textbf{M}(n - 1, s)\texttt{\}}
Running Time

\textbf{ShortestPath} \((G, s, t)\).

\* Let \(n = |V| \*\)

Define 2-D Array \(M[0, \ldots, n - 1, s, v_1, v_2, \ldots, t]\)

Initialise \(M[0, t] = 0\), and \(M[0, v] = \infty\) for all other \(v \in V\).

\(O(1)\) \hspace{2cm} \(O(n)\)

For \(i = 1, 2, \ldots, n - 1\)

\(n\) \hspace{1cm} For \(v \in V\)

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\(M(i, v) = \min\{M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w))\}\)

Return \(M(n-1, s)\)
Running Time

**ShortestPath** \((G, s, t)\).

\[\text{\textbackslash* Let } n = |V| \text{ \textbackslash*}\]
Define 2-D Array \(M[0, \ldots, n - 1, s, v_1, v_2, \ldots, t]\)
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\(O(1)\) \hspace{2cm} \(O(n)\)

\(n - 1\) For \(i = 1, 2, \ldots, n - 1\)
\(n\) For \(v \in V\)

\[M(i,v) = \min\{M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w))\} O(n)\]

Return \(M(n-1, s)\)
Running Time

\textbf{ShortestPath} \((G, s, t)\).

\* Let \(n = |V|\) \*\n
Define 2-D Array \(M[0, \ldots, n-1, s, v_1, v_2, \ldots, t]\)

Initialise \(M[0, t] = 0\), and \(M[0, v] = \infty\) for all other \(v \in V\).

\(O(1)\) \hspace{2cm} \(O(n)\)

For \(i = 1, 2, \ldots, n-1\)

\(O(n)\)

For \(v \in V\)

\(M(i, v) = \min\{M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w))\}\)

\(O(n)\)

Return \(M(n-1, s)\)

\(O(n)\)

Overall: \(O(n^3)\)
Improved Analysis

\[ M(i,v) = \min \{ M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w)) \} \]
Improved Analysis

$M(i,v) = \min\{M(i-1,v), \min_{w\in N(v)} (c_{vw} + M(i-1,w))\}$

Suffices to only check $w$ such that $(v, w) \in E.$
Improved Analysis
Improved Analysis

Recall that $N(v)$ be the set of nodes $w$ for which there is an edge $(v, w)$, and let $n_v = |N(v)|$ be their number.
Improved Analysis

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$$M(i, v) = \min \{ M(i-1, v), \min_{w \in N(v)} (c_{vw} + M(i-1, w)) \} \quad O(n_v)$$
Improved Analysis

Recall that $N(v)$ be the set of nodes $w$ for which there is an edge $(v, w)$, and let $n_v = |N(v)|$ be their number.

$$M(i, v) = \min \{ M(i-1, v) , \min_{w \in N(v)} (c_{vw} + M(i-1, w)) \} \quad O(n_v)$$

We have to compute an entry for every $v \in V$ and every index $i \in [0, n - 1]$, so in total we need time $O\left(n \sum_{v \in V} n_v\right)$.
Improved Analysis

How large is \( \sum_{v \in V} n_v \)?
Improved Analysis

How large is $\sum_{v \in V} n_v$?

Each node $v$ contributes exactly as many terms as the number of its outgoing edges $(v, w)$. 
Improved Analysis

How large is $\sum_{v \in V} n_v$?

Each node $v$ contributes exactly as many terms as the number of its outgoing edges $(v, w)$.

$\sum_{v \in V} n_v = m$
Improved Analysis

Let $N(v)$ be the set of nodes $w$ for which there is an edge $(v, w)$, and let $n_v = |N(v)|$ be their number.

$$M(i,v) = \min\{ M(i-1,v) , \min_{w \in N(v)} (c_{vw} + M(i-1,w)) \} \quad O(n_v)$$

We have to compute an entry for every $v \in V$ and every index $i \in [0,n-1]$, so in total we need time $O\left( n \sum_{v \in V} n_v \right)$
Improved Analysis

Let $N(v)$ be the set of nodes $w$ for which there is an edge $(v, w)$, and let $n_v = |N(v)|$ be their number.

$$M(i,v) = \min \{ M(i-1,v), \min_{w \in N(v)} (c_{vw} + M(i-1,w)) \} \quad O(n_v)$$

We have to compute an entry for every $v \in V$ and every index $i \in [0, n - 1]$, so in total we need time $O\left( n \sum_{v \in V} n_v \right)$.

Overall: $O(nm)$
Improved Memory Implementation
Improved Memory Implementation

- We need to store the 2D array $M$ in memory, which takes space $\Theta(n^2)$.
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Memory usage is a common problem with many dynamic programming algorithms.
Improved Memory Implementation

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- Memory usage is a common problem with many dynamic programming algorithms.

- Here, we can instead use an 1D array $M'$ of space $O(n)$. 
Improved Memory Implementation

- We need to store the 2D array $M$ in memory, which takes space $\Theta(n^2)$.

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- Here, we can instead use an 1D array $M'$ of space $O(n)$.

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M[v] = \min\{M[v], \min_{w \in N(v)} (c_{vw} + M[w])\}
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- That works, however more work is needed to recover the shortest paths!
Shortest Paths in Graphs

• **Input:** A directed graph \( G = (V, E) \), and designated nodes \( s, t \) in \( V \). We also assume that every node \( u \) in \( V \) is reachable from \( s \), and *that the graph does not have any negative cycles*. We are also given a cost \( c_e \in \mathbb{R} \) for every edge \( e \) in \( E \).

• **Output:** A shortest path \( s \sim t \) from \( s \) to \( t \). In other words, a path \( P \) that minimises \( \sum_{(u,v)\in P} c_{uv} \).
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*What if the graph has negative cycles? Can we at least detect that?*
Detecting Negative Cycles

- Can be done in time $O(mn)$.
- It is in fact usually included as a part of the Bellman-Ford algorithm.
- We will not cover this here, see KT 6.10 for the details if you are interested.
Reading and References

- Kleinberg and Tardos 6.8.
- CLRS 22.1.
- Roughgarden 18.1, 18.2.
- The Bellman-Ford visualiser:
  https://algorithms.discrete.ma.tum.de/graph-algorithms/spp-bellman-ford/index_en.html