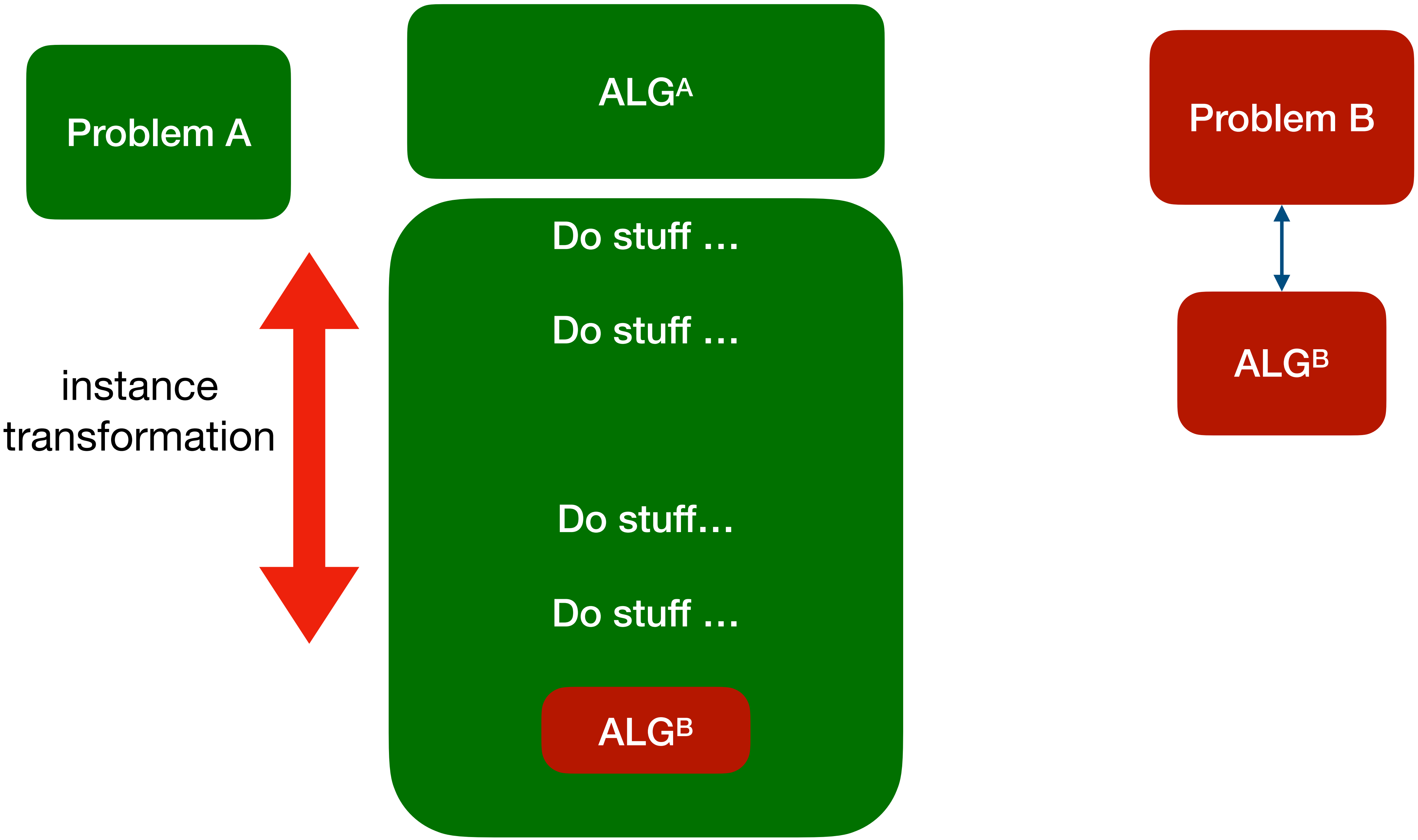
Introduction to Algorithms and Data Structures

Vertex Cover and Other NP-complete problems

Polynomial Time Reduction

- We are given a problem A that we want to solve.
- We can reduce solving problem A to solving some other problem B.
- Assume that we had an algorithm ALG^B for solving problem B, which we can use at cost **O(*1*)**.
- We can construct an algorithm ALG^A for solving problem A, which uses calls to the algorithm ALG^B as a subroutine.
- If ALG^A is a polynomial time algorithm, then this is a *polynomial time reduction*.

Pictorially



Types of reductions

- **Turing reduction:**
  - Argument: Here is an algorithm which runs in polynomial time solving problem A, using polynomially many calls to an oracle for problem B.
- **Many-one reduction:**
  - Argument:
    - If z is a solution to instance I of problem A, then z’ is a solution of instance f(I) to problem B.
    - If z is not a solution to instance I of problem A, then z’ is not a solution of instance f(I) to problem B.
    - Equivalently: If z’ is a solution of instance f(I) to problem B, then z is a solution to instance I of problem A.

How to work with reductions

- Positive: Assume that I want to solve problem A and I know how to solve problem B in polynomial time.
  - I can try to come up with a polynomial time reduction $A \leq^p B$, which will give me a polynomial time algorithm for solving A.
- Contrapositive: Assume that there is a problem A for which it is unlikely that there is a polynomial time algorithm that solves it.
  - If I come up with a polynomial time reduction $A \leq^p B$, it is also unlikely that there is a polynomial time algorithm that solves B.
  - B is "*at least as hard to solve as*" A, because if I could solve B, I could also solve A.

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3 SAT

- A CNF formula with m clauses and k literals.

  $\phi = (x_1 \vee x_5 \vee x_3) \wedge (x_2 \vee x_6 \vee \neg x_5) \wedge \ldots \wedge (x_3 \vee x_8 \vee x_{12})$

- ("An AND of ORs").

- Each clause has three literals.

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- Computational problem 3SAT : Decide if the input formula ϕ has a satisfying assignment.

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- Remarks:
  - The first problem shown to be NP-complete was the SAT problem (more general than 3 SAT), and this reduces to 3SAT.
  - Several textbooks start from Circuit SAT, a version of the SAT problem defined on circuits with boolean gates AND, OR or NOT.

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- First, prove that A is in NP.
  - Usually by observing that a solution is efficiently checkable.
- Then prove that A is NP-hard.
  - Construct a polynomial time reduction from some NP-complete (or just NP-hard) problem P.

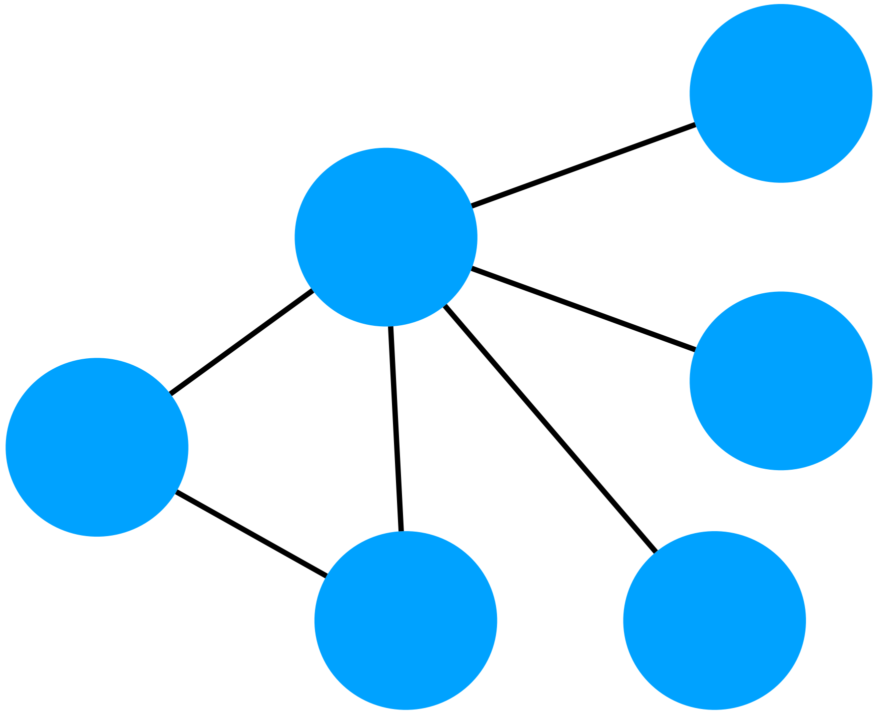
Enough with the definitions. Let's see how it works.

- We will prove that a well-known problem on graphs, called Vertex Cover is NP-complete.

Vertex Cover

- Definition: A vertex cover C of a graph G=(V, E) is a subset of the nodes such that every edge e in the graph has at least one endpoint in C.
- Definition: A minimum vertex cover is a vertex cover of the smallest possible size.
- Vertex Cover
  Input: A graph G=(V, E)
  Output: A minimum vertex cover.

Example



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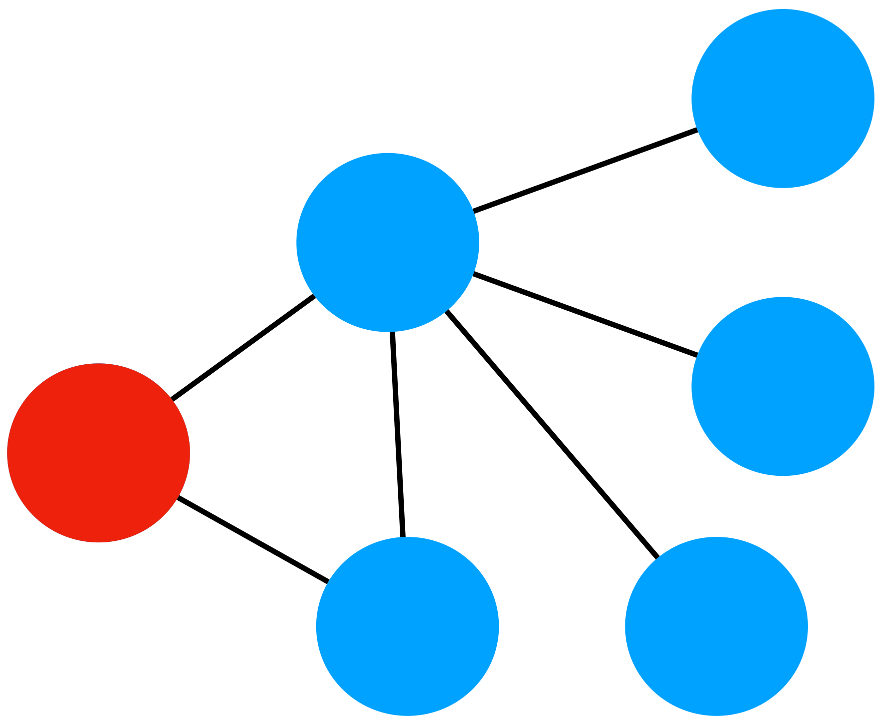

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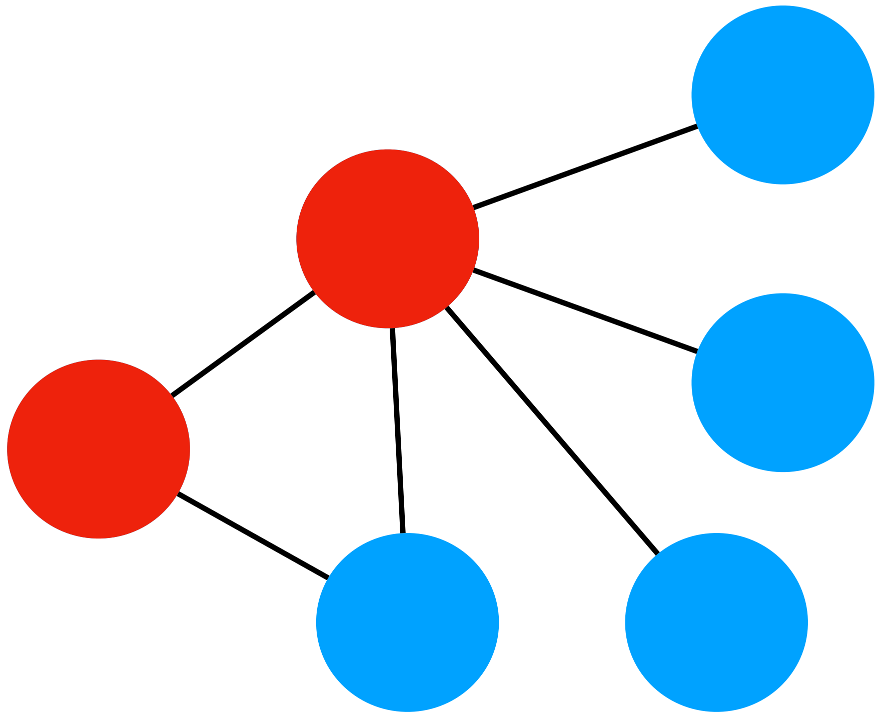

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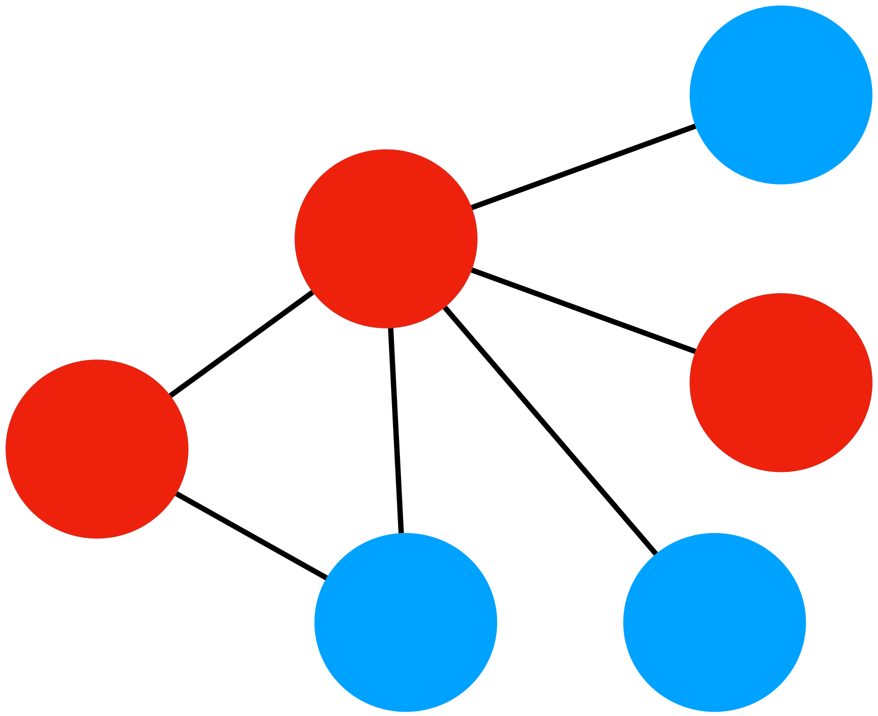

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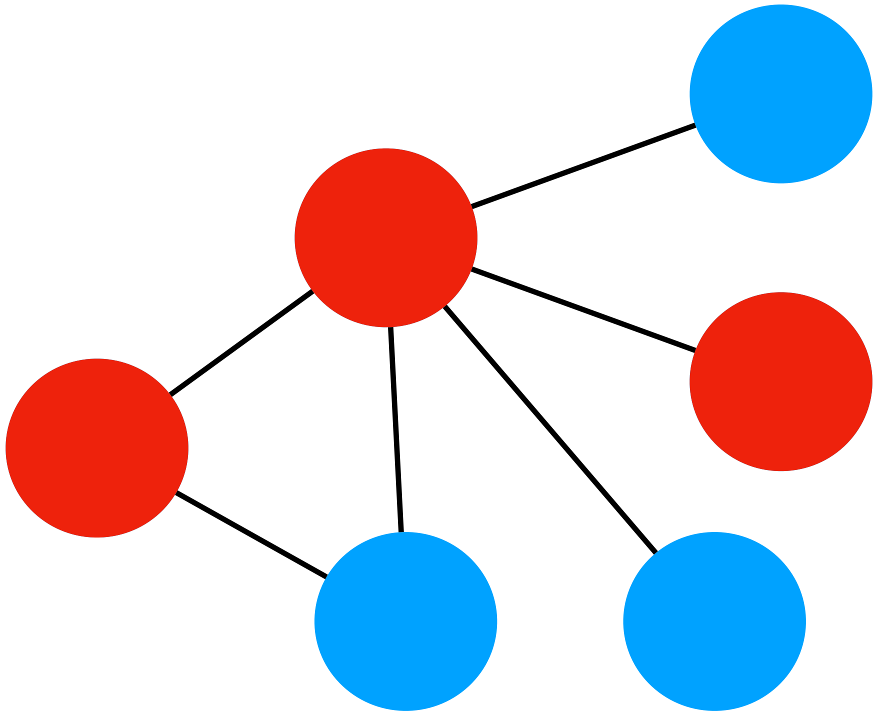
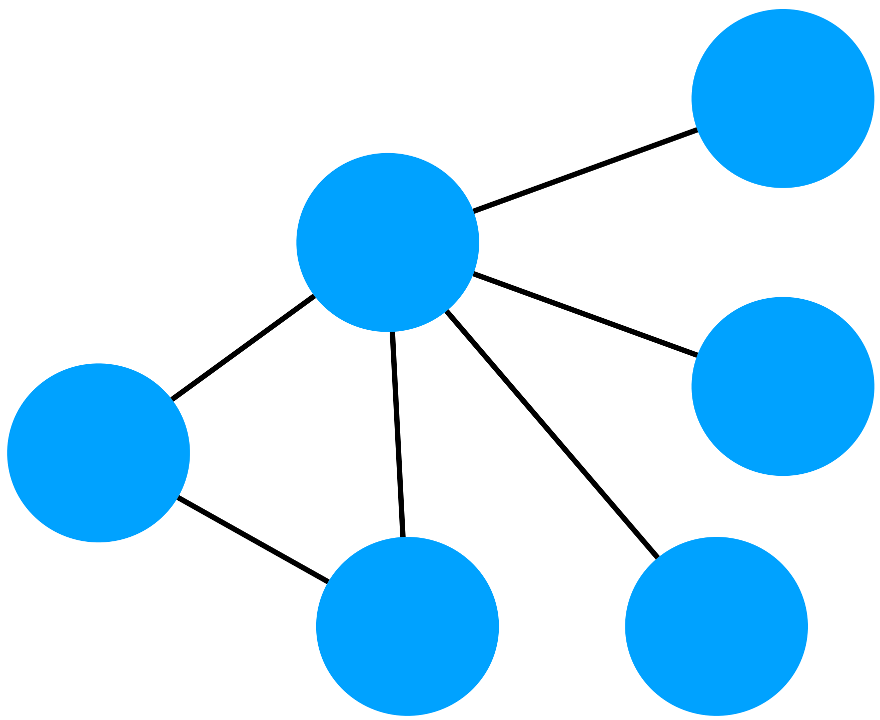


A vertex cover

Example



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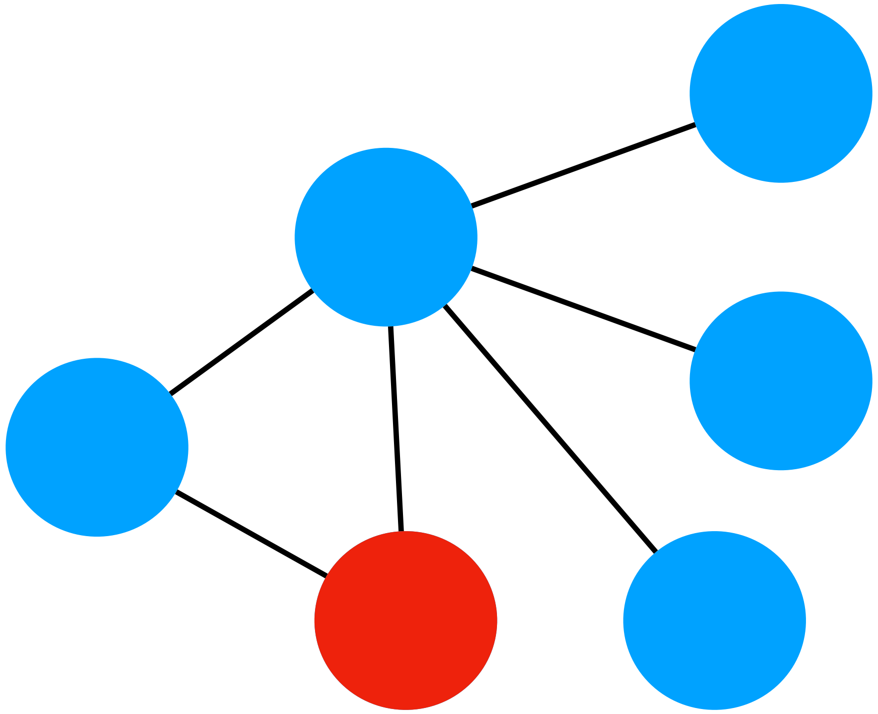

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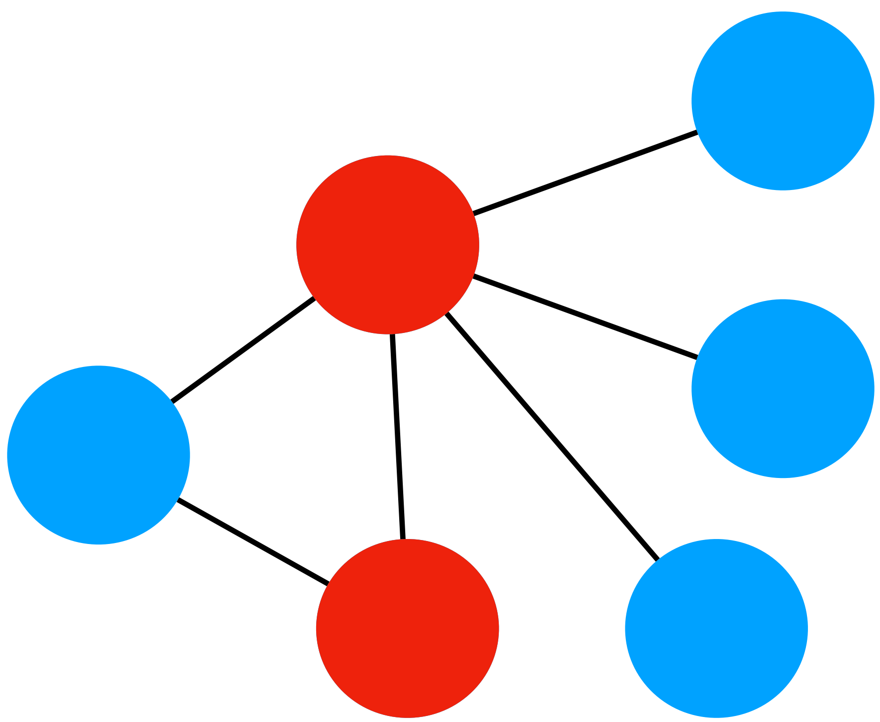

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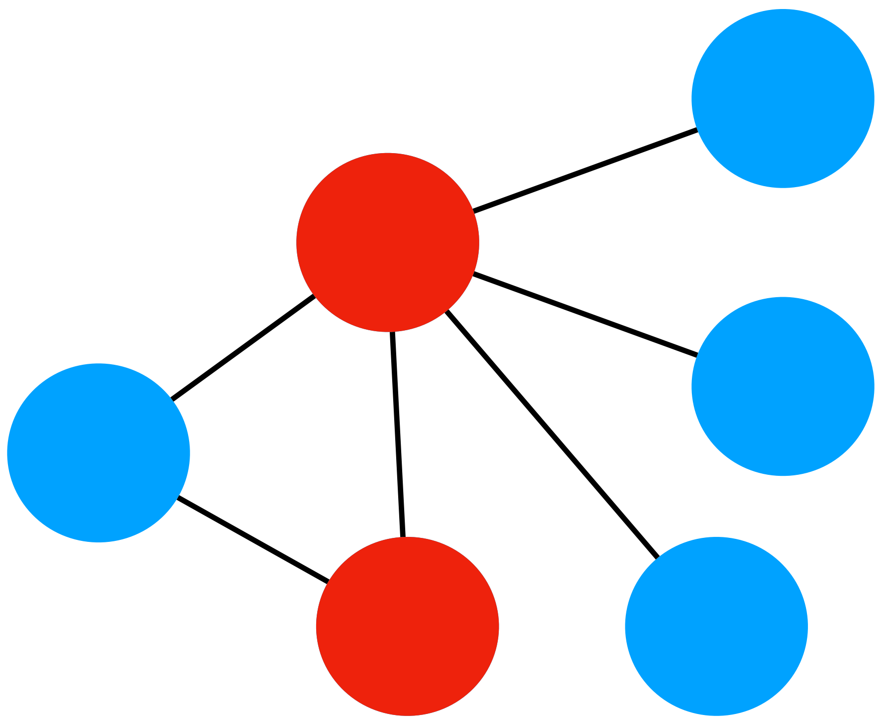

A minimum vertex cover

Vertex Cover

- Definition: A vertex cover C of a graph G=(V, E) is a subset of the nodes such that every edge e in the graph has at least one endpoint in C.

- Definition: A minimum vertex cover is a vertex cover of the smallest possible size.

- Vertex Cover
  Input: A graph G=(V, E)
  Output: A minimum vertex cover.

Vertex Cover decision version

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- Definition: A minimum vertex cover is a vertex cover of the smallest possible size.

- Vertex Cover
  Input: A graph G=(V, E) and a number k
  Output: Is there a vertex cover of size ≤ k?.

Vertex cover

Vertex cover

- Vertex Cover is in NP.

Vertex cover

- Vertex Cover is in NP.
- Assume that we are given a vertex cover.
    - We can check that is has size k and that it is a vertex cover in polynomial time.

Vertex cover

Vertex cover

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Vertex cover

- Vertex Cover is in NP-hard.
- We will construct a polynomial time reduction from 3SAT.
  - i.e., we will prove that 3SAT $\leq^p$ Vertex Cover.

The reduction

- Let ϕ be a 3-CNF formula with m clauses and d variables.
- We construct, in polynomial time, an instance <G, k> of Vertex Cover such that
    - If ϕ is satisfiable => G has a vertex cover of size at most k.
    - If ϕ is not satisfiable => G does not have any vertex cover of size at most k.

The reduction

- For every variable x in ϕ, we create two nodes x and $\neg x$ in G and we connect them with an edge $e = (x , \neg x)$.

Running example: $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_2) \wedge (\neg x_1 \vee x_2 \vee x_2)$

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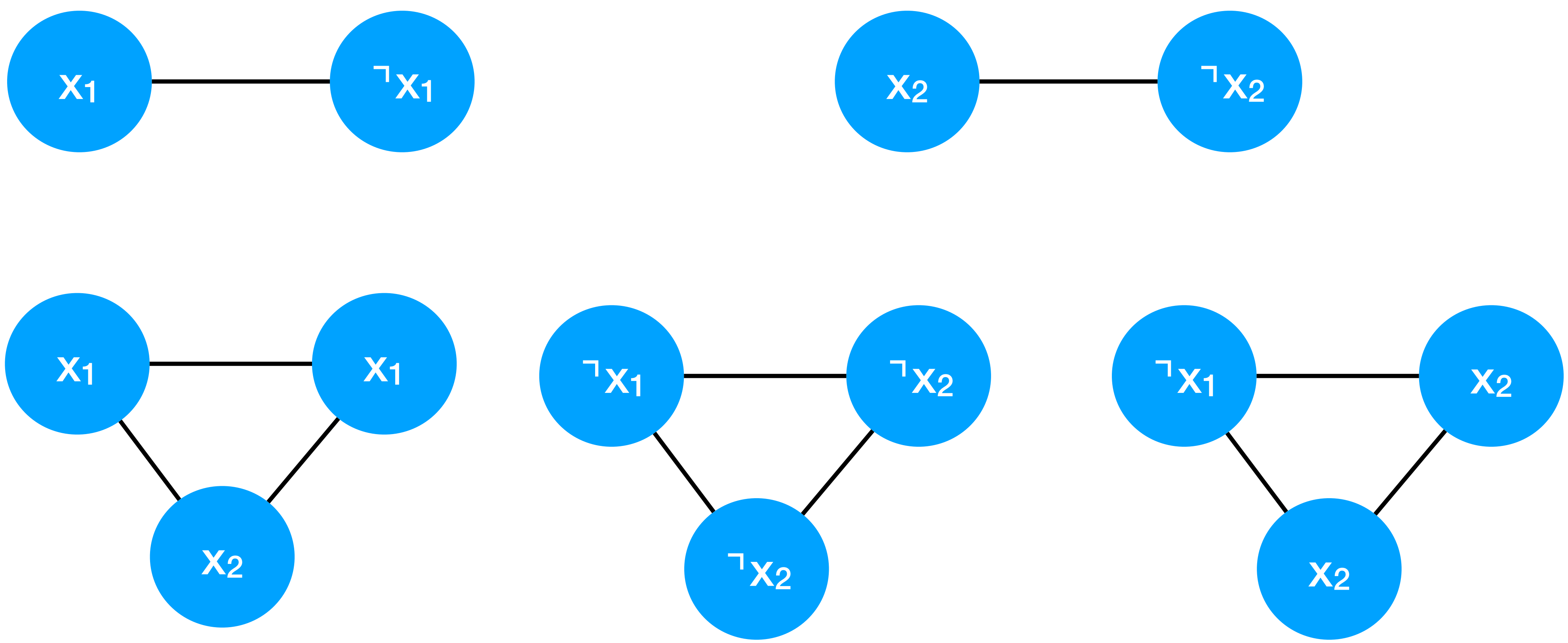
The reduction

- For every clause $\ell = (\ell_1, \ell_2, \ell_3)$ in ϕ, we create three nodes ℓ_1, ℓ_2, ℓ_3 in G and we connect them all with each other.



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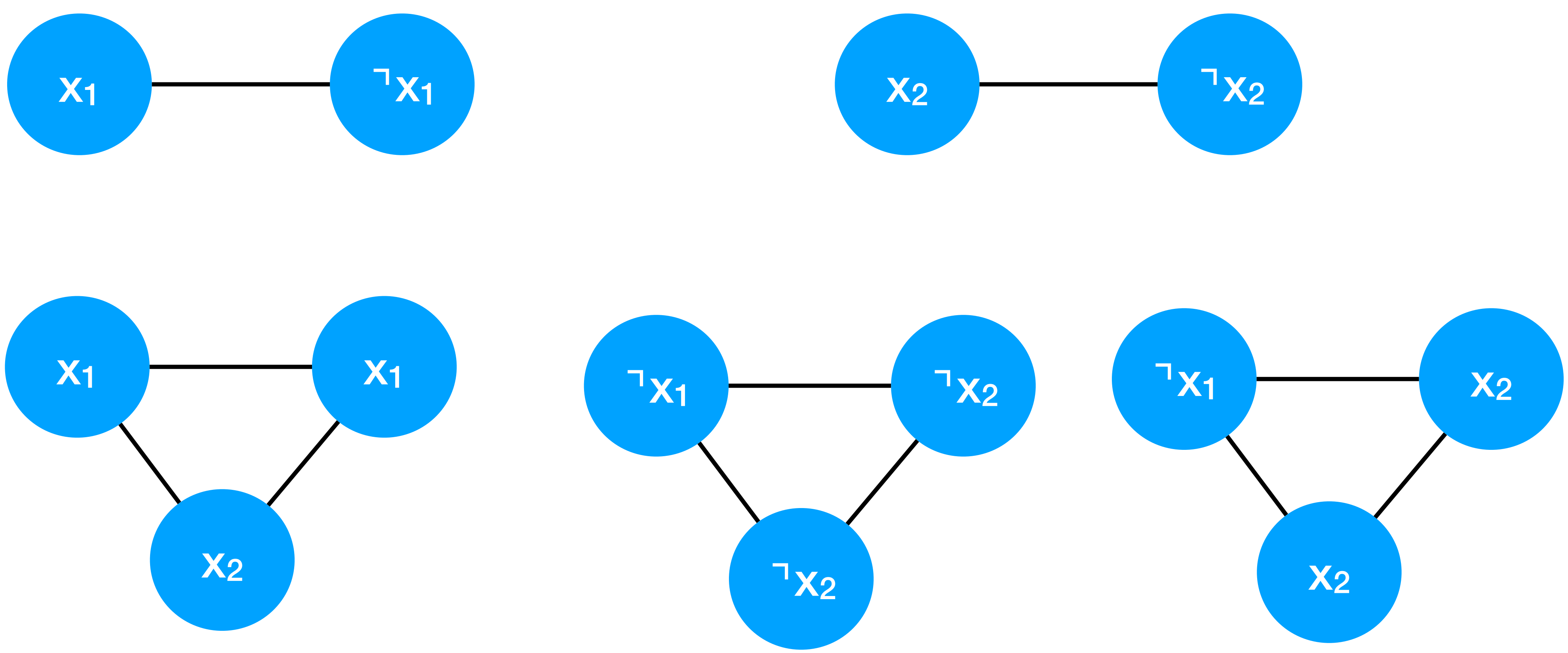
The reduction

- We add an edge between all nodes with the same label on the top and on the bottom.



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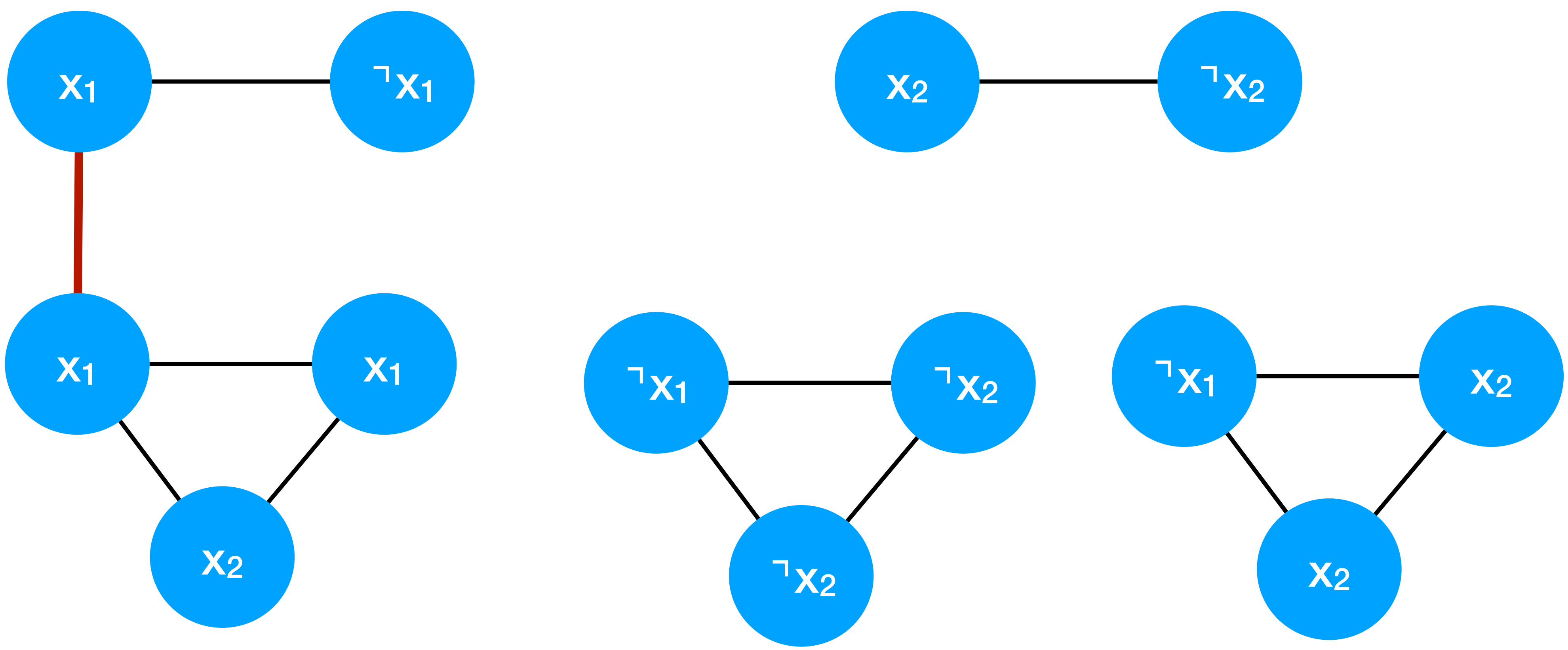

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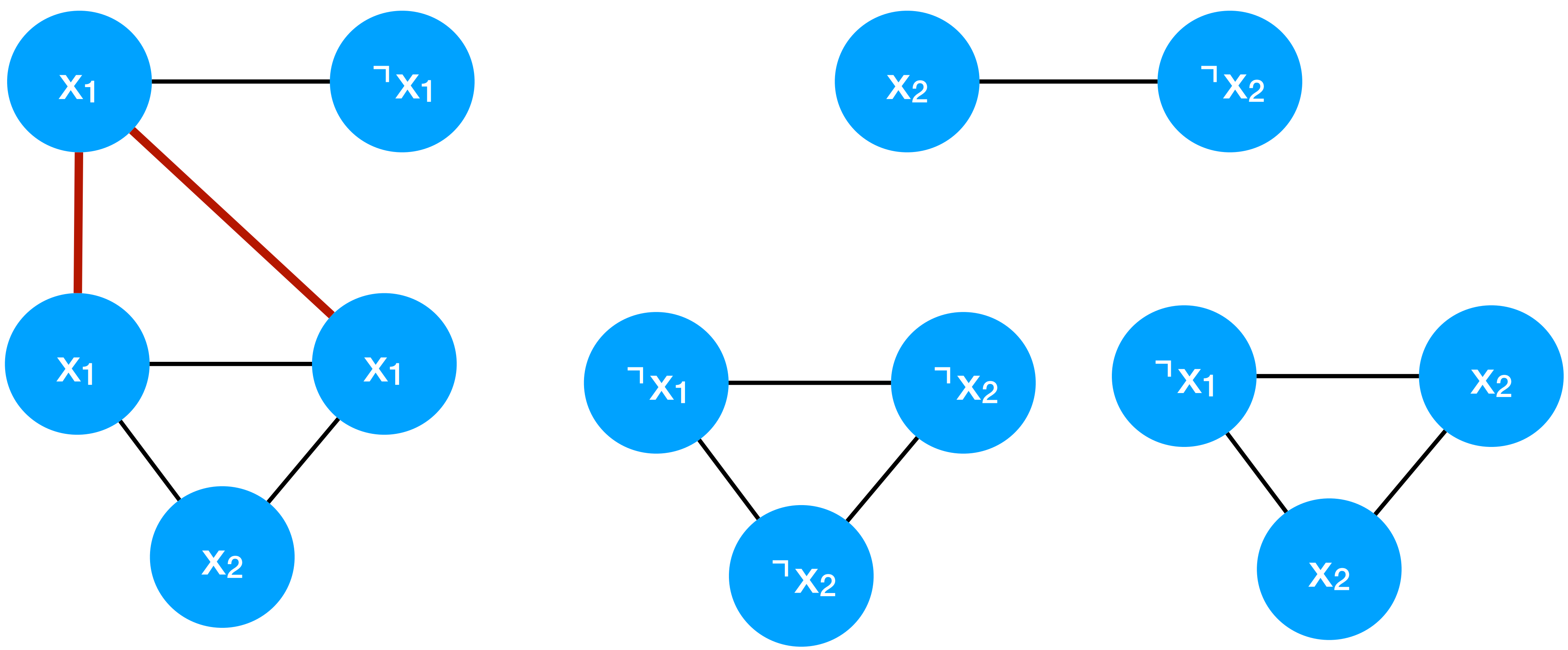

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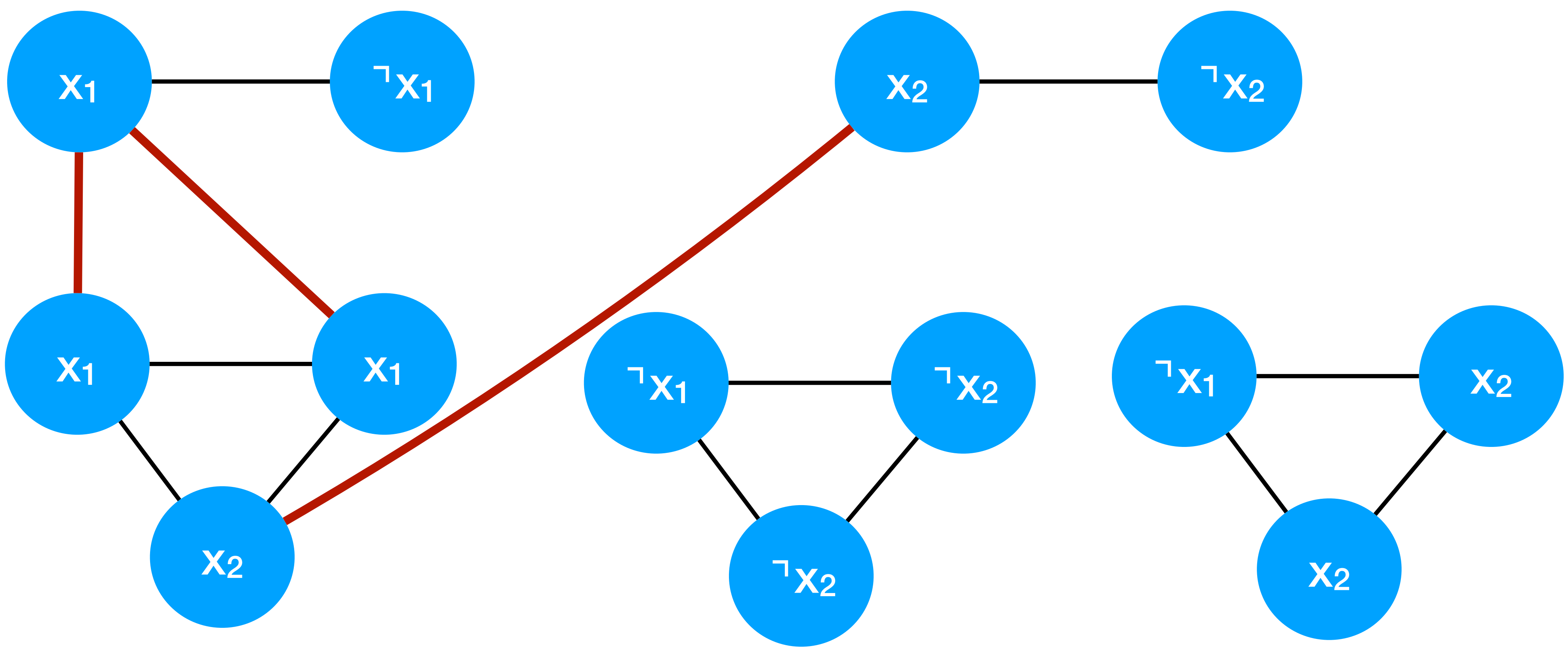

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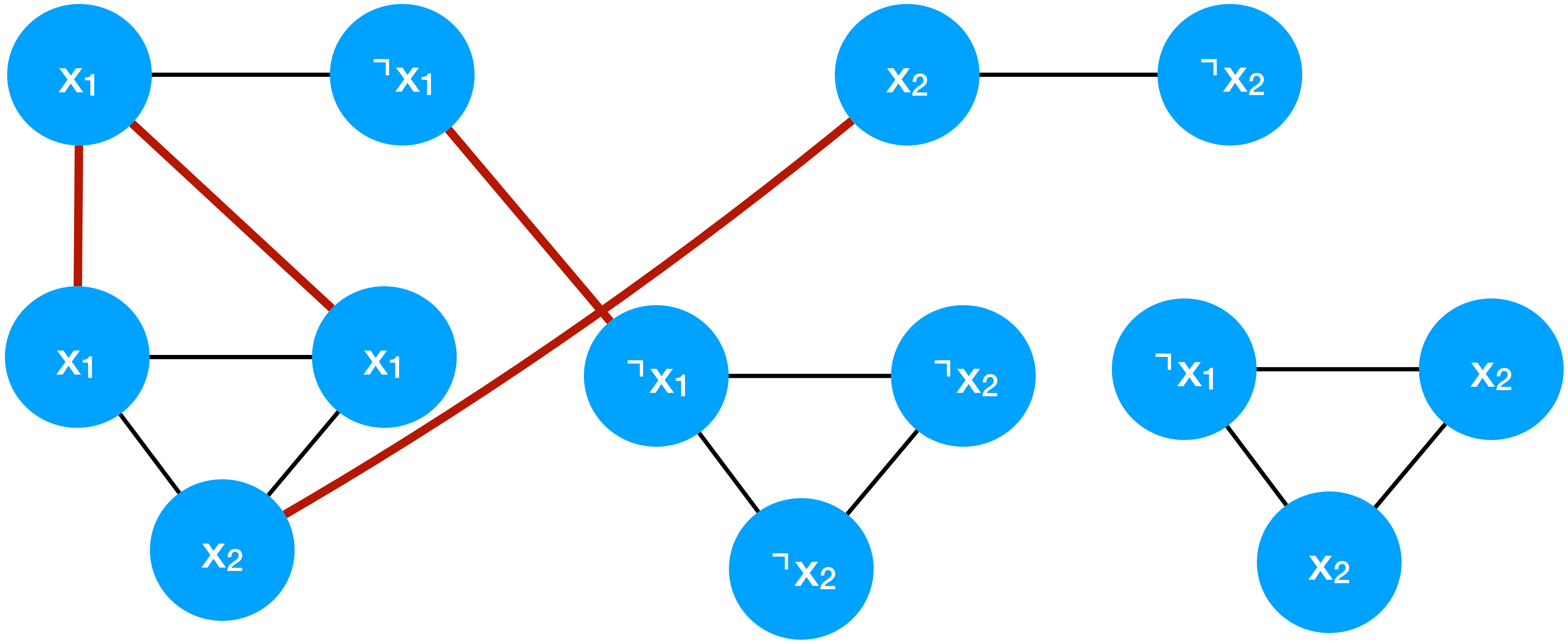

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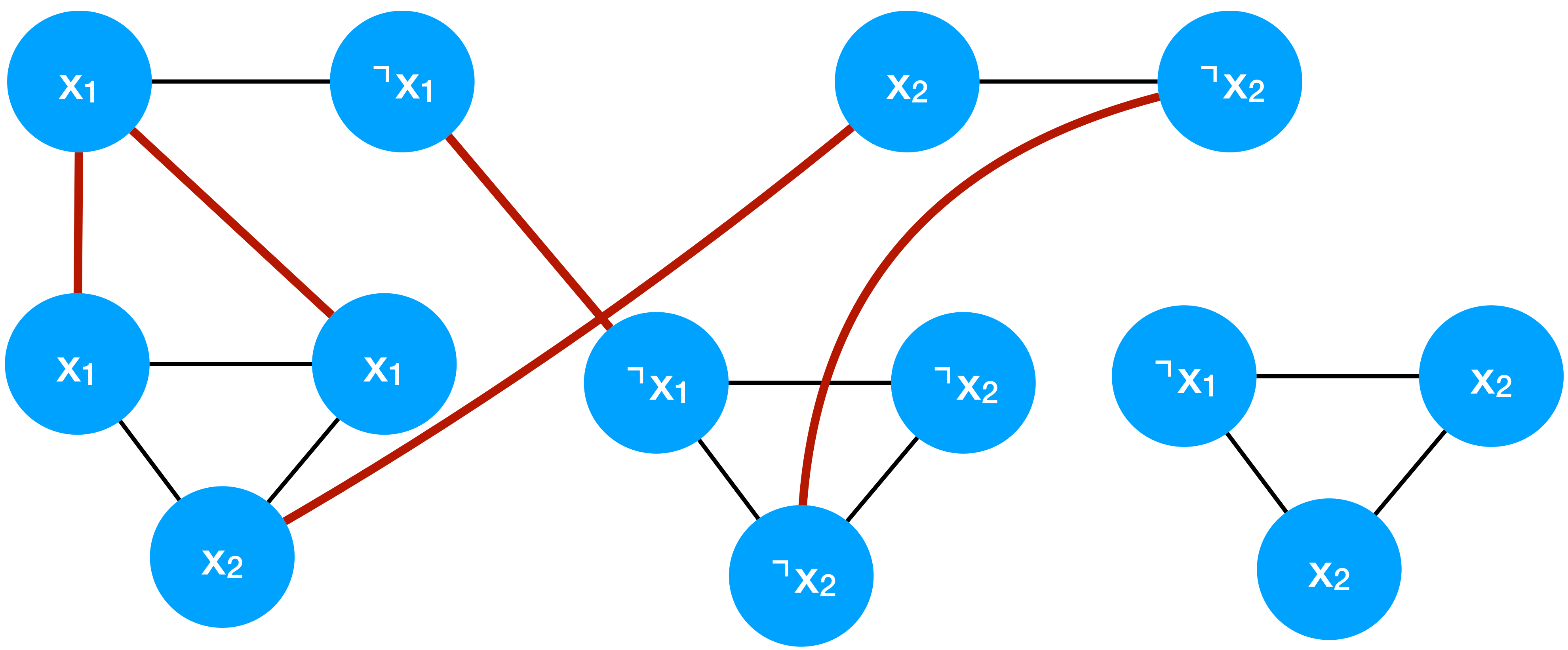

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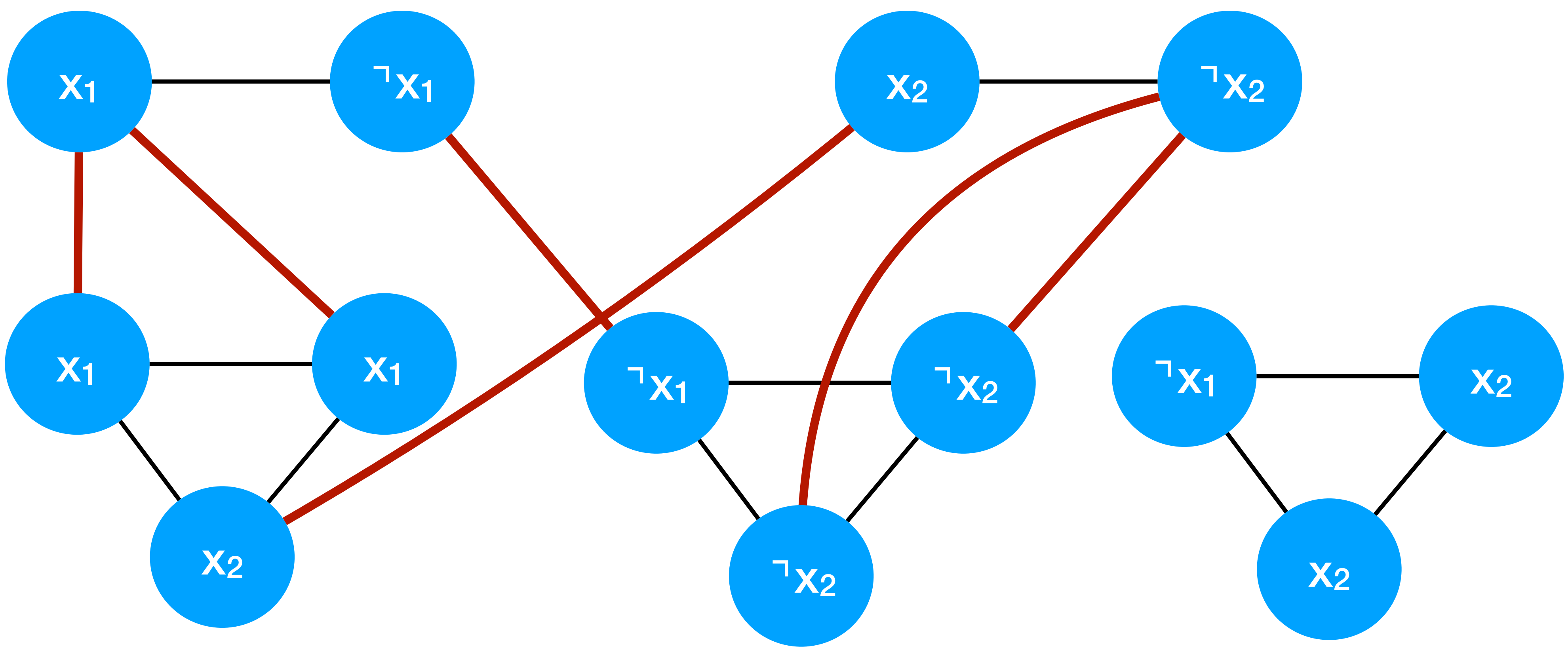

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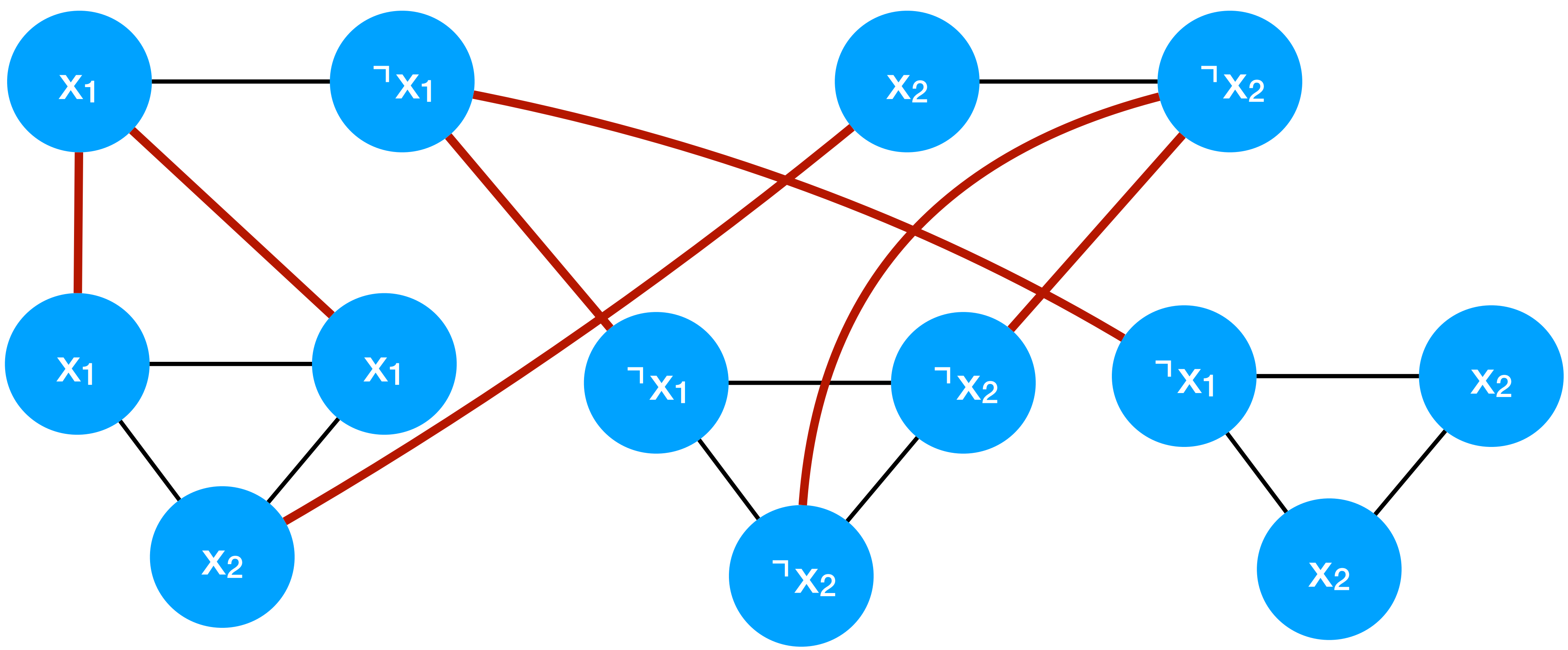

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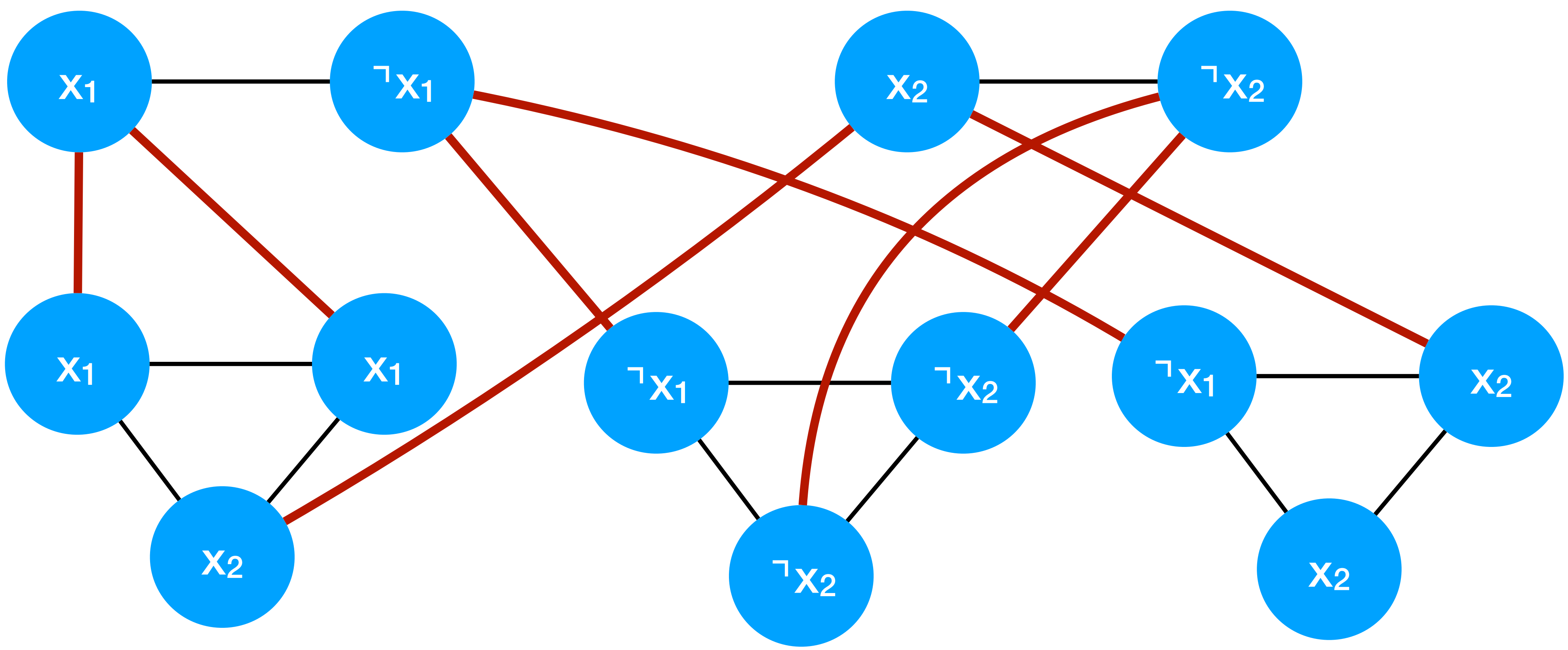

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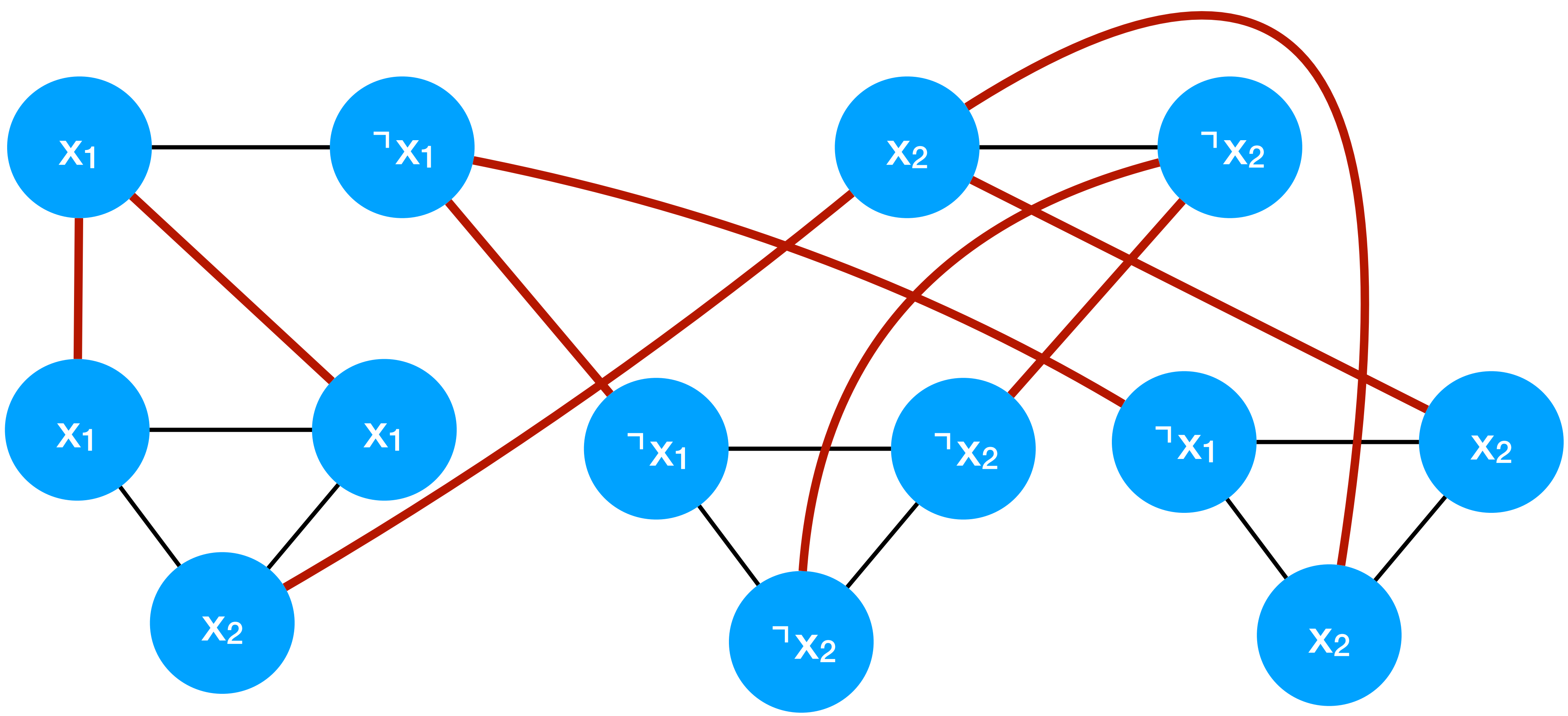

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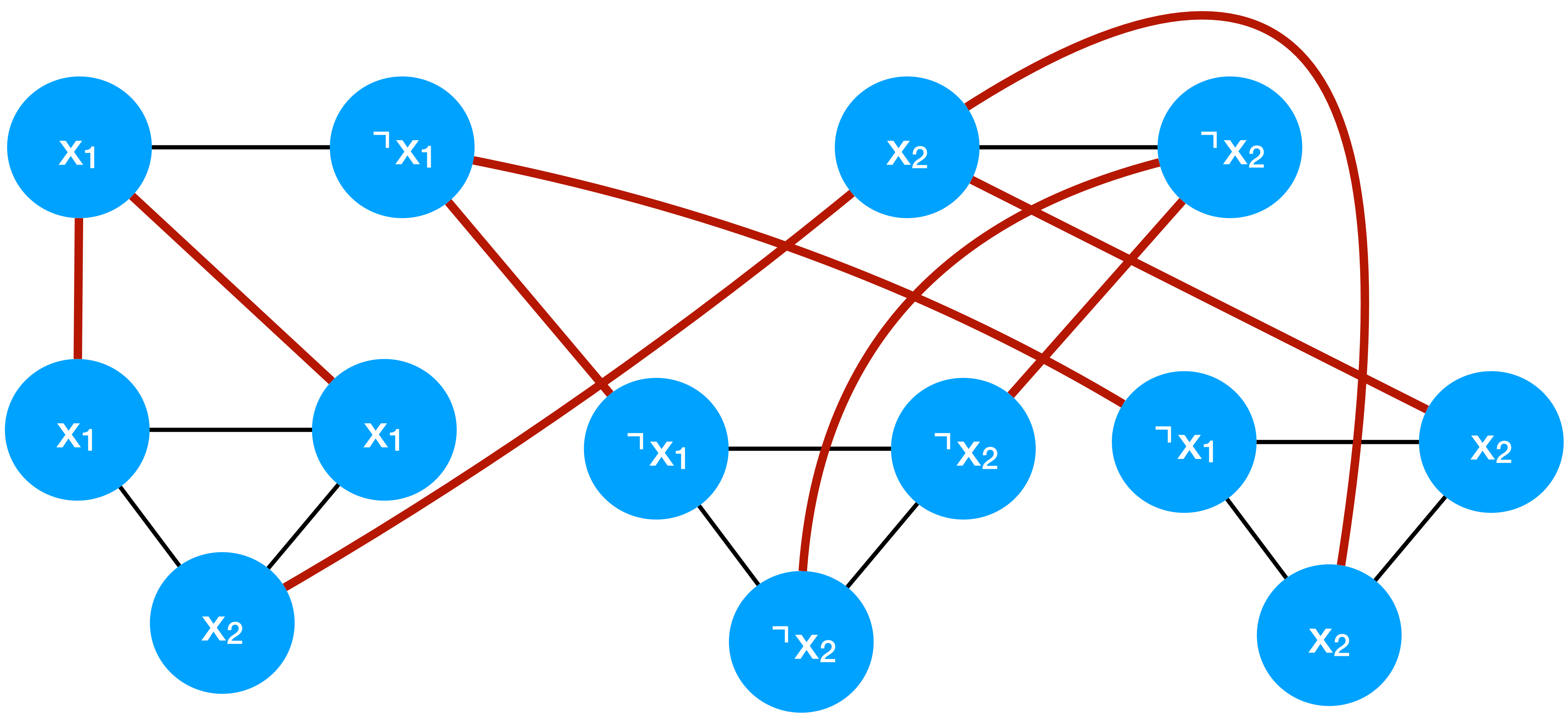
- For the nodes on the bottom: In each triangle, choose a note x_i that has been picked on the top and do not include it in the vertex cover. Include the other two nodes.

Example

- For the nodes on the top: If $y_i = 1$, include node x_i in the vertex cover C, otherwise, include node $\neg x_i$.
  - Assume $y_1 = 0$, $y_2 = 1$.



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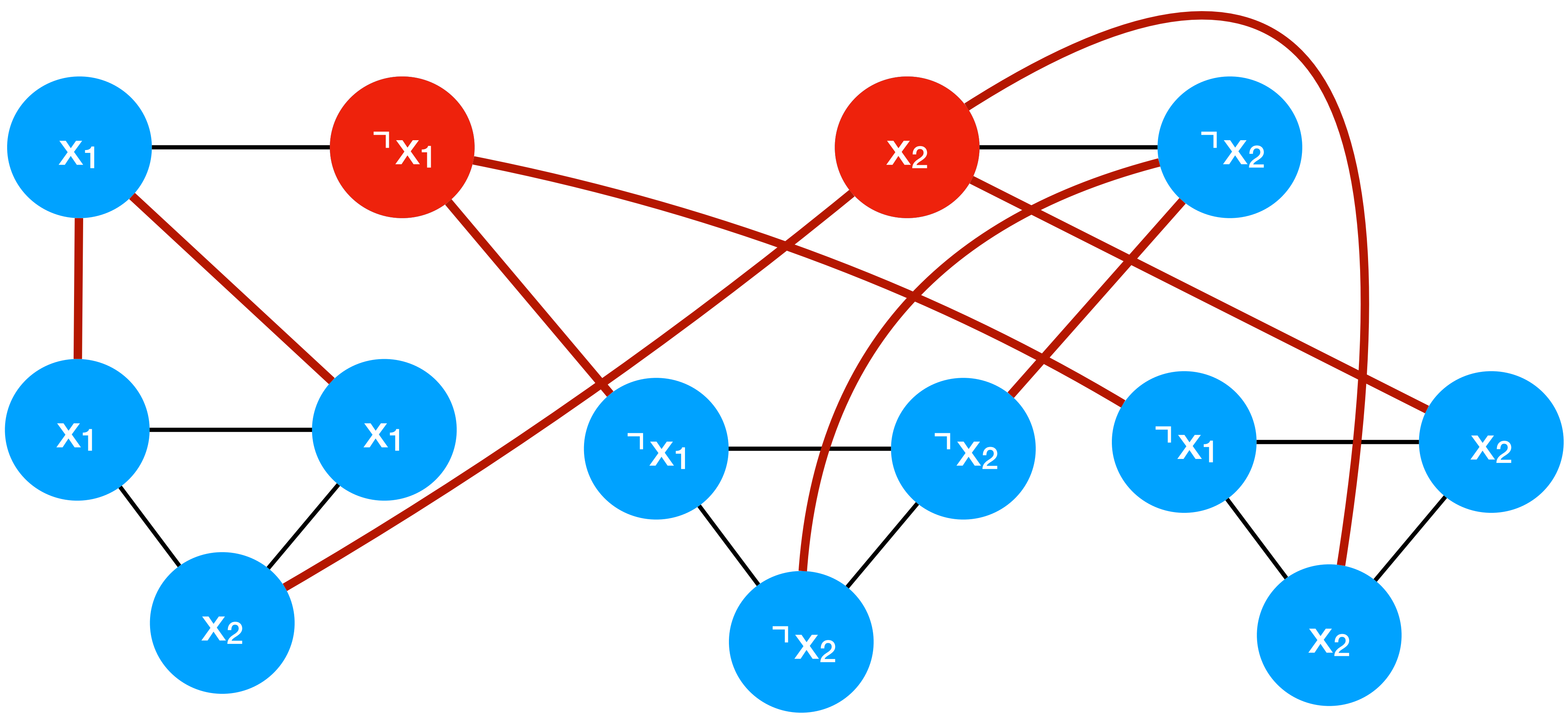

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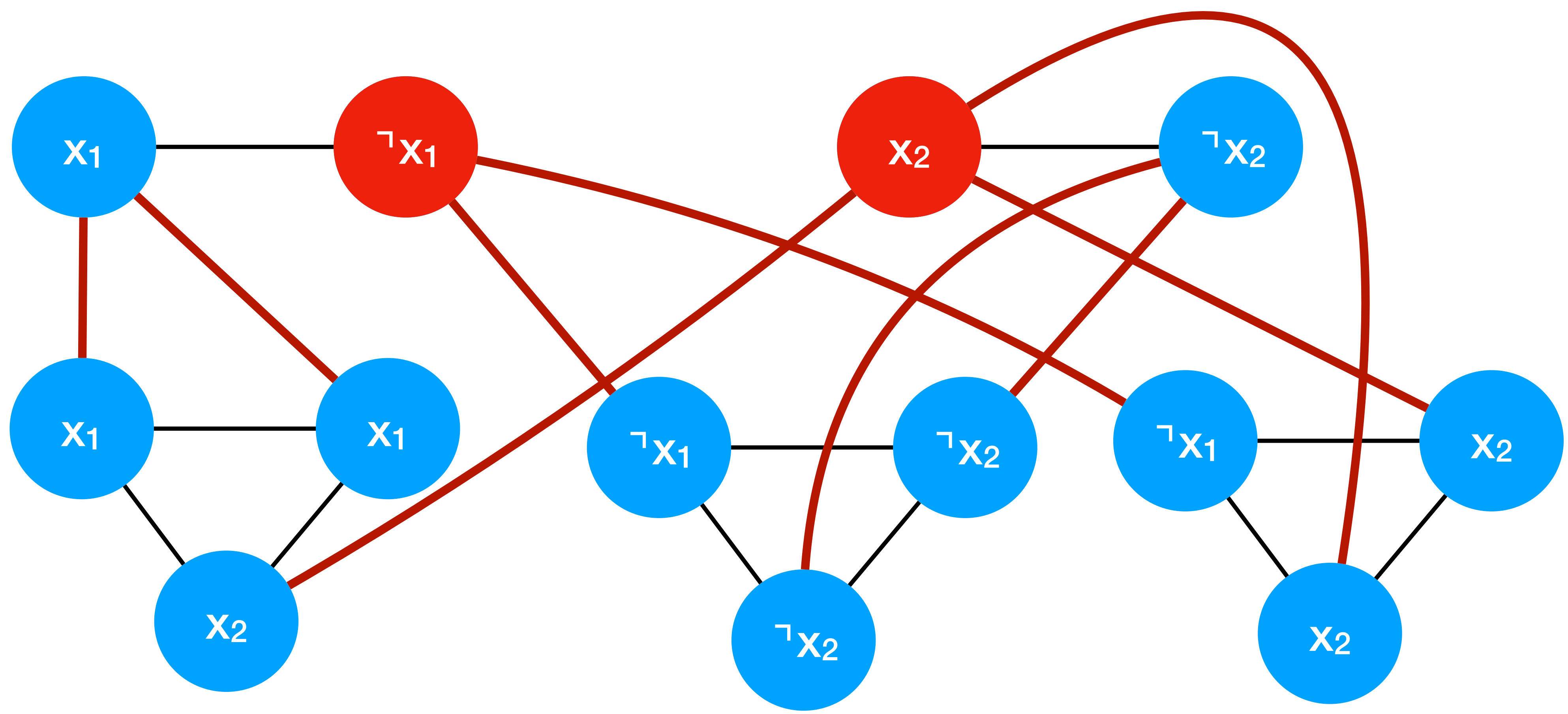

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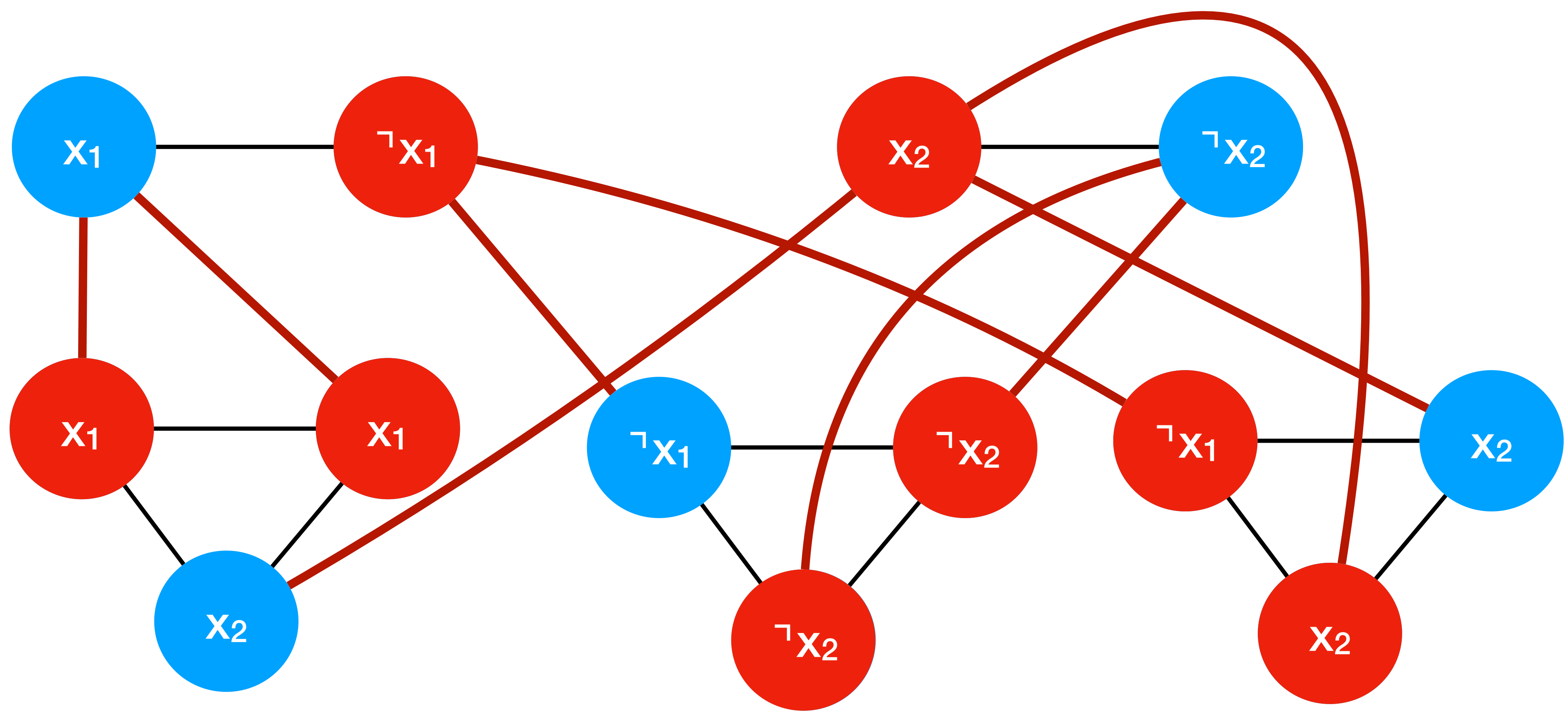

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One direction

- Claim: The set of nodes we have chosen is a vertex cover.
  - Every edge on the top is incident to either node x_i or node $\neg x_i$.
  - Every edge on the bottom is incident to some node in the set, since we select two out of three nodes.
  - Every edge between the top and to bottom is incident to some node.

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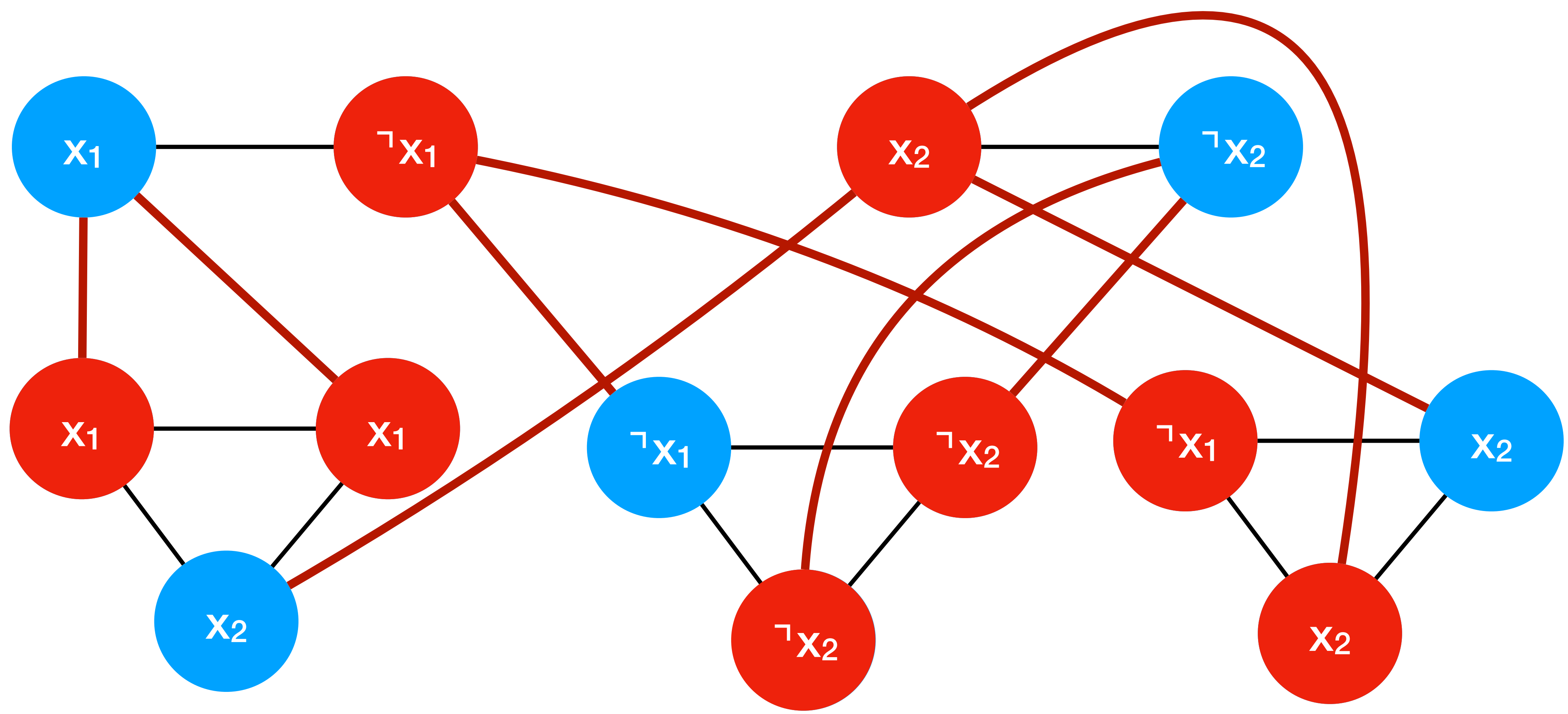

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One direction

- Claim: The vertex cover has size $k = d + 2m$
    - Each variable is selected at the top (either as x_i or as $\neg x_i$).
    - For each clause, we select two nodes at the bottom.

Other direction

- If ϕ is not satisfiable => G does not have any vertex cover of size at most k.

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- Since it is a vertex cover, it must include at least two out of three nodes in each “clause gadget” at the bottom.

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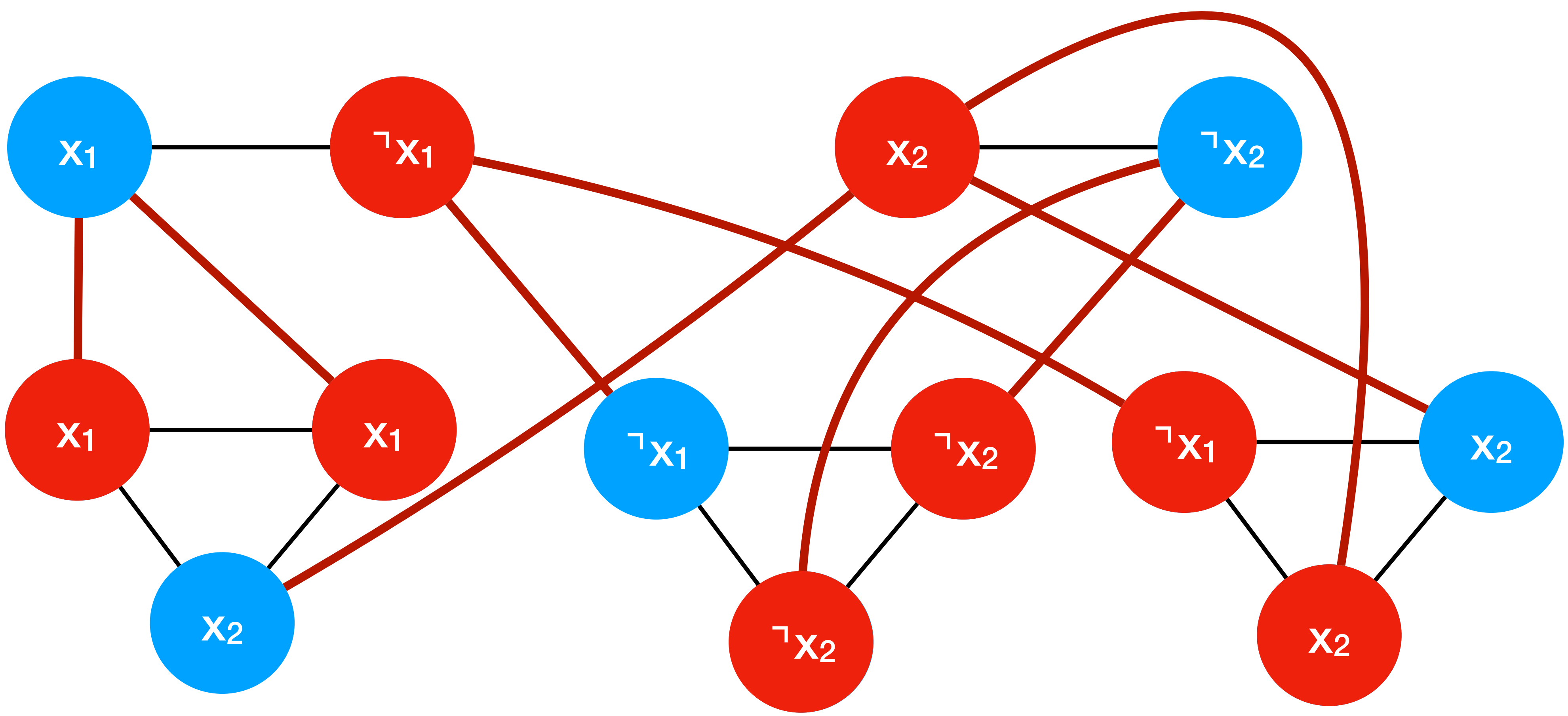

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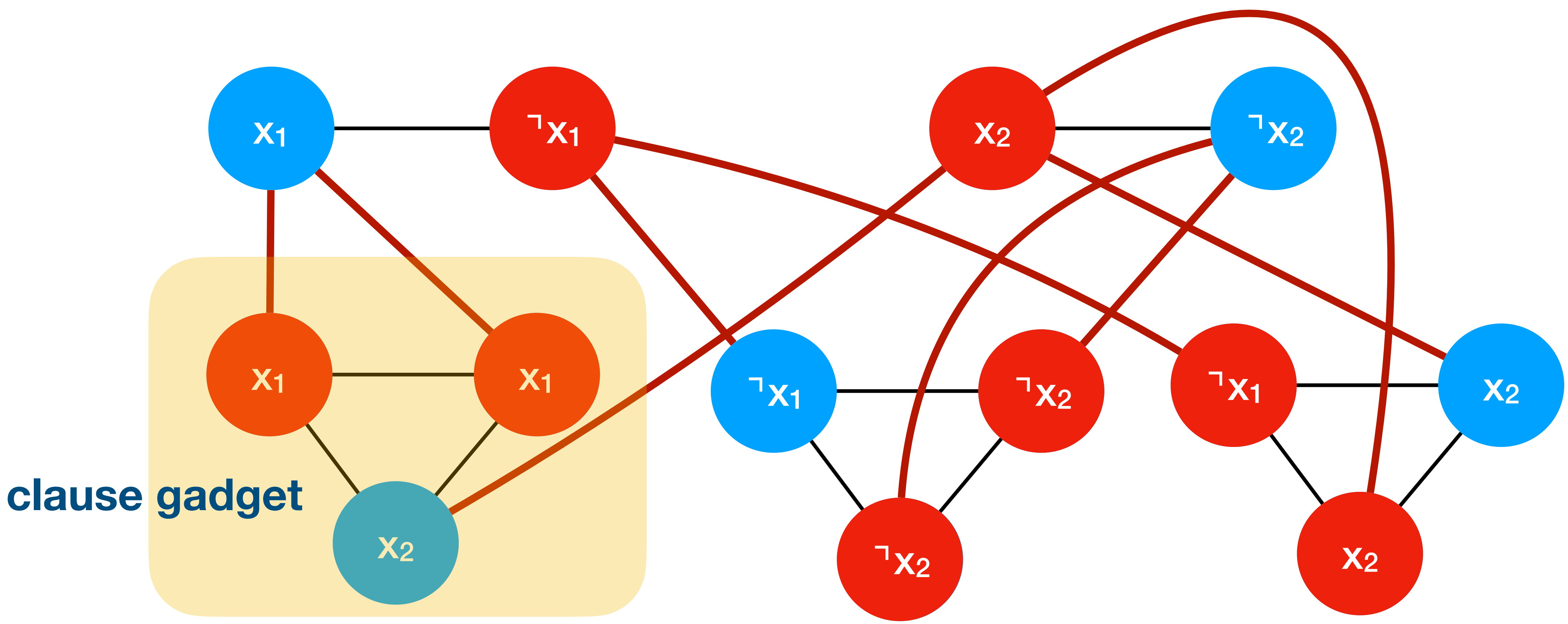

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Other direction

- G has a vertex cover of size at most k. => ϕ is satisfiable.
- Let C be a vertex cover of size k = d + 2m in G.
- Since it is a vertex cover, it must include at least two out of three nodes in each “clause gadget” at the bottom.

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  - This means that at least 2m nodes of C are at the bottom.
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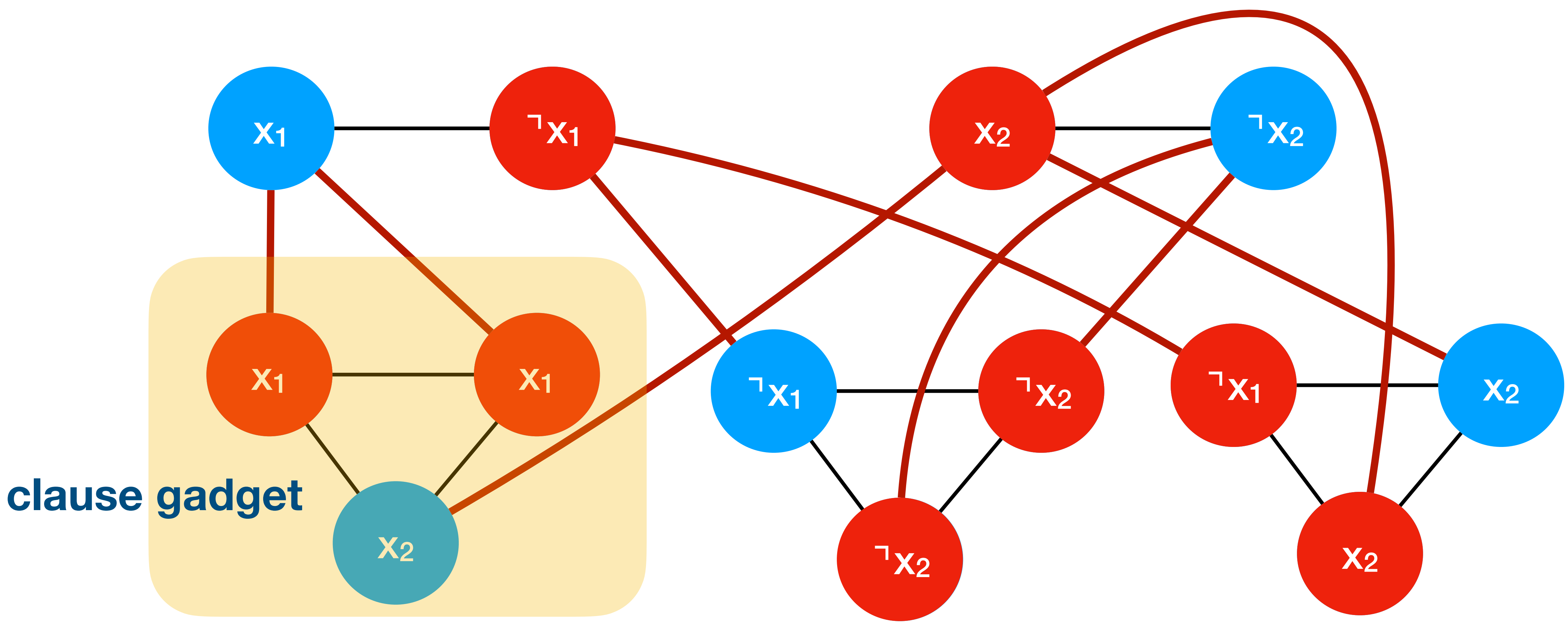

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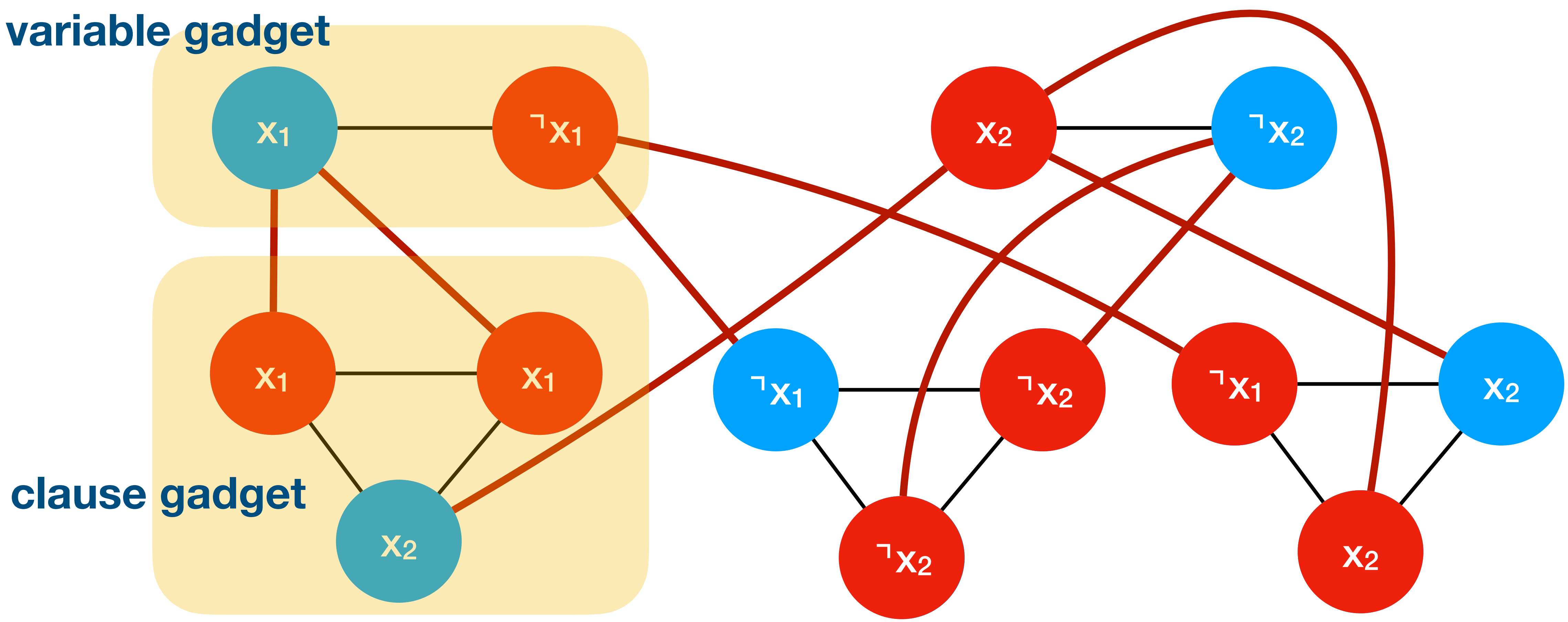

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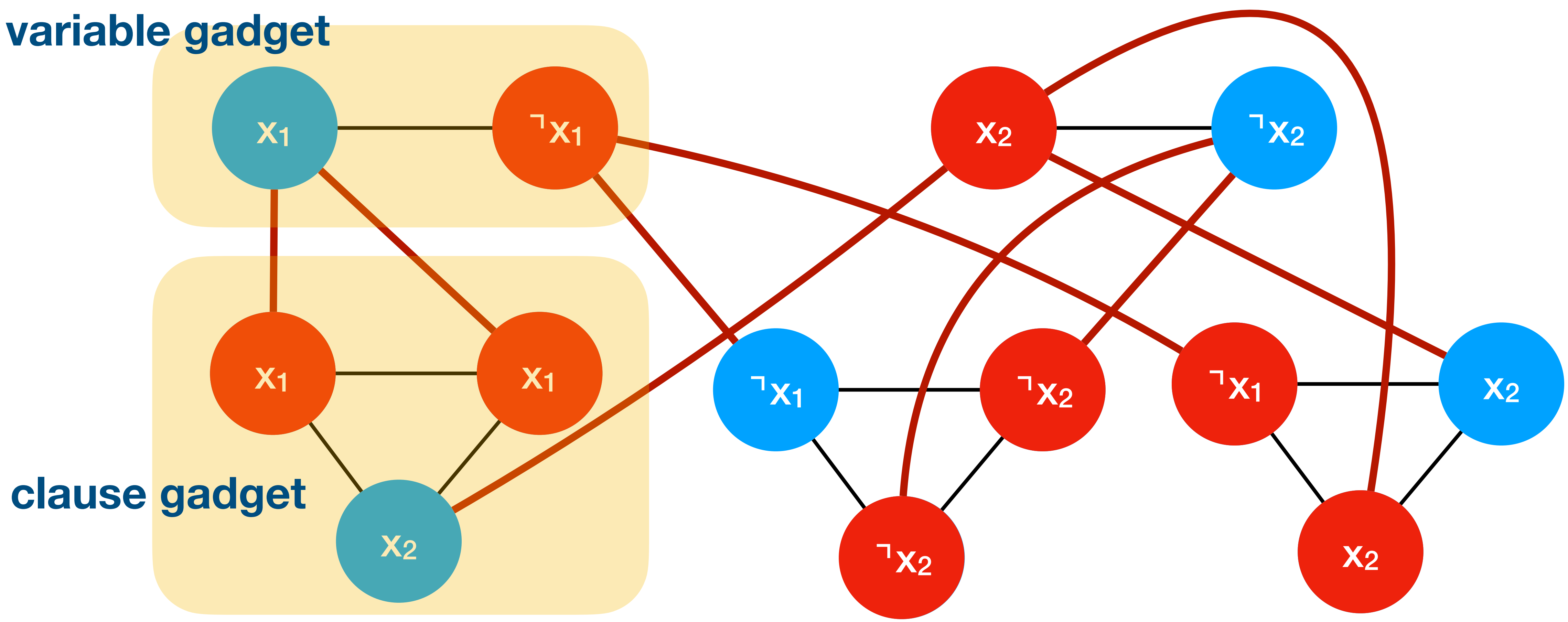
  - This means that there is one variable of the clause that is set to 1.
  - Thus the clause is satisfied.

Example

- To satisfy the edges at the top, in each "variable gadget", at least one node must be included in C.



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Vertex Cover

- Definition: A vertex cover C of a graph G=(V, E) is a subset of the nodes such that every edge e in the graph has at least one endpoint in C.

- Definition: A minimum vertex cover is a vertex cover of the smallest possible size.

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  Input: A graph G=(V, E)
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Vertex Cover decision version

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- Definition: A minimum vertex cover is a vertex cover of the smallest possible size.

- Vertex Cover
  Input: A graph $G=(V, E)$ and a number k
  Output: Is there a vertex cover of size $\leq k$?.

From optimisation to decision

- We are given an optimisation problem P (assume minimisation).
    - E.g., find the minimum vertex cover.
- We introduce a threshold k.
- The decision version P_d becomes: Given an instance of P and the threshold k as input, is there a solution to P of value at most k?
    - E.g., is there a vertex cover of size at most k?

Optimisation vs decision

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- If we can solve P in polynomial time, we can solve P_d in polynomial time. (why?)

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- Vertex Cover (Optimisation)
  Input: A graph $G=(V, E)$
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Optimisation vs decision

- Vertex Cover Size (Optimisation)
  Input: A graph $G=(V, E)$
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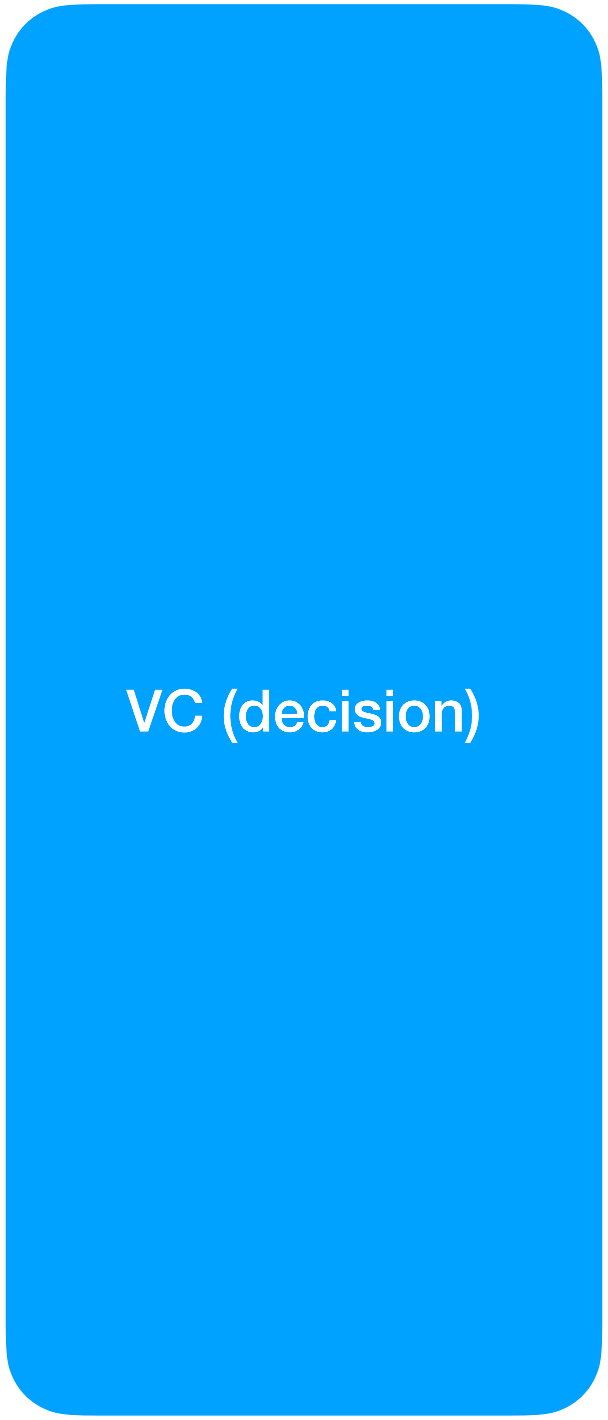

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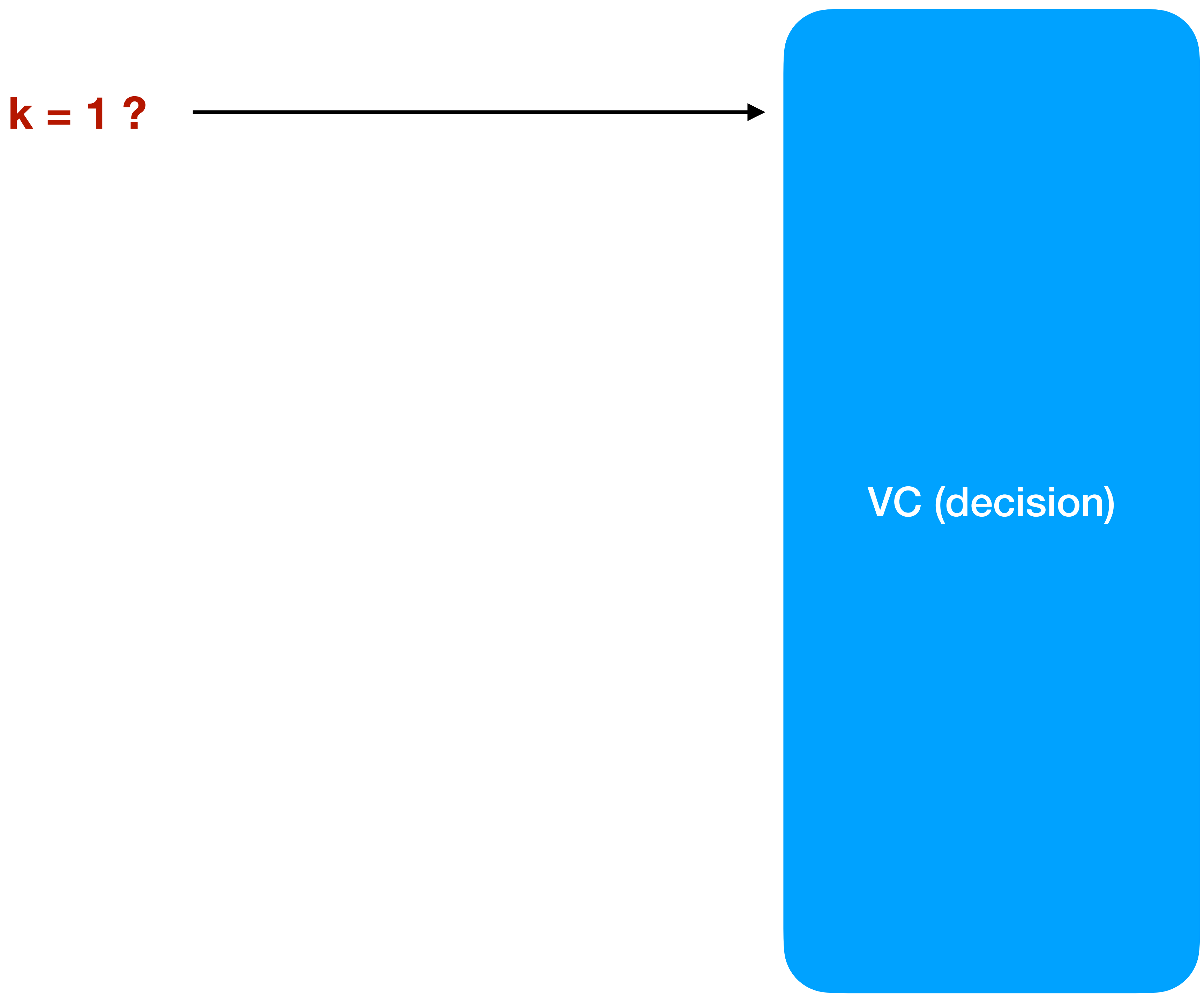

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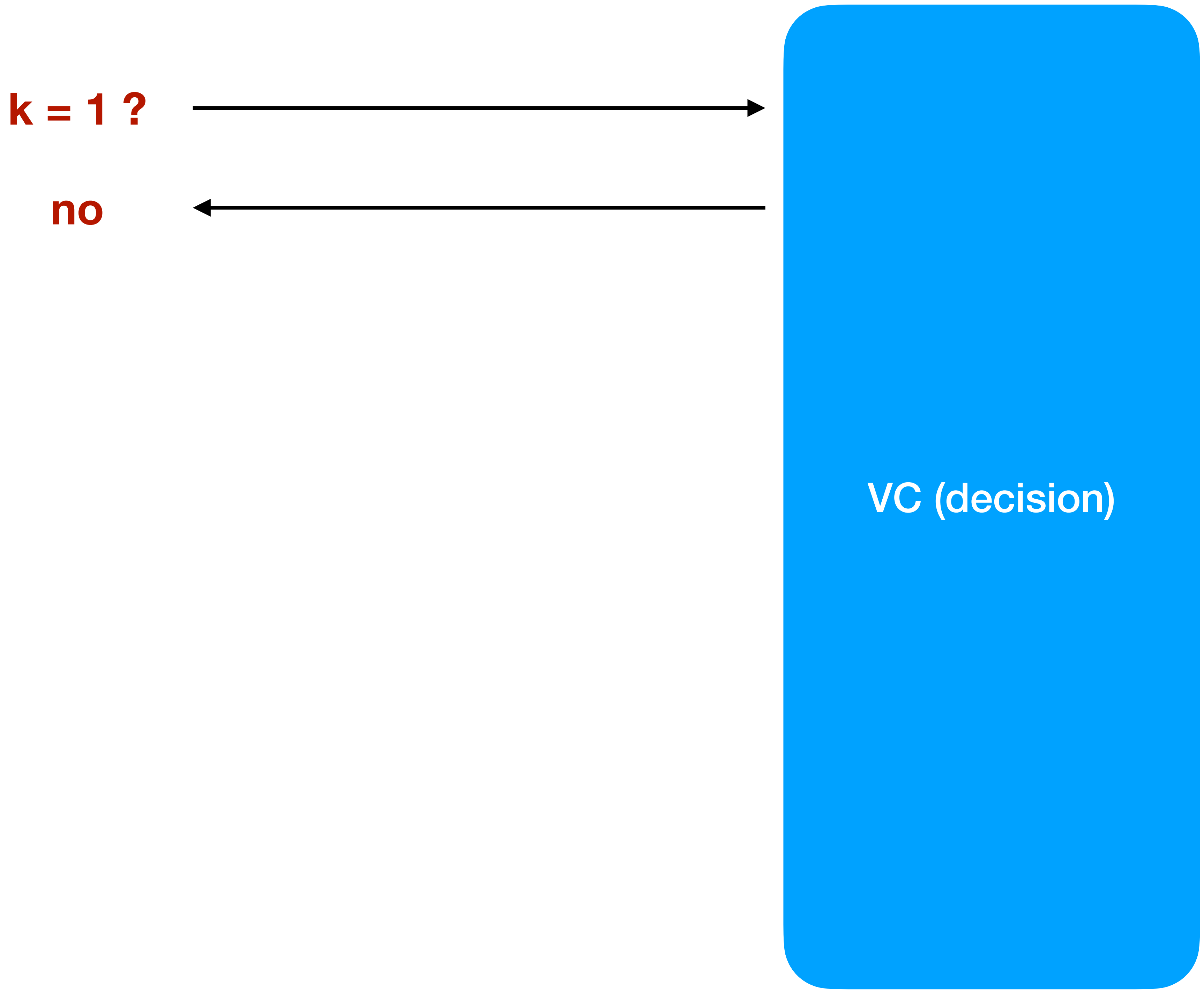

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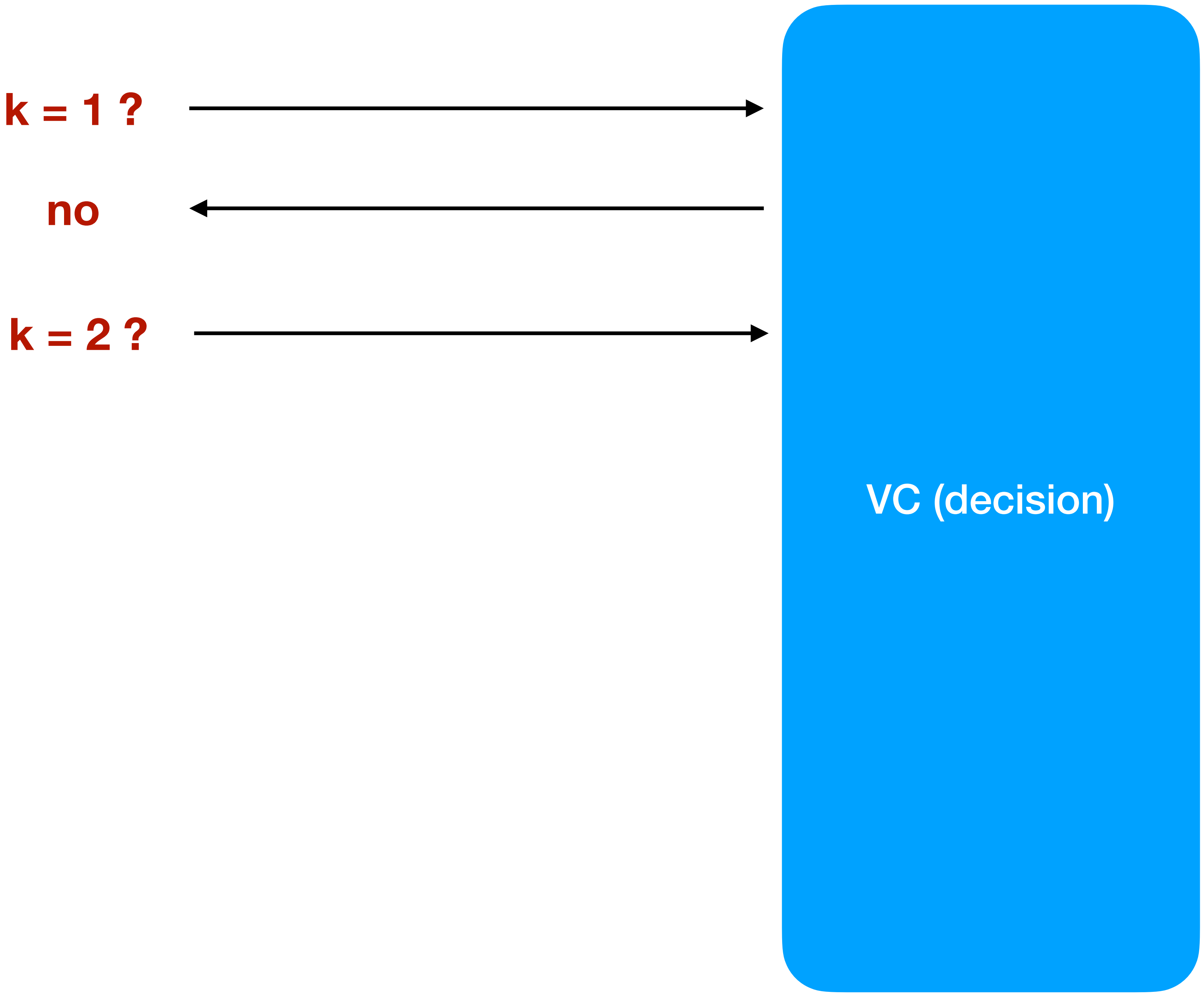

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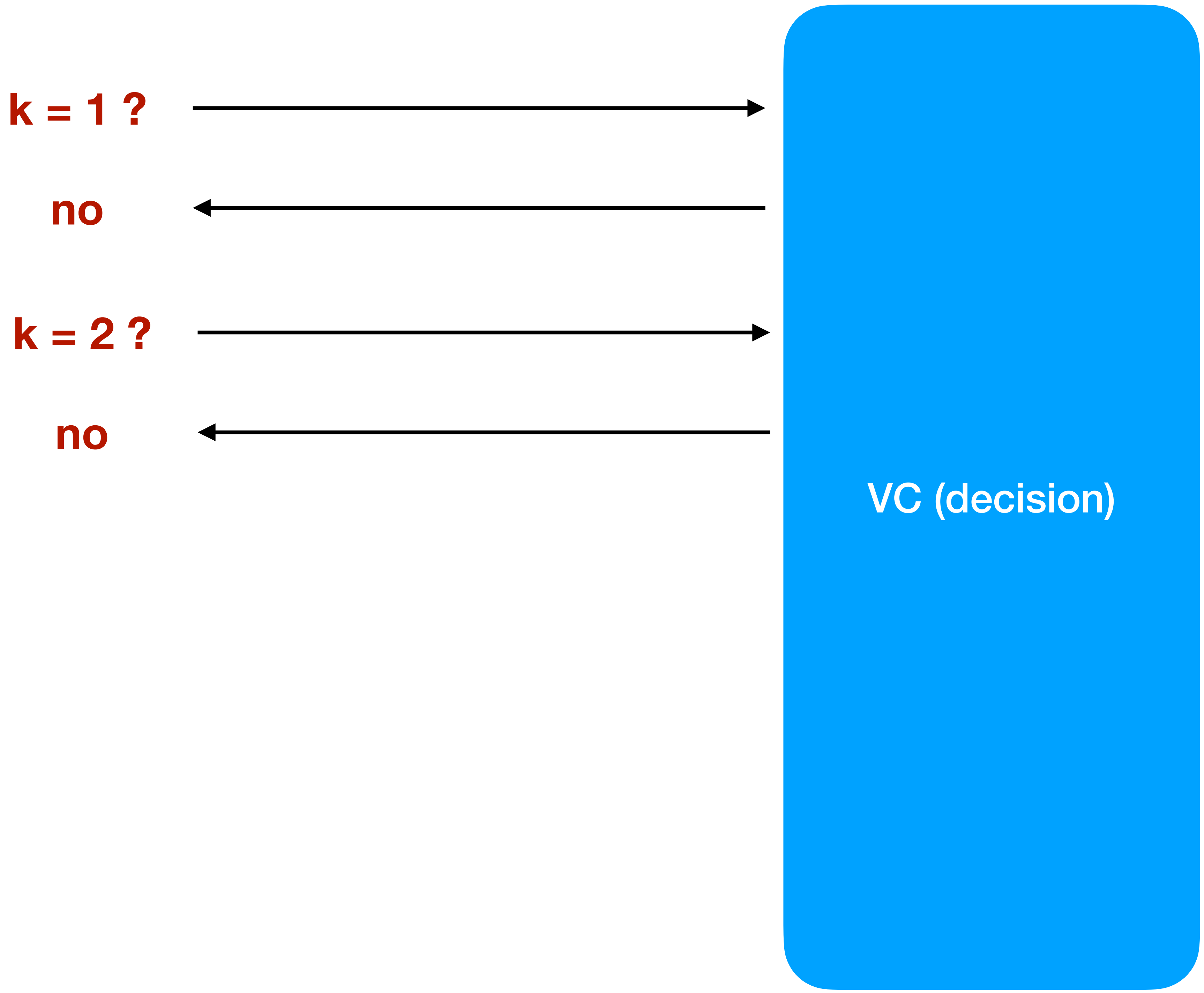

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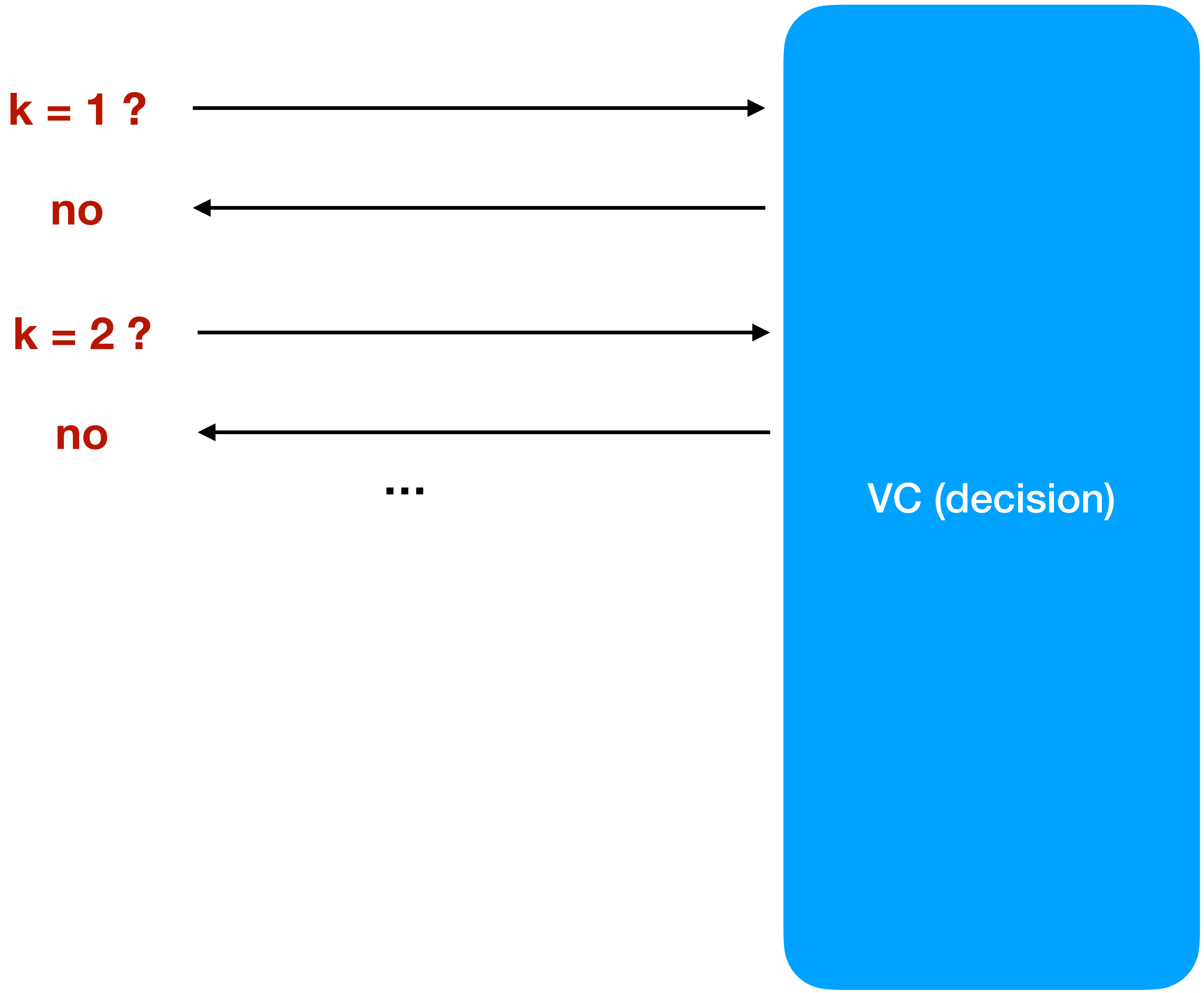

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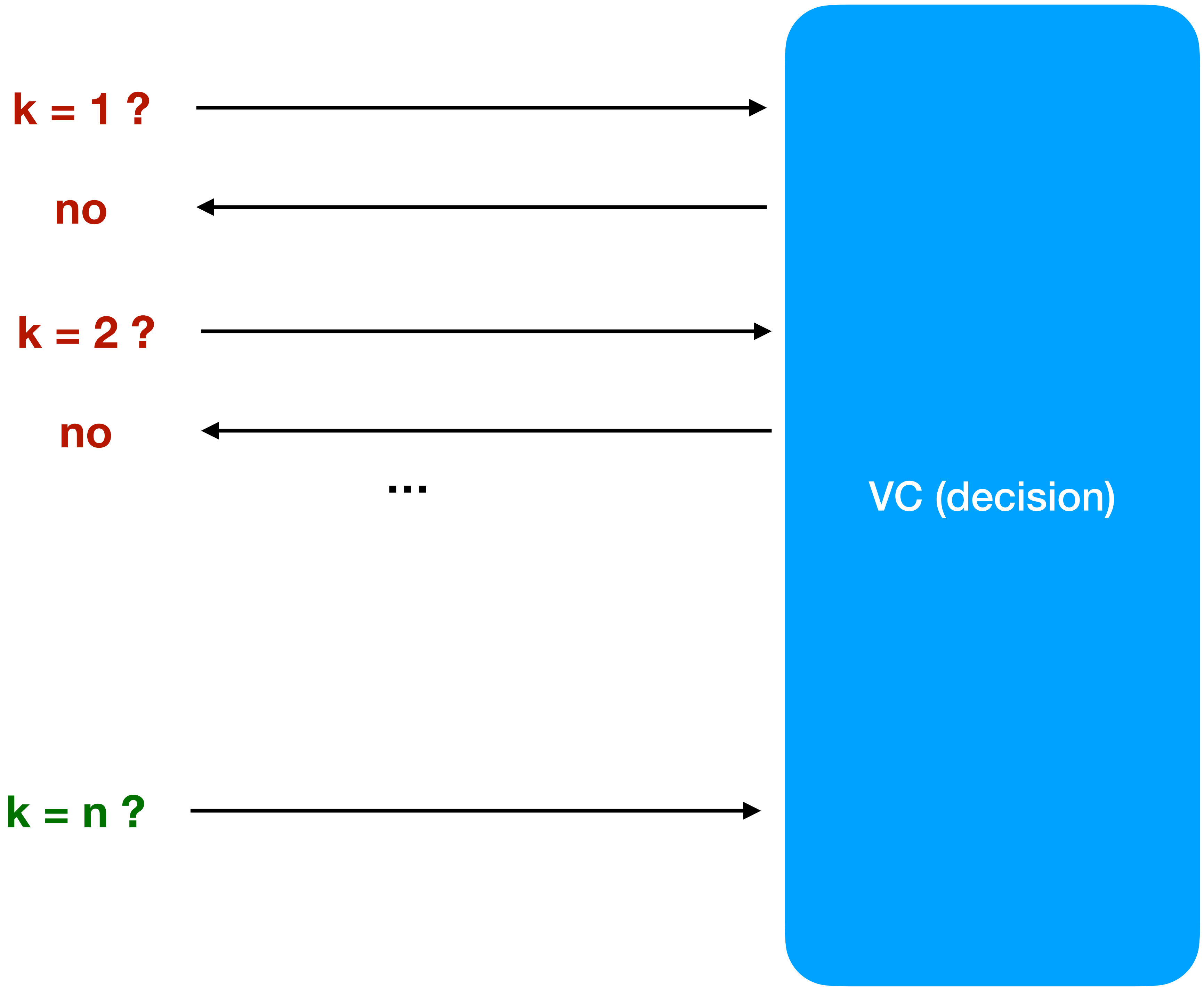

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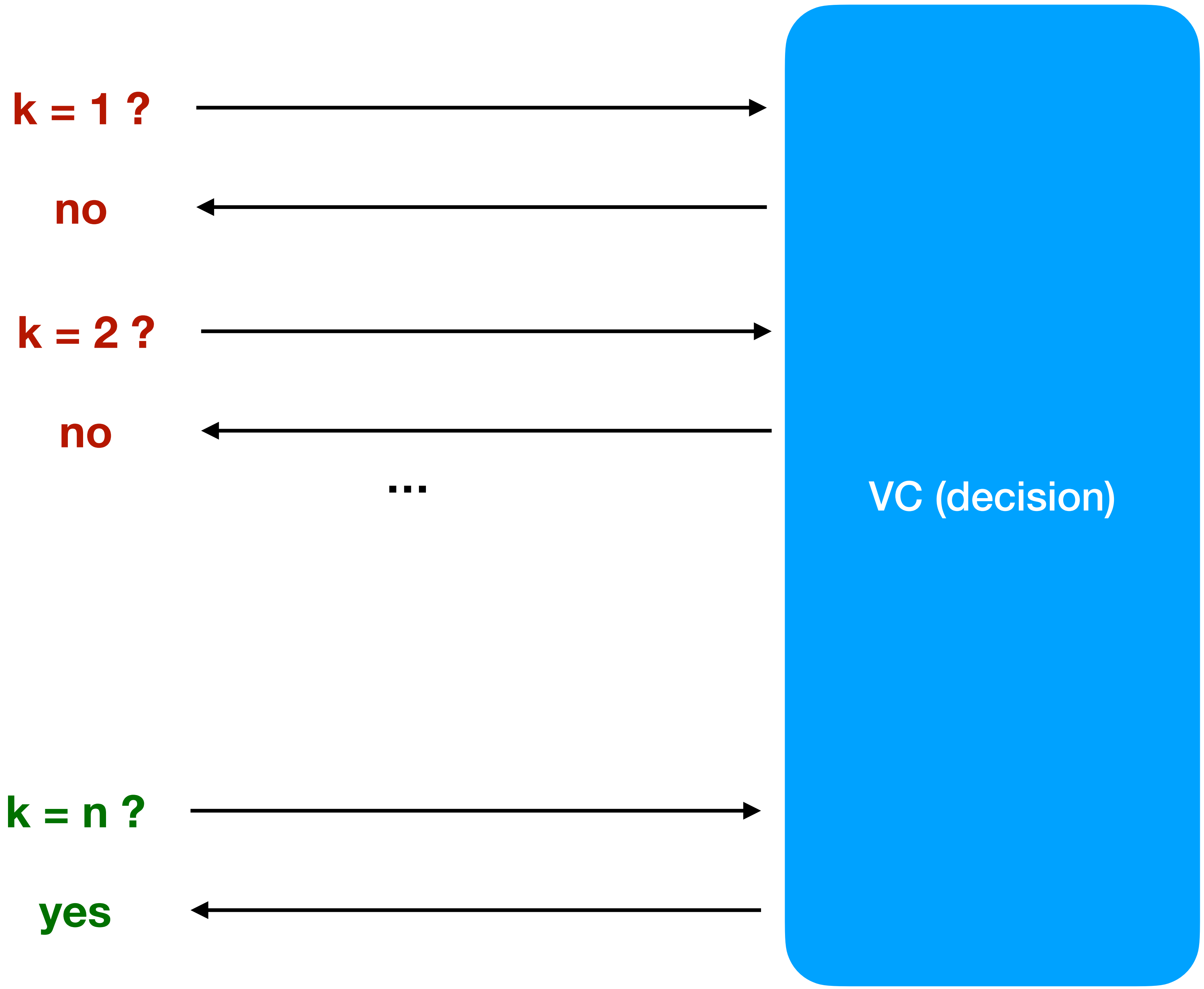

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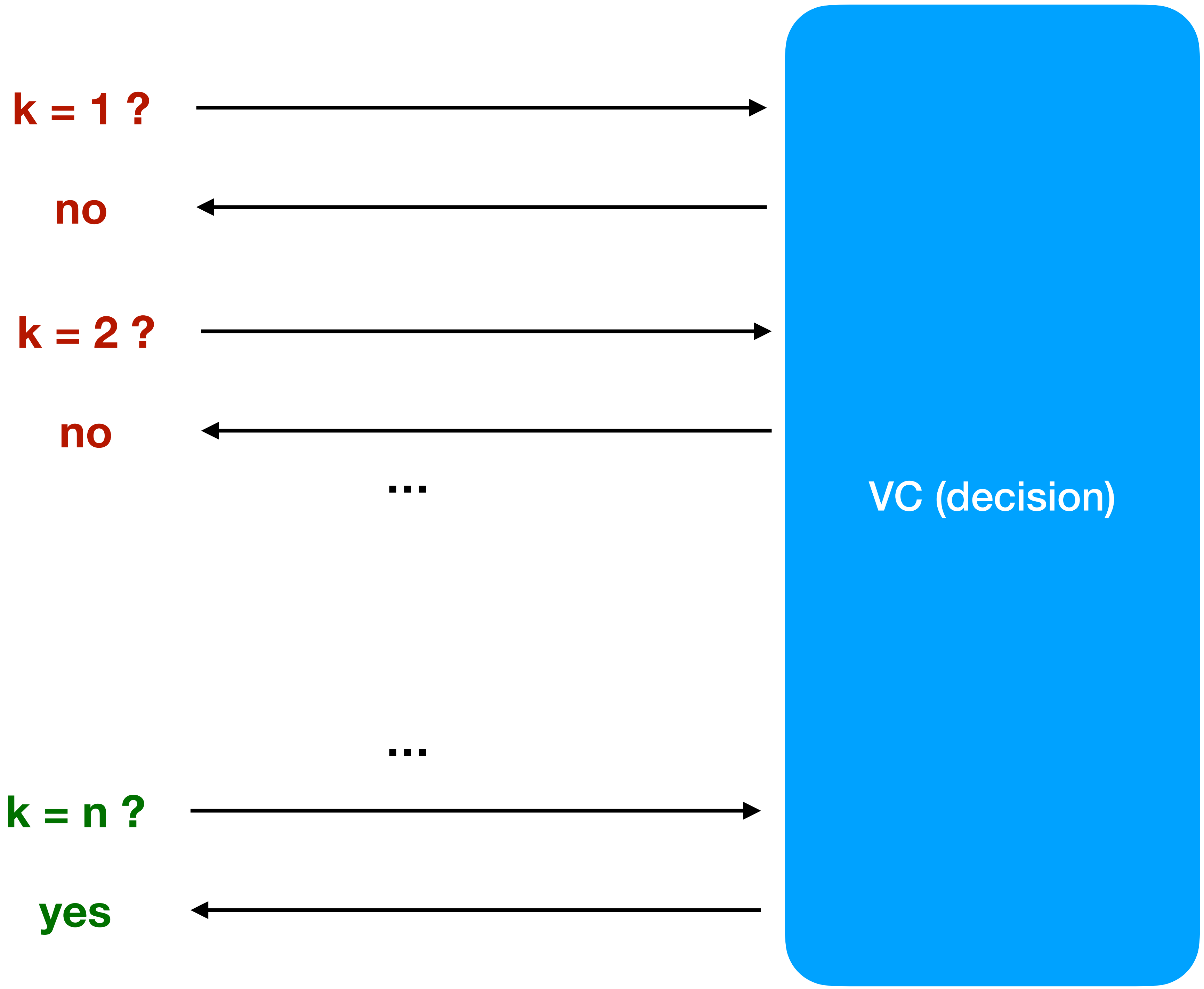

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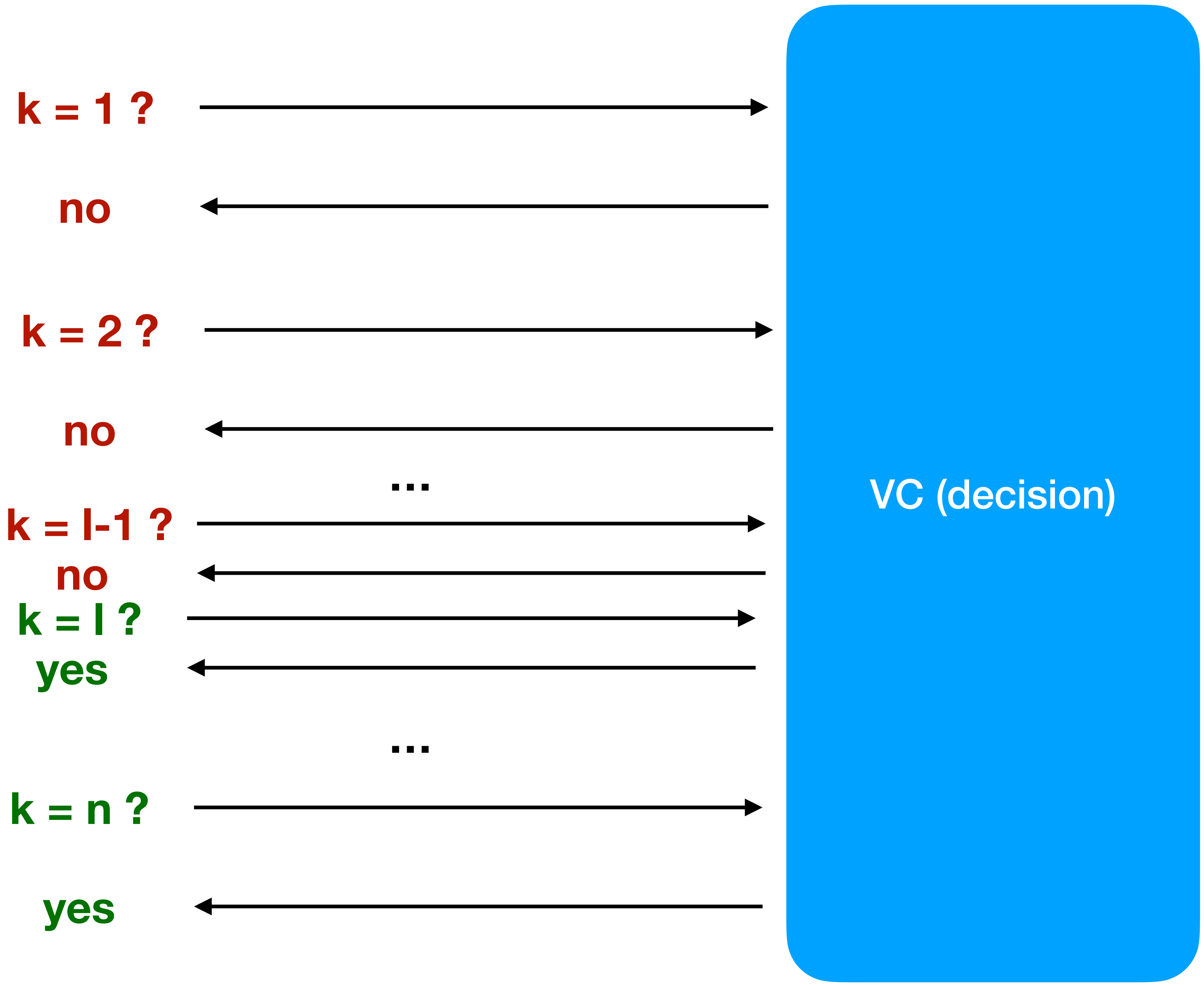

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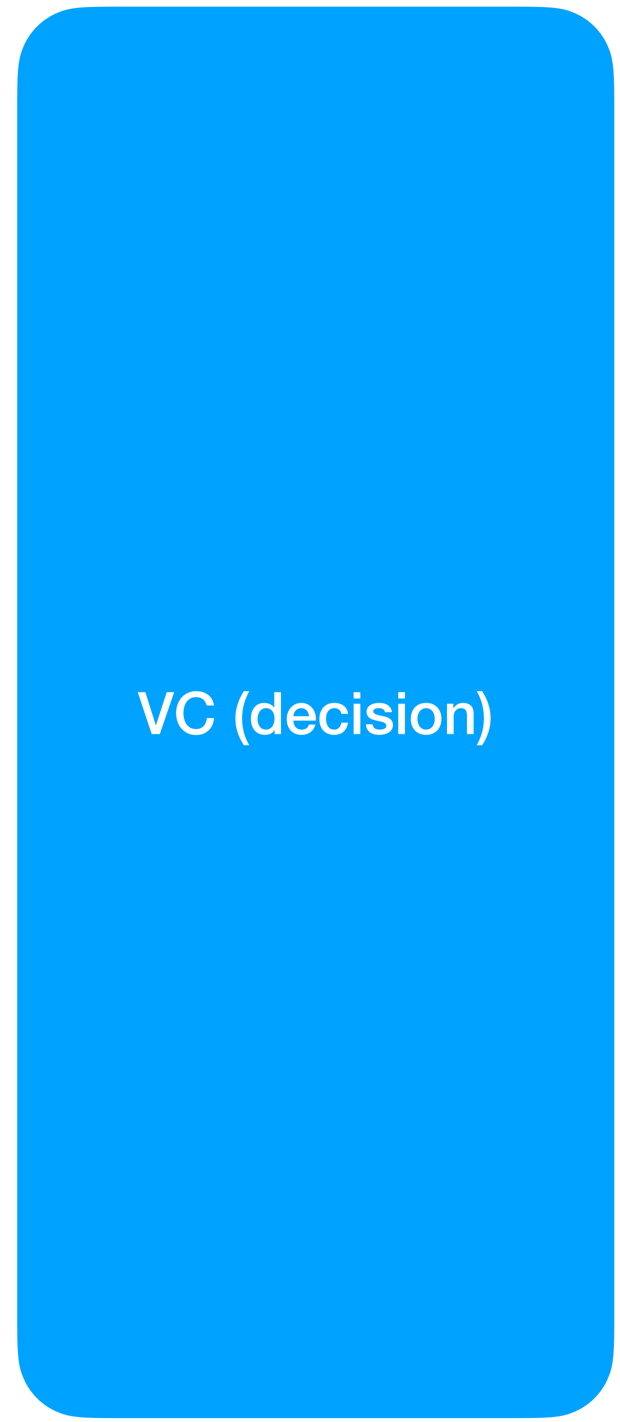

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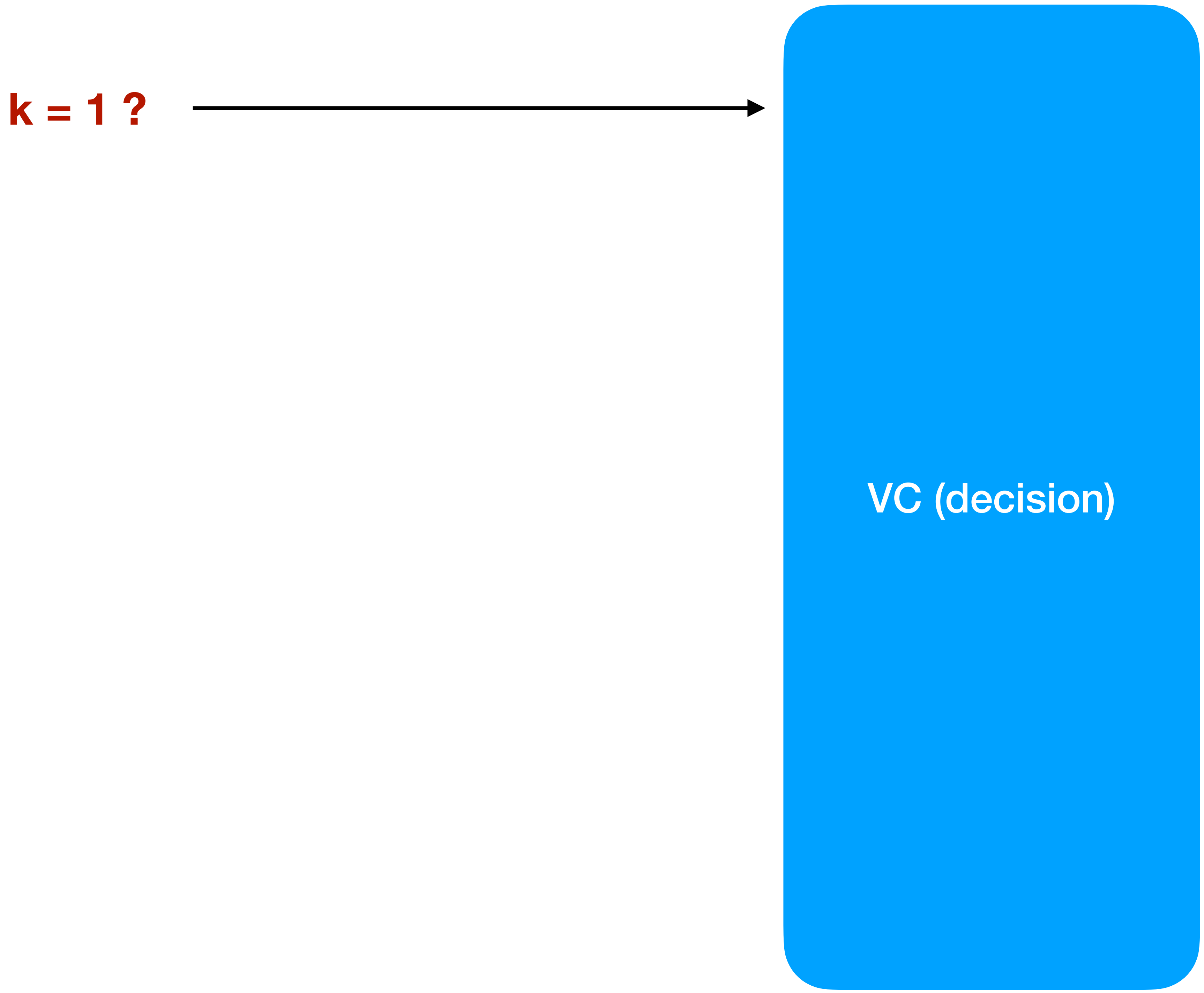

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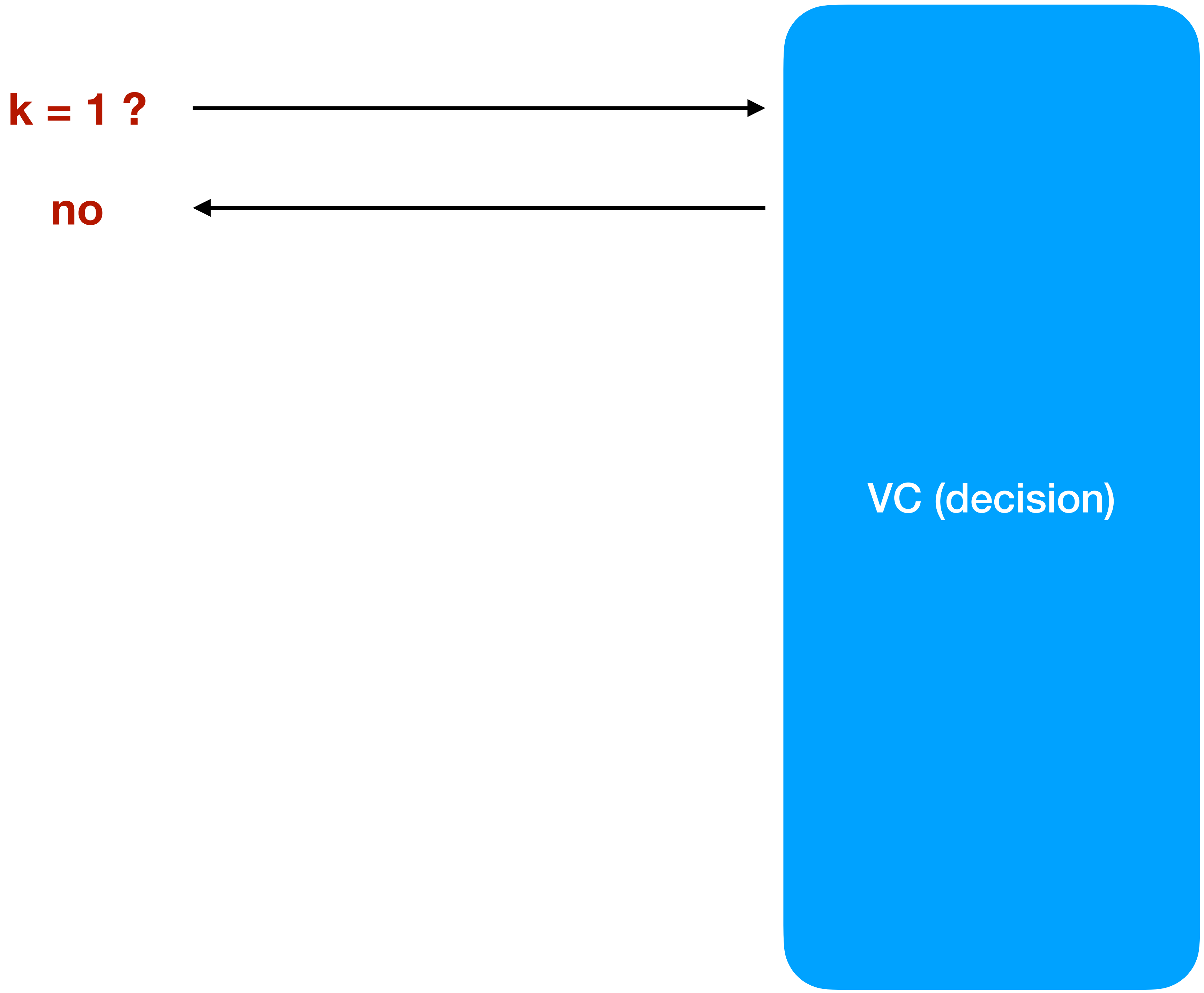

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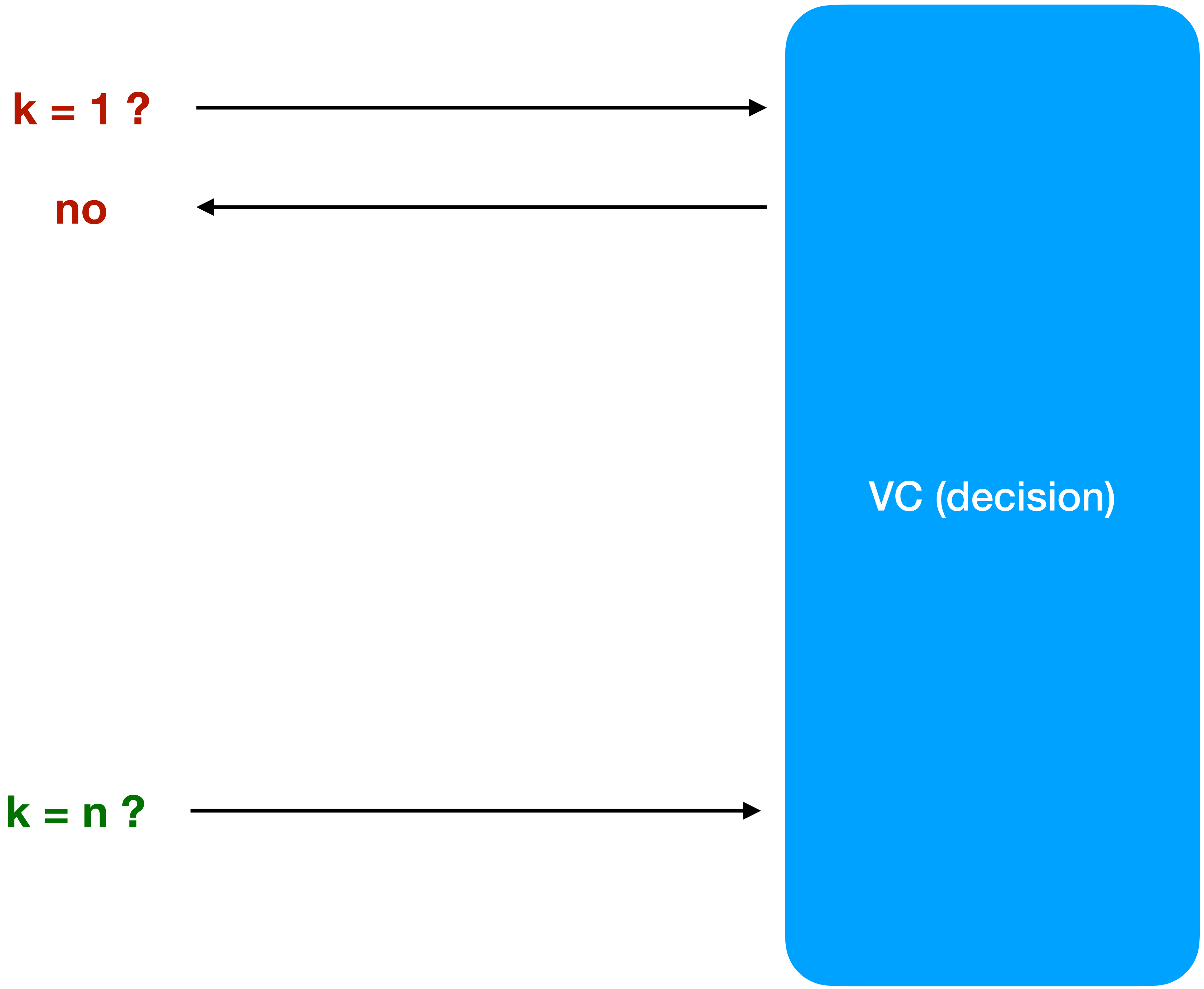

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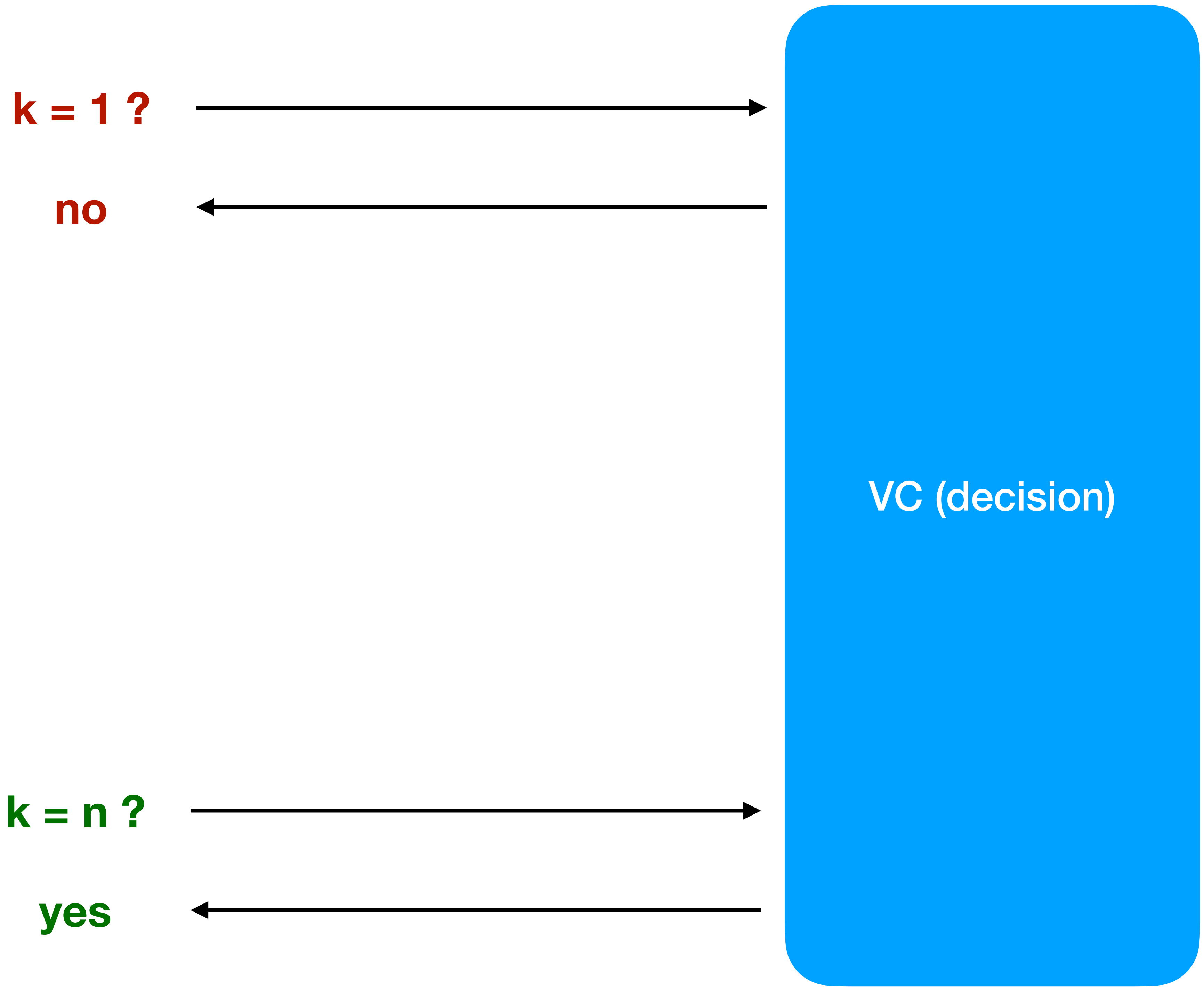

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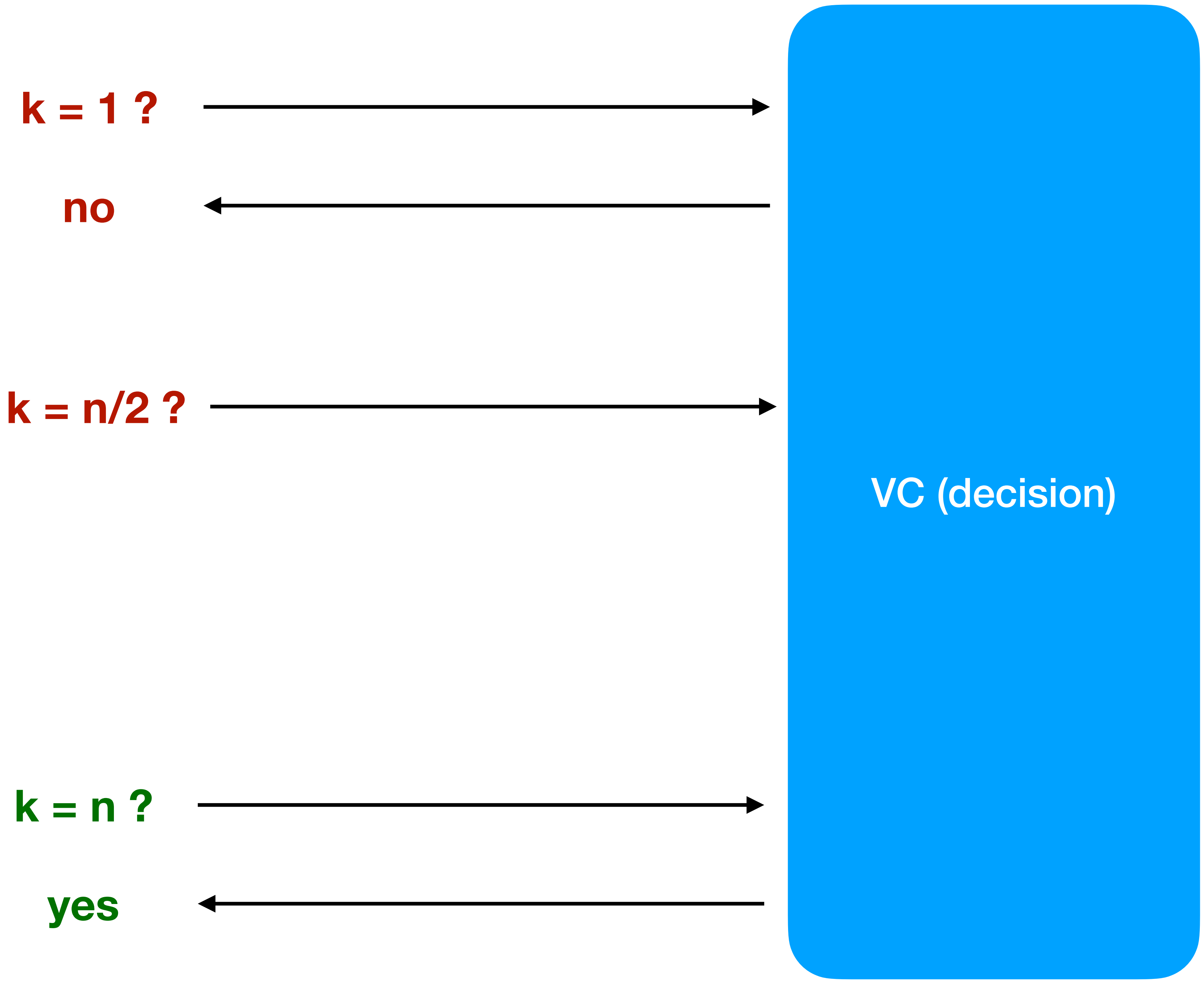

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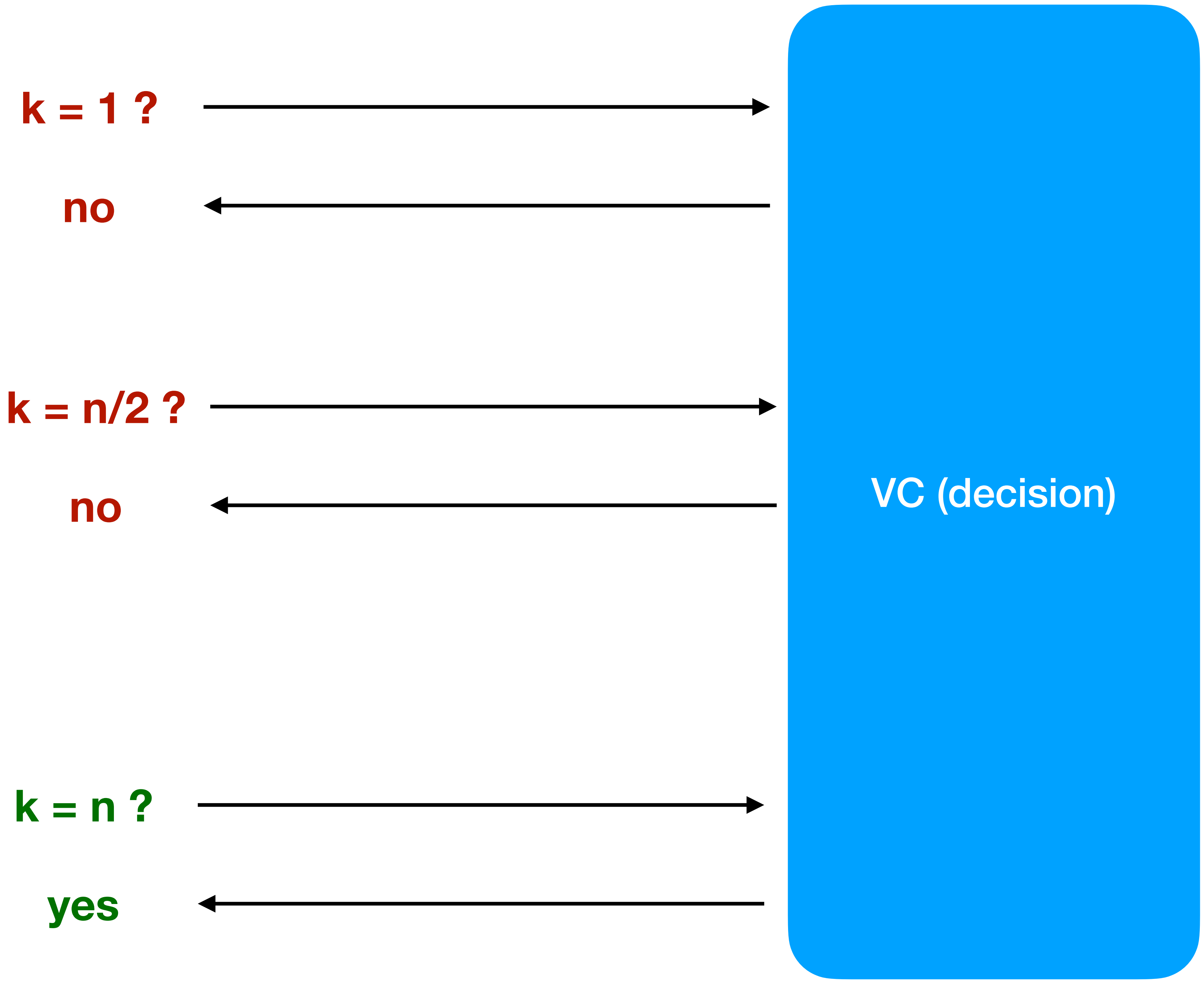

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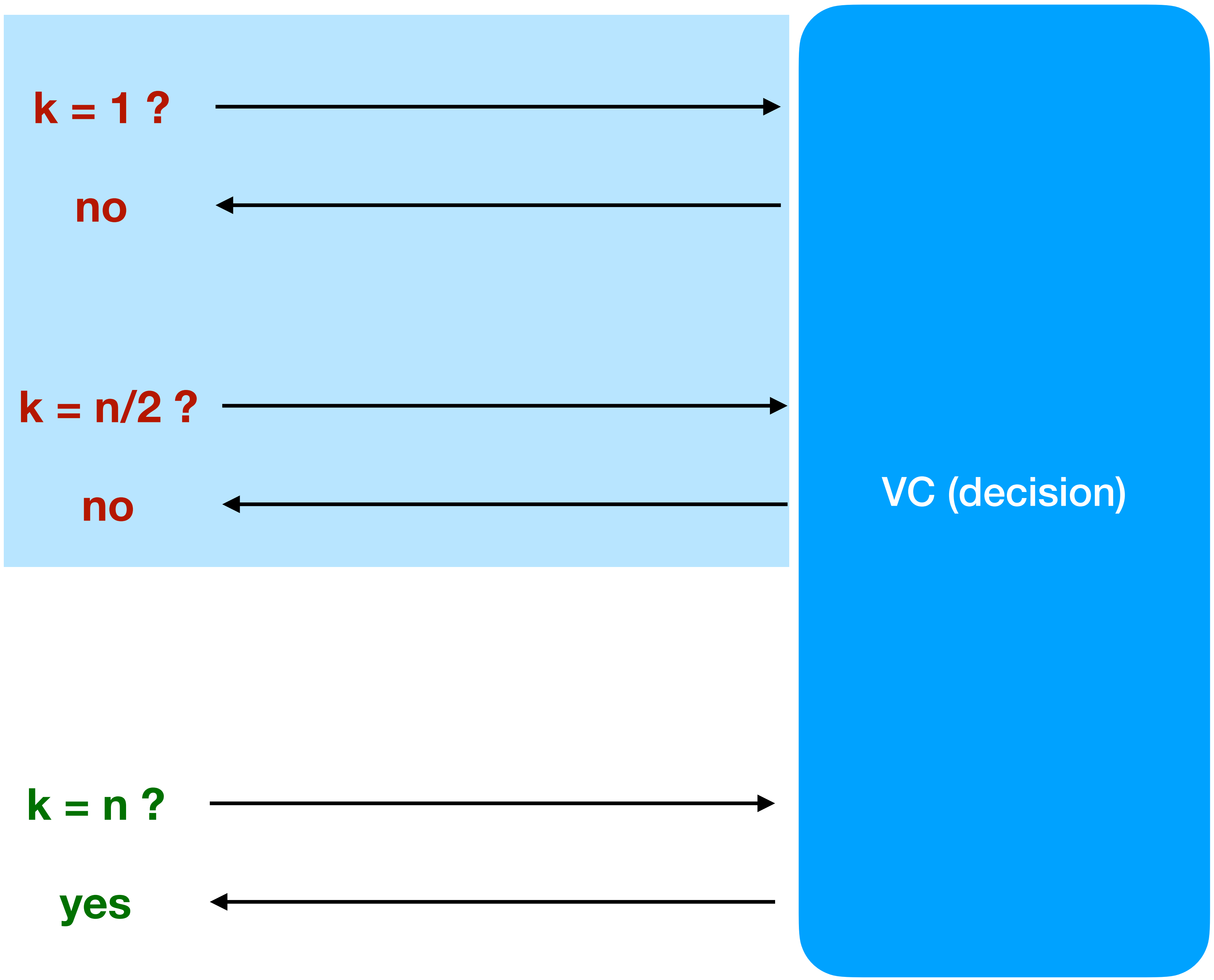

Optimisation vs decision

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    - Yes: Include v in the vertex cover.
    - No: Do not include v in the vertex cover.
    - Then move to the next vertex.

The subset sum problem

- We are given a set of n items {*1*, *2*, … , *n*}.
- Each item *i* has a non-negative integer weight w_i.
- We are given an integer bound W.
- Goal: Select a subset S of the items such that $\sum_{i \in S} w_i \leq W$

  and $\sum_{i \in S} w_i$ is maximised.

Equivalent formulation decision version

- We are given a set T of n items {*1*, *2*, … , *n*}.
- Each item *i* has a non-negative integer weight w_i.
- We are given an integer bound W.
- Goal: Decide if there exists a subset S of the items such that

$$\sum_{i \in S} w_i = W$$

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We did this for VC. Can we also do it for SS?