A type system is a tractable syntactic method for proving the absence of certain program behaviors by classifying phrases according to the kinds of values they compute.

*Benjamin Pierce* Types and Programming Languages
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That's what we call Types
ChocoPy Type System

- ChocoPy is a *statically typed* language
  - We verify that programs are well-typed by analyzing the program’s syntax without executing it

- ChocoPy is a *strongly typed* language
  - Programs with type errors are rejected and there are no implicit type conversions

- ChocoPy has *subtyping*
  - Types form a hierarchy and a value of a subtype can safely be used in a context where a value of the supertype is expected
Types of ChocoPy

- The grammar of ChocoPy contains the syntax of the following types:
  - `int` - representing integer values
  - `bool` - representing the two values `True` and `False`
  - `str` - representing strings
  - `object` - the *top type*, i.e., every value has this type
  - `[T]` - representing a list with elements of type `T`, where `T` is itself a type

- ChocoPy defines three more types that cannot be written by the user:
  - `<Empty>` - representing an empty list
  - `<None>` - representing the value `None`
  - `⊥` - the *bottom type*, i.e., the type that has no value

```plaintext
... type := type_name | `[` type `]`
type_name := int | bool | str | object
...
```
Type Hierarchy of ChocoPy

- The types form a hierarchy:

```
object
│   └── int ─ bool ─ str ─ [int] ─ … ─ [[bool]] ─ <Empty> ─ <None>
           └── ⊥
```

- The type hierarchy is precisely defining by a subtyping relationship (≤), where:
  - \( T \leq T \) for all types \( T \)
  - \( T \leq \text{object} \) for all types \( T \)
  - \( \bot \leq T \) for all types \( T \)
  - If none of the three cases above apply, then the types are not related by subtyping, for example: \([\text{int}]\) and \([\text{bool}]\) are not related by subtyping
Type Checking

- The type checking process verifies and enforces the type system

- The type system is defined by a set of formal typing rules that describe under what conditions a syntactic construct is well-typed (“has a valid type”)

- To perform type checking, we process a syntactically well-formed program and apply the typing rules to check if we can justify that every definition, statement, and expression is well-typed
Typing rules are inference rules

- A typing rule is a form of a logical inference rule

- We write a typing rule like this:

- Each typing rule contains:
  - a \([\text{name}]\),
  - zero, one, or multiple premises above the line,
  - a conclusion below the line.

- The rule states, that if the premises are true, then the conclusion is true as well.
  In other words: to check the conclusion, we must check that all premises are true
Typing Judgement

- \( 0 \vdash e : T \) is a **typing judgement**, where the turnstile symbol (\( \vdash \)) separates the **typing environment** on the left from the proposition on the right.

- This judgement should be read as:

  "In the type environment 0 the expression e is well typed and has type T"

- Why do we need a typing environment?
- Why can’t we just say: “The expression e is well typed and has type T”? 
Typing Environment

- Consider: \( \vdash x + 4 : \text{int} \) Is this a valid typing judgement?

- We cannot say without knowing the type of \( x \! \):
  - If \( x \) has type \( \text{int} \), then the judgement seems valid
  - If \( x \) has not the type \( \text{int} \), then the judgement seems wrong

- The *typing environment* records the type of all variables and functions that are in scope when type checking a definition, statement, or expression

- Given a typing environment, we can always conclude if a typing judgement is valid:
  \( \{x: \text{int}\} \vdash x + 4 : \text{int} \)
Typing Environment of ChocoPy

- For type checking ChocoPy, we use a local environment $O$ that contains:
  - The types of all variables in scope
    We write $O(v) = T$ to indicate that variable $v$ is in the local environment and has type $T$.
  - Information about all functions in scope
    We write $O(f) = \{T_1 \times \ldots \times T_n \rightarrow T_0; x_1, \ldots, x_n; v_1: T'_1, \ldots, v_m: T'_m\}$ to indicate that function $f$ is in the local environment and
      - has a function type with
        - $T_1, \ldots, T_n$ the types of the function parameters
        - $T_0$ the function return type
      - has function parameters with names $x_1, \ldots, x_n$
      - has identifiers and types $v_1: T'_1, \ldots, v_m: T'_m$ of variables declared in the body of $f$
  - We also record the return type $R$ of the current function in the environment.
First ChocoPy Typing Rules

\[ 0, R \vdash \text{False} : \text{bool} \]
First ChocoPy Typing Rules

\[
\frac{}{O, R \vdash \text{False} : \text{bool}}
\]  

“\text{There is no premise that must be true, so we can directly conclude that in the type environment } O \text{ and } R \text{ the expression } \text{False} \text{ is well typed and has type } \text{bool}”
First ChocoPy Typing Rules

16

--------------------- [BOOL-TRUE]
O, R ⊢ True : bool

“There is no premise that must be true, so we can directly conclude that in the type environment O and R the expression False is well typed and has type bool”

--------------------- [BOOL-FALSE]
O, R ⊢ False : bool

--------------------- [BOOL-TRUE]
O, R ⊢ True : bool
First ChocoPy Typing Rules

\[\begin{align*}
&\text{--------------------- } [\text{BOOL-FALSE}] \\
&\text{0, R }\vdash \text{False : bool}
\end{align*}\]

“There is no premise that must be true, so we can directly conclude that in the type environment 0 and R the expression \text{False} is well typed and has type \text{bool}”

\[\begin{align*}
&\text{--------------------- } [\text{BOOL-TRUE}] \\
&\text{0, R }\vdash \text{True : bool}
\end{align*}\]

“There is no premise that must be true, so we can directly conclude that in the type environment 0 and R the expression \text{True} is well typed and has type \text{bool}”
First ChocoPy Typing Rules

O, R ⊢ e₁ : bool
O, R ⊢ e₂ : bool
-------------------------- [AND]
O, R ⊢ e₁ and e₂ : bool

“If e₁ has type bool in the type environment 0 and R, and if e₂ has type bool in the same type environment 0 and R, then we can conclude that in the same type environment 0 and R the expression e₁ and e₂ is well typed and has type bool”
Example of Type Checking

\[
\begin{align*}
&\text{Example of Type Checking} \\
&\text{--------------------- [BOOL-FALSE]} \\
&O, R \vdash \text{False} : \text{bool} \\
&\text{--------------------- [BOOL-TRUE]} \\
&O, R \vdash \text{True} : \text{bool} \\
&\text{----------------------------------------------------[?]}
&O, R \vdash \text{False and (True and False)} : \text{bool} \\
&\text{------------------------------------- [AND]} \\
&O, R \vdash e_1 : \text{bool} \\
&O, R \vdash e_2 : \text{bool} \\
&O, R \vdash e_1 \text{ and } e_2 : \text{bool}
\end{align*}
\]
Example of Type Checking

0, R ⊢ False : bool
0, R ⊢ True and False : bool
0, R ⊢ False and (True and False) : bool
Example of Type Checking

\[
\begin{align*}
0, R &\vdash False : bool \\
O, R &\vdash True : bool \\
O, R &\vdash e_1 : bool \\
O, R &\vdash e_2 : bool \\
&\vdash \text{AND} \\
O, R &\vdash e_1 \text{ and } e_2 : bool
\end{align*}
\]
Example of Type Checking

O, R ⊢ False : bool

O, R ⊢ True and False : bool

O, R ⊢ False and (True and False) : bool
Example of Type Checking

\[
\begin{align*}
0, R & \vdash \text{False} : \text{bool} \quad \text{[BOOL-FALSE]}\\
0, R & \vdash \text{True} : \text{bool} \quad \text{[BOOL-TRUE]}\\
0, R & \vdash e_1 : \text{bool}\\
0, R & \vdash e_2 : \text{bool}\\
0, R & \vdash \text{False and (True and False)} : \text{bool} \quad \text{[AND]}
\end{align*}
\]

\[
\begin{align*}
0, R & \vdash \text{False} : \text{bool} \\
0, R & \vdash \text{True} : \text{bool} \\
0, R & \vdash e_1 : \text{bool}\\
0, R & \vdash e_2 : \text{bool}\\
0, R & \vdash e_1 \text{ and } e_2 : \text{bool} \quad \text{[AND]}
\end{align*}
\]
Example of Type Checking

---

\[
O, R \vdash \text{False} : \text{bool}
\]

---

\[
O, R \vdash \text{True} : \text{bool}
\]

---

\[
O, R \vdash e_1 : \text{bool}
\]

\[
O, R \vdash e_2 : \text{bool}
\]

---

\[
O, R \vdash e_1 \text{ and } e_2 : \text{bool}
\]

---

\[
O, R \vdash \text{False} : \text{bool}
\]

---

\[
O, R \vdash \text{True} : \text{bool}
\]

---

\[
O, R \vdash \text{False} : \text{bool}
\]

---

\[
O, R \vdash \text{False and (True and False)} : \text{bool}
\]
Example of Type Checking

--------------------- [BOOL-FALSE]
0, R ⊢ False : bool

--------------------- [BOOL-TRUE]
0, R ⊢ True : bool

O, R ⊢ e₁ : bool
O, R ⊢ e₂ : bool

-------------------------- [AND]
O, R ⊢ e₁ and e₂ : bool

-------------------- [BOOL-FALSE]
O, R ⊢ False : bool

O, R ⊢ True : bool

O, R ⊢ False : bool

-------------------------- [AND]
O, R ⊢ True and False : bool

----------------------------------- [AND]
O, R ⊢ False and (True and False) : bool

----------------------------------- [AND]
O, R ⊢ e₁ and e₂ : bool
Example of Type Checking

\[
\begin{align*}
0, R ⊢ False : bool & \quad \text{[BOOL-FALSE]} \\
\end{align*}
\]

\[
\begin{align*}
0, R ⊢ True : bool & \quad \text{[BOOL-TRUE]} \\
\end{align*}
\]

\[
\begin{align*}
0, R ⊢ e₁ : bool \\
0, R ⊢ e₂ : bool \\
\end{align*}
\]

\[
\begin{align*}
0, R ⊢ e₁ \text{ and } e₂ : bool & \quad \text{[AND]} \\
\end{align*}
\]

\[
\begin{align*}
0, R ⊢ False : bool \\
0, R ⊢ True : bool \\
\end{align*}
\]

\[
\begin{align*}
0, R ⊢ False : bool \\
\end{align*}
\]

\[
\begin{align*}
0, R ⊢ True \text{ and } False : bool \\
\end{align*}
\]

\[
\begin{align*}
0, R ⊢ False \text{ and } (True \text{ and } False) : bool \\
\end{align*}
\]
Example of Type Checking

\[
\begin{align*}
&\text{[BOOL-\text{FALSE}]} \\
&0, R \vdash \text{False} : \text{bool} \\
&\text{[BOOL-\text{TRUE}]} \\
&0, R \vdash \text{True} : \text{bool} \\
&\text{[\?]} \\
&0, R \vdash \text{False} : \text{bool} \\
&\text{[AND]} \\
&0, R \vdash \text{True and False} : \text{bool} \\
&\text{[AND]} \\
&0, R \vdash \text{False and (True and False)} : \text{bool}
\end{align*}
\]
Example of Type Checking

\[
\begin{align*}
0, R & \vdash False : bool \\
0, R & \vdash True : bool \\
0, R & \vdash e_1 : bool \\
0, R & \vdash e_2 : bool \\
0, R & \vdash e_1 \text{ and } e_2 : bool
\end{align*}
\]