Introduction to Algorithms and Data Structures

Introduction to NP-completeness

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- We argued about the correctness of ALG^A (sometimes).
- We argued about its running time.

Running time hierarchy

$O(\log n)$	$O(n)$	$O(n \log n)$	$O(n^2)$	$O(n^\alpha)$	$O(c^n)$
logarithmic	linear		quadratic	polynomial	exponential
The algorithm does not even read the whole input.	The algorithm accesses the input only a constant number of times.	The algorithm splits the inputs into two pieces of similar size, solves each part and merges the solutions.	The algorithm considers pairs of elements.	The algorithm performs many nested loops.	The algorithm considers many subsets of the input elements.

constant	$O(1)$	superlinear	$\omega(n)$
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- If we were not interested in efficiency, we could solve all of these problems in exponential time $O(c^n)$ using brute force.
- If its possible to design an efficient algorithm for a problem, we shouldn’t be satisfied with brute force.

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Efficient algorithms

- Is it possible to design a polynomial-time algorithm for *every* problem?
- Are there problems for which polynomial-time algorithms do not exist?

Reductions

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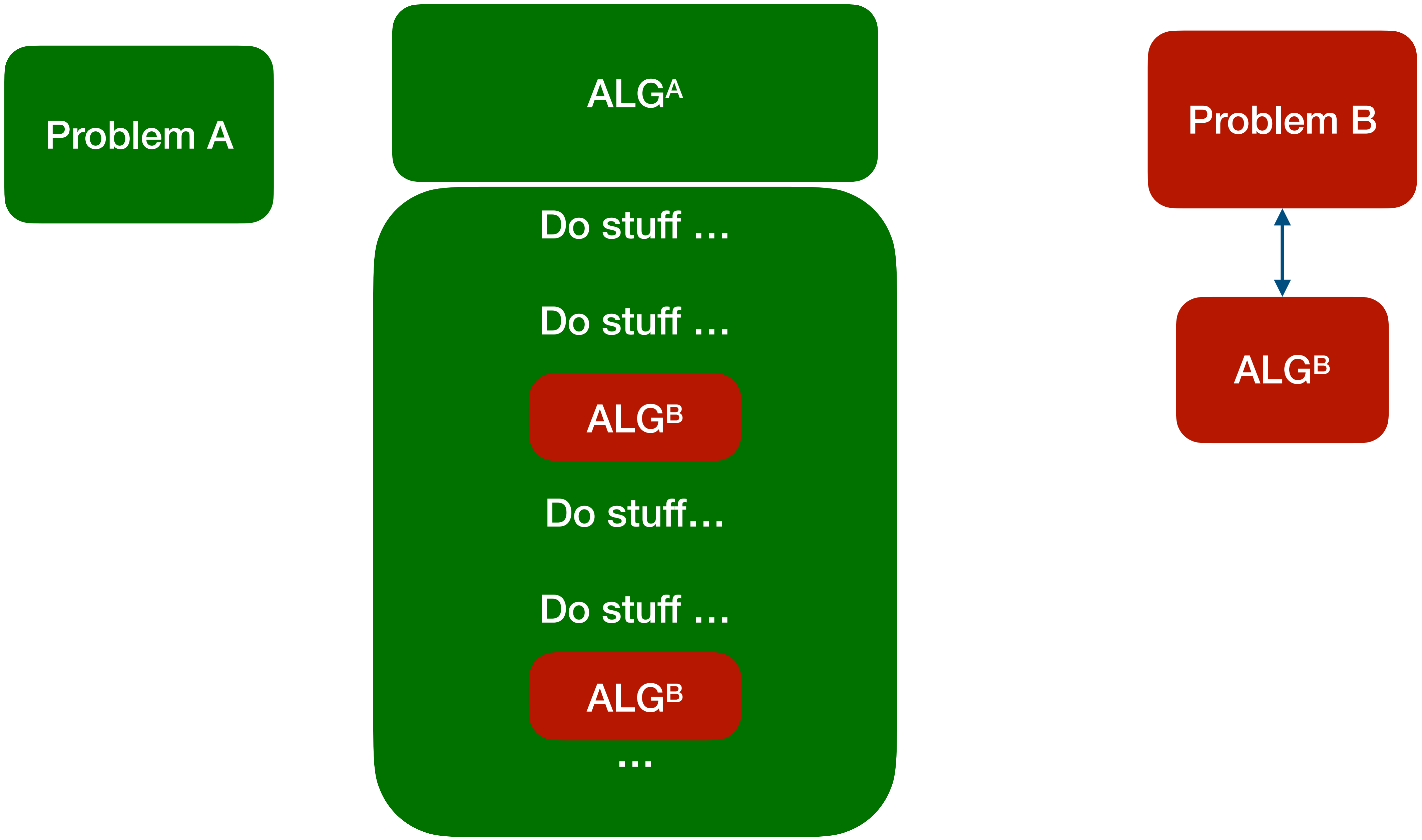
- We can construct an algorithm ALG^A for solving problem A, which uses calls to the algorithm ALG^B as a subroutine.
- If ALG^A is a polynomial time algorithm, then this is a *polynomial time reduction*.

Pictorially



Notation

- When problem A reduces to problem B in polynomial time, we write

  $A \leq^p B$

  We often say "there is a polynomial time reduction *from* A *to* B".

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Types of reductions

- **Turing reduction:**
  - Notation: $A \leq_T B$
  - A reduction which solves problem A using (polynomially) many calls to an oracle (an algorithm) for solving problem B.
  - (Also known as Cook reduction).

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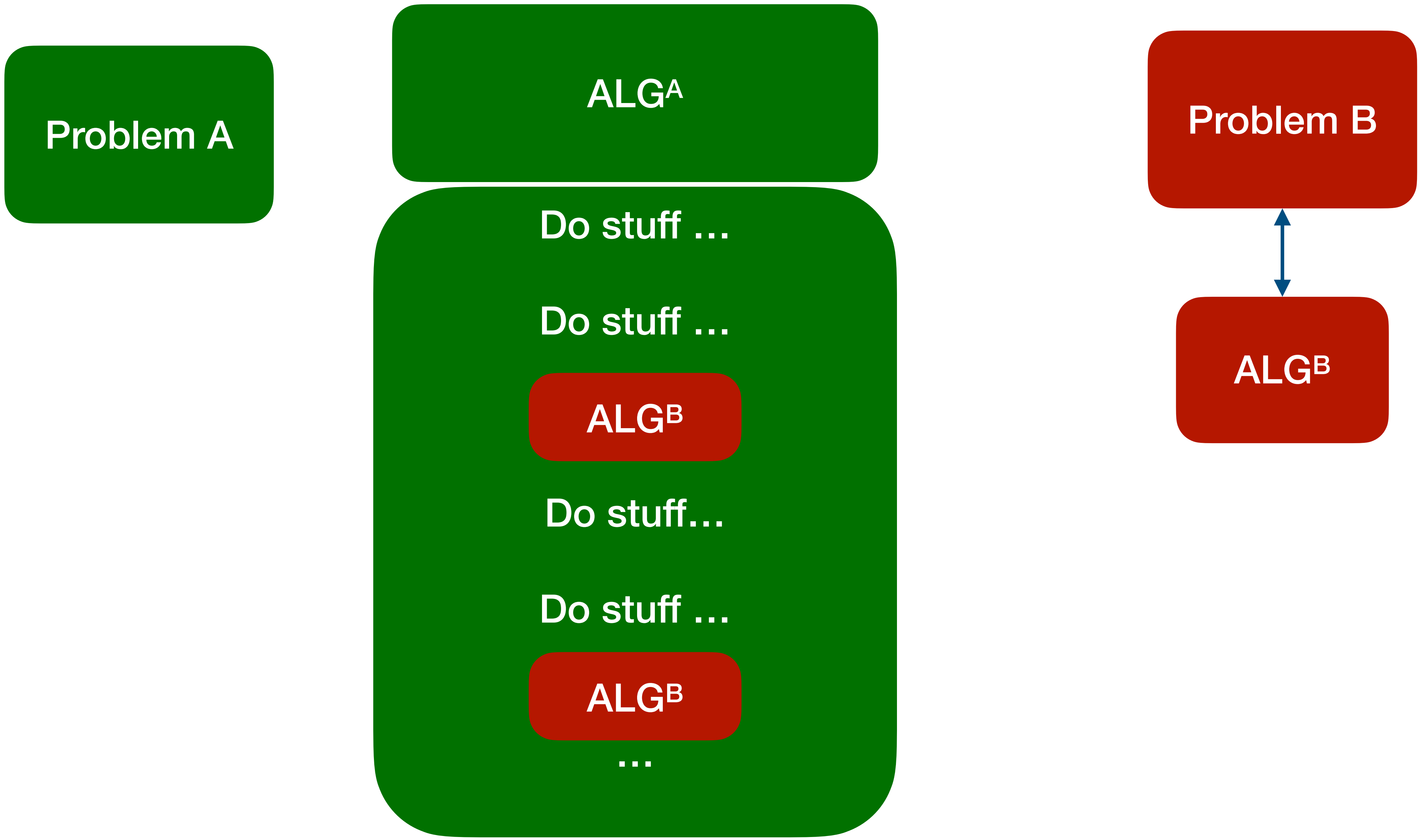

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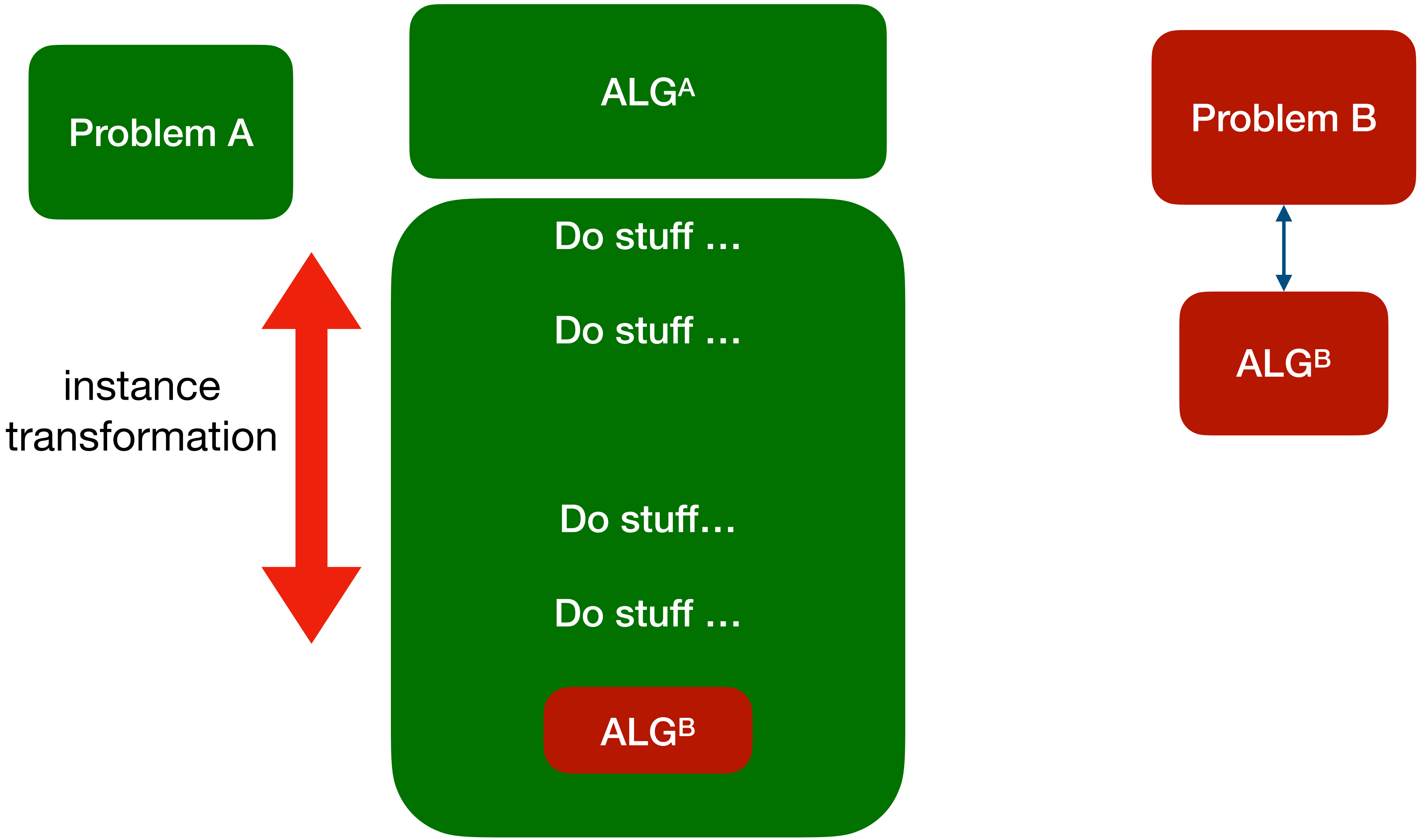
- **Many-one reduction:**
  - Notation: $A \leq_m B$
  - A reduction which *converts instances* of problem A to *instances* of problem B.
  - (Also known as Karp reduction).

Pictorially



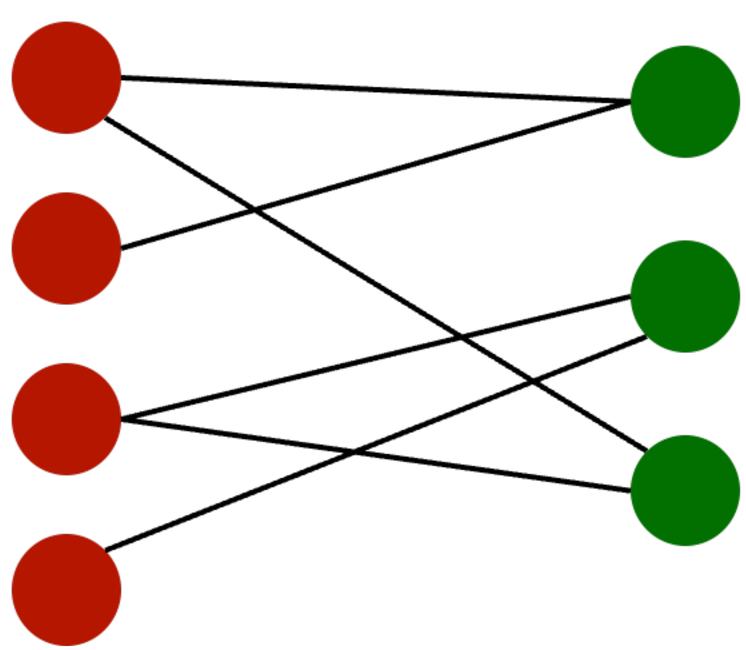
Types of reductions

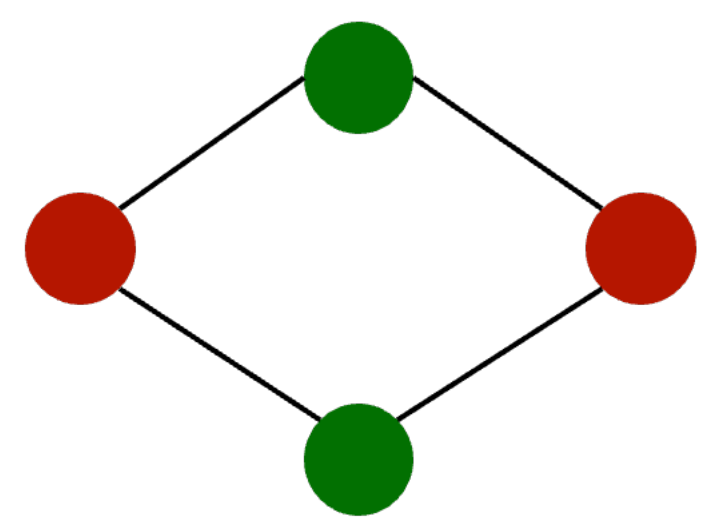
- **Turing reduction:**
  - Argument: Here is an algorithm which runs in polynomial time solving problem A, using polynomially many calls to an oracle for problem B.
- **Many-one reduction:**
  - Argument:
    - If z is a solution to instance I of problem A, then z' is a solution of instance f(I) to problem B.
    - If z is not a solution to instance I of problem A, then z' is not a solution of instance f(I) to problem B.
    - Equivalently: If z' is a solution of instance f(I) to problem B, then z is a solution to instance I of problem A.

Examples of reductions?

Bipartite graphs

- A graph G=(V,E) is bipartite *if any only if* it can be partitioned into sets A and B such that each edge has one endpoint in A and one endpoint in B.
  - Often, we write G=(A U B,E).





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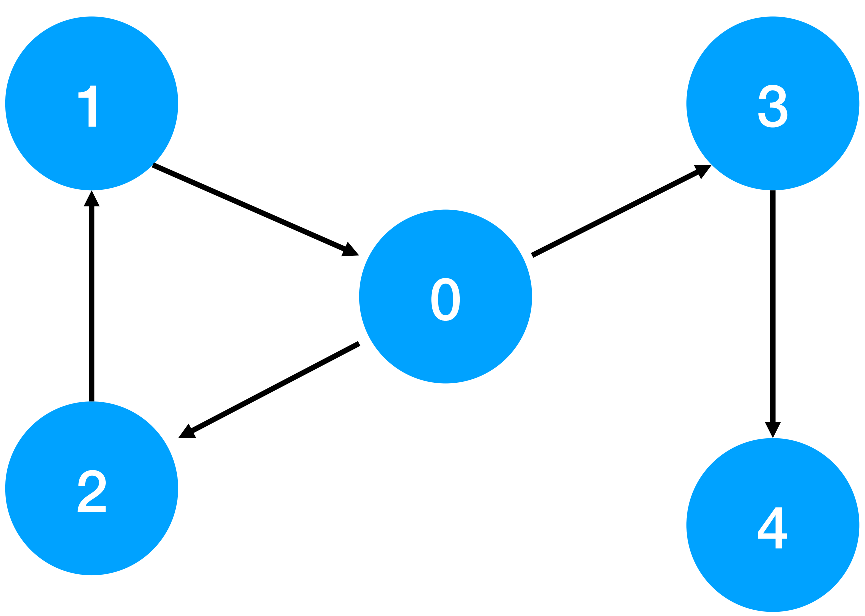
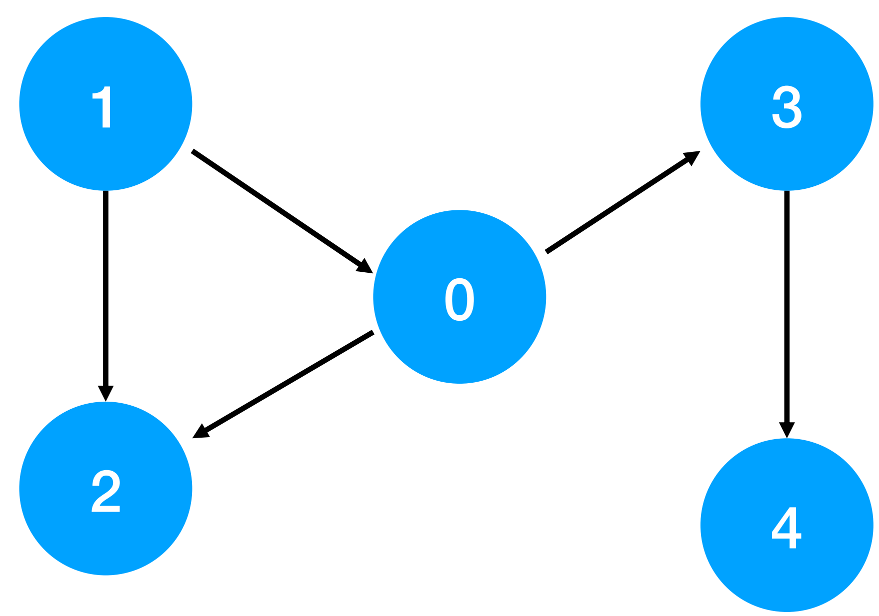
    - We reduced it to checking whether BFS colours two adjacent nodes with the same colour.

Directed Acyclic Graphs

- A directed acyclic graph (DAG) G is a graph that does not have any cycles.



not a DAG



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  - And how can we solve that?
  - We can develop an algorithm that finds a topological ordering, or returns that there is none.

Computational classes

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- Every problem for which there is a known polynomial time algorithm is in the computational class P.
  - Searching, sorting, interval scheduling, graph traversal, …
  - The class P contains computational problems *that can be solved in polynomial time*.
    - We also say that they can be solved *efficiently*.

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  - Knapsack?

The landscape of complexity

contains all problems that
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- More intuitive definition:
  - Problems such that, *if a solution is given*, it can be *checked* that it is indeed a solution in polynomial time.
  - *Efficiently verifiable*.

The subset sum problem

- We are given a set of n items {*1*, *2*, … , *n*}.
- Each item *i* has a non-negative integer weight w_i.
- We are given an integer bound W.
- Goal: Select a subset S of the items such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} w_i$ is maximised.

Equivalent formulation decision version

- We are given a set of n items {*1*, *2*, … , *n*}.
- Each item *i* has a non-negative integer weight w_i.
- We are given an integer bound W.
- Goal: Decide if there exists a subset S of the items such that

$$\sum_{i \in S} w_i = W$$

Subset Sum is in NP

- If we are given a candidate solution S, we can easily check whether the following holds or not:

$$\sum_{i \in S} w_i = W$$

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    - Searching, sorting, minimum spanning tree, graph traversal, Weighted Interval Scheduling, …
- Problems in NP:
    - Subset Sum, Knapsack, Weighted Interval Scheduling, searching, sorting, graph traversal, Weighted Interval Scheduling, …

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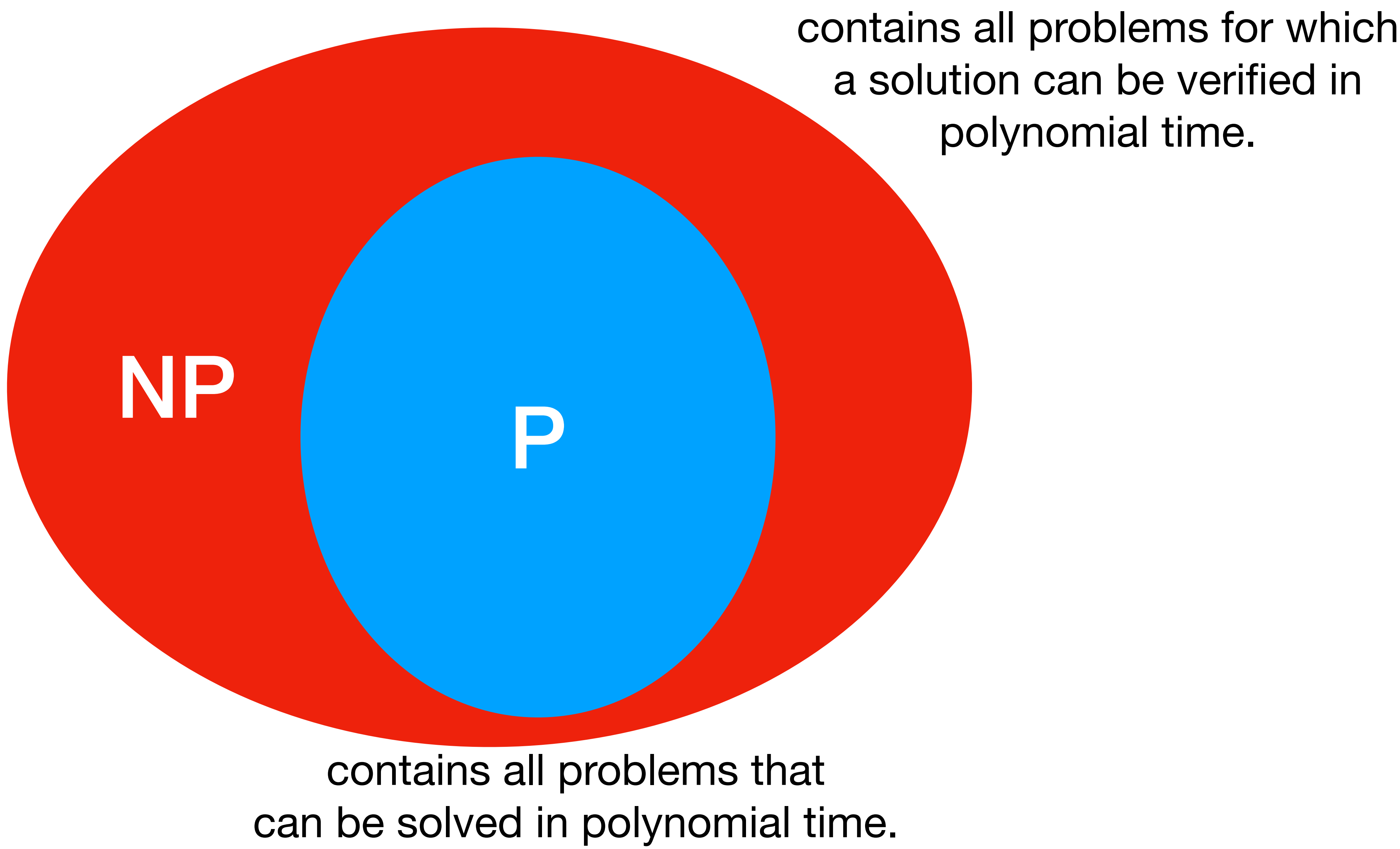

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NP-hardness

- A problem B is NP-hard if for every problem A in NP, it holds that A $\leq^p$ B.
  - If every problem in NP is “polynomial time reducible to B”.
  - This captures the fact that B is *at least as hard as the hardest problems* in NP.

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- To prove NP-hardness, it seems that we have to construct a reduction from every problem A in NP.

  - This is not very useful!

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    - i.e., every problem in NP can be efficiently reduced to it.

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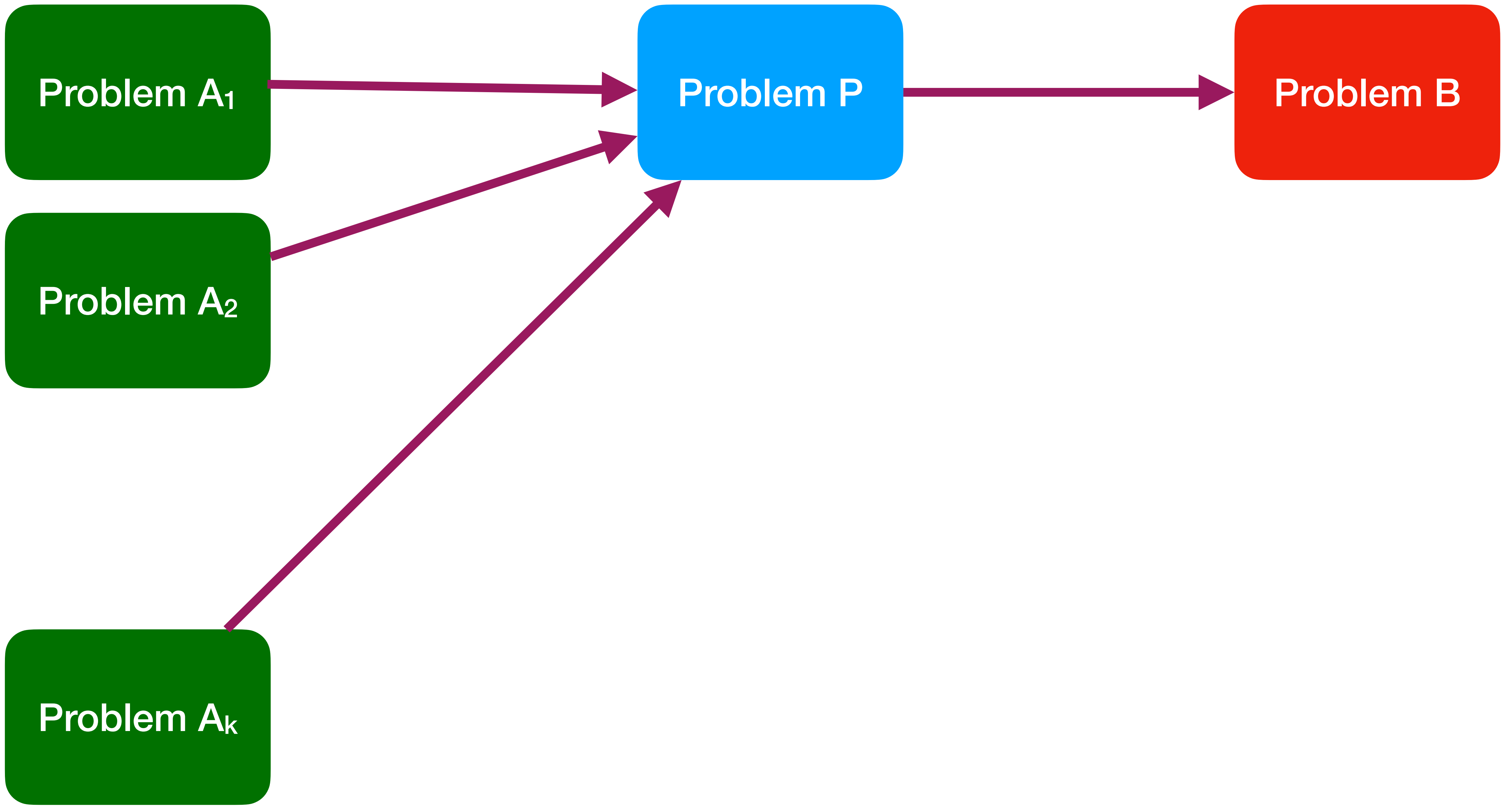
  - A reduction from any other problem A to B goes “via” P.

NP-hardness via P



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- This all works if we have an NP-complete problem to start with.

3 SAT

- A CNF formula with m clauses and k literals.

  $\phi = (x_1 \vee x_5 \vee x_3) \wedge (x_2 \vee x_6 \vee \neg x_5) \wedge \ldots \wedge (x_3 \vee x_8 \vee x_{12})$

- ("An AND of ORs").

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- Computational problem 3SAT : Decide if the input formula ϕ has a satisfying assignment.

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- Remarks:
  - The first problem shown to be NP-complete was the SAT problem (more general than 3 SAT), and this reduces to 3SAT.
  - Several textbooks start from Circuit SAT, a version of the SAT problem defined on circuits with boolean gates AND, OR or NOT.

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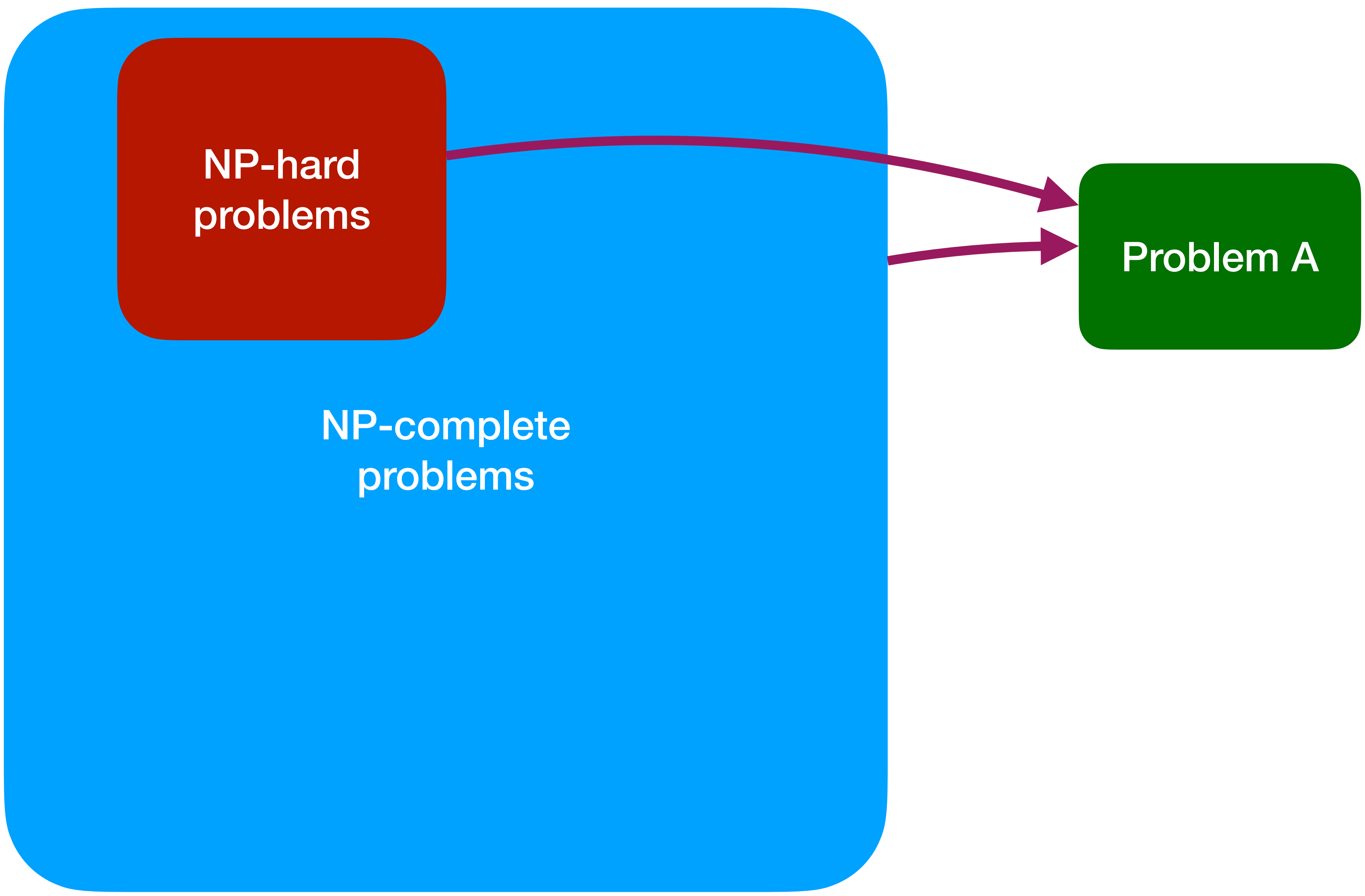

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Next time!