Introduction to Algorithms and Data Structures

Greedy Approximation Algorithms

NP-hardness

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NP-hardness is a worst-case impossibility

- Sometimes we can provably design polynomial algorithms on certain *input structures*.

- For example, a minimum Vertex Cover on *trees* can be found in polynomial time using Dynamic Programming.

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- What if the instances to our problem do not have any good input structure?

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  - Compute a solution that is "*close*" to the optimal.

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- How do we make such an argument, if we cannot really find the optimal?
- How do we know if our algorithm is the best possible? Can we get “*closer*” to the optimal?

Methods for approximation algorithms

- Greedy algorithms
- Pricing method (also known as the Primal-Dual method)
- Linear Programming and Rounding
- Dynamic Programming on rounded inputs

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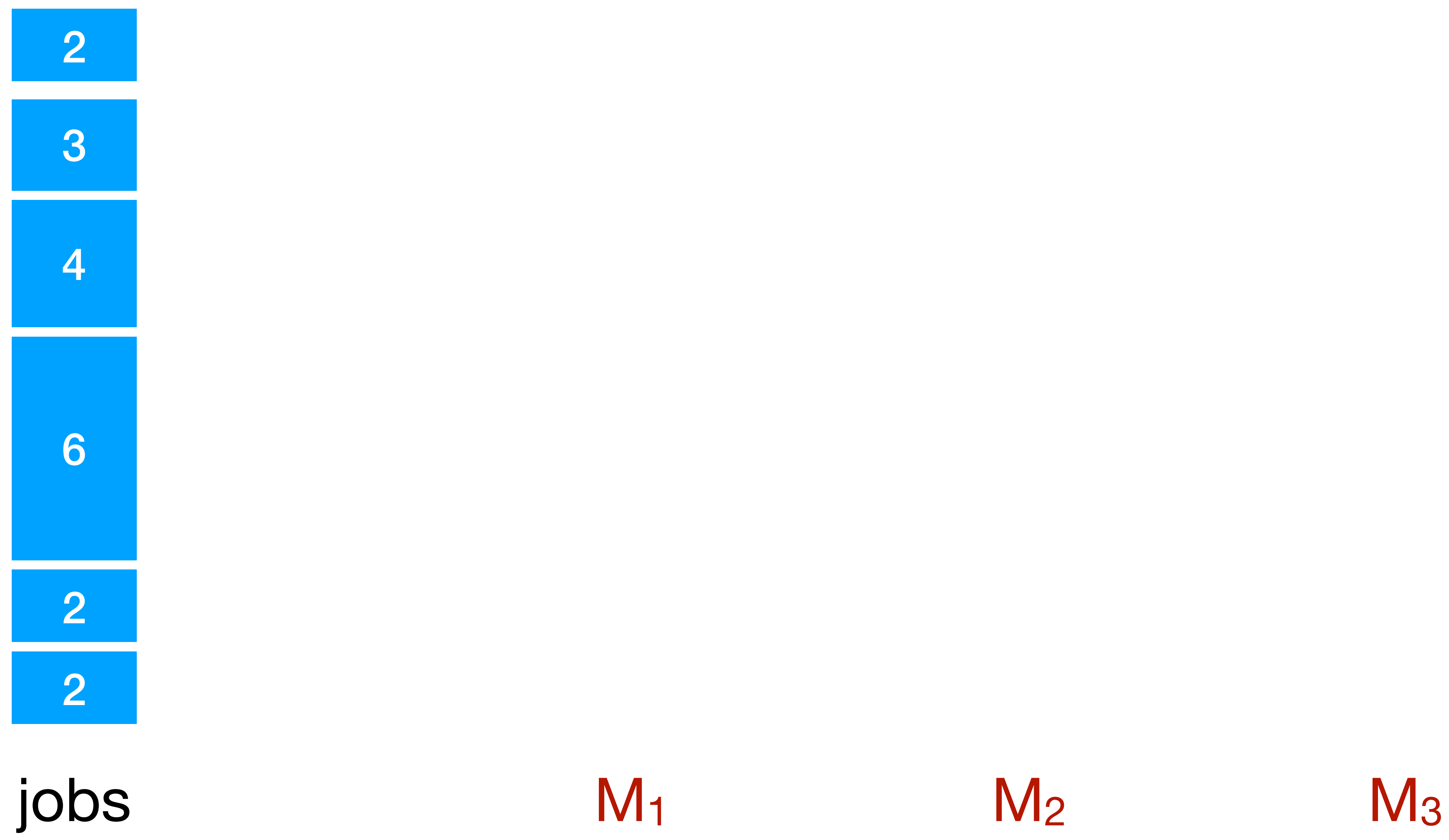
- Let $A(i)$ be the set of jobs assigned to machine i.
- The load of machine i is $T_i = \sum_{j \in A(i)} t_j$
- The goal is to minimise the *makespan*, i.e.,

$$T = \max_{i \in M} T_i$$

Example



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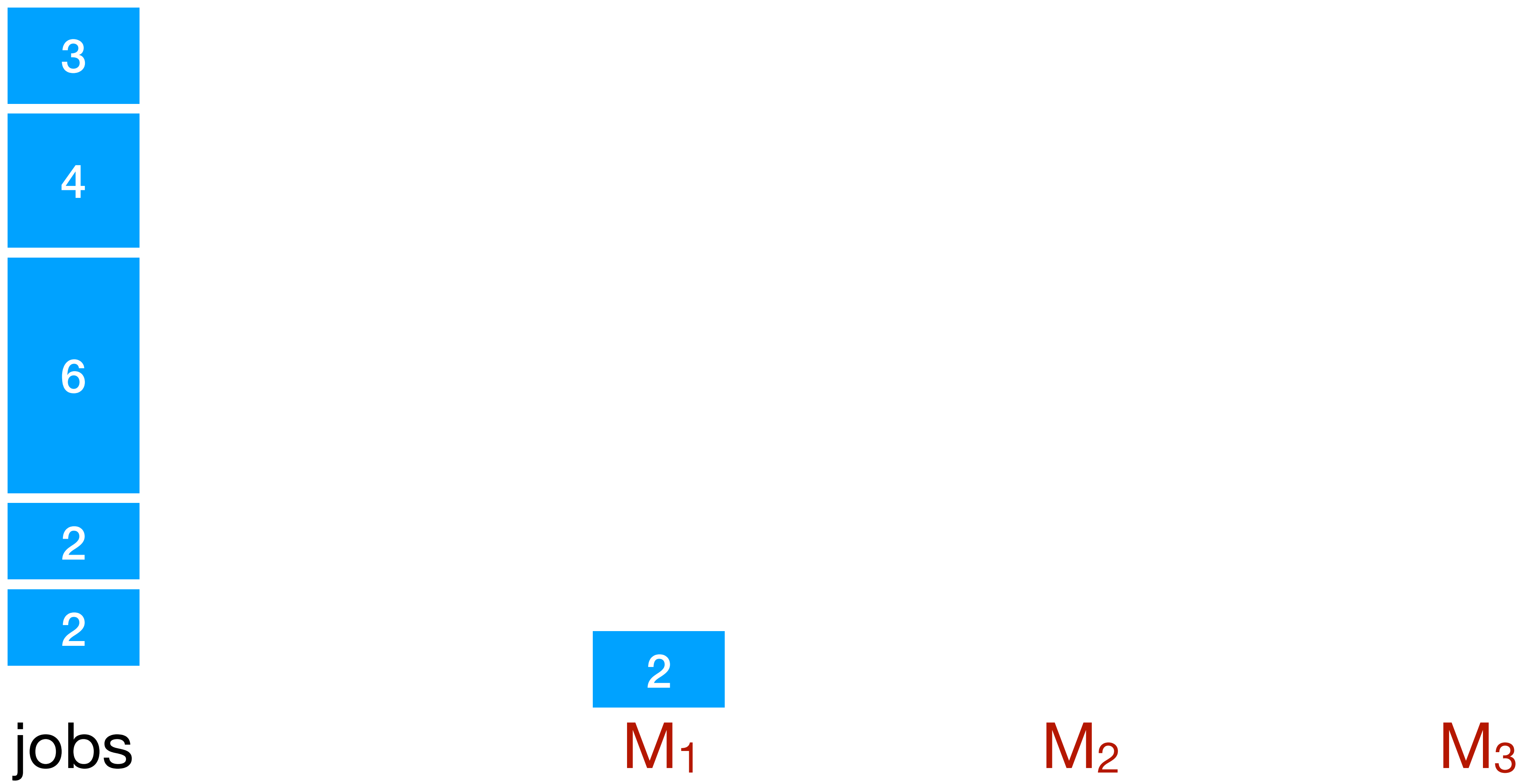

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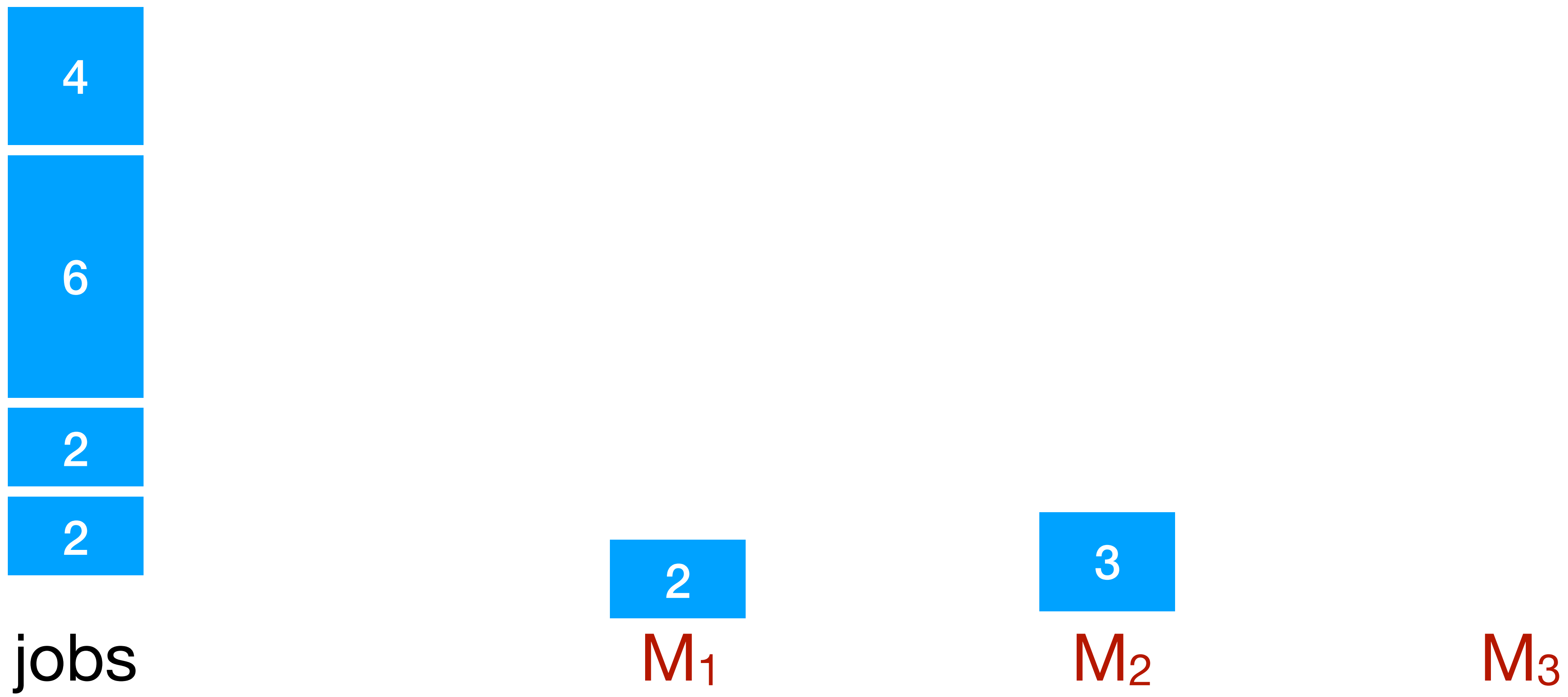

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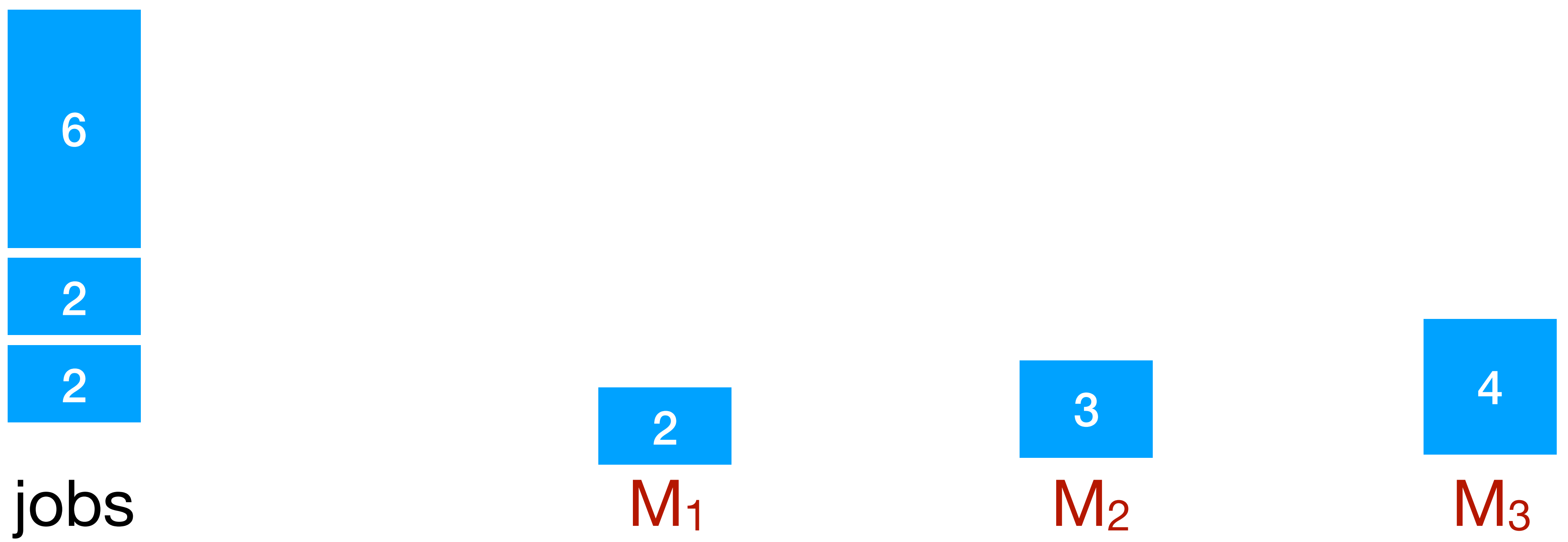

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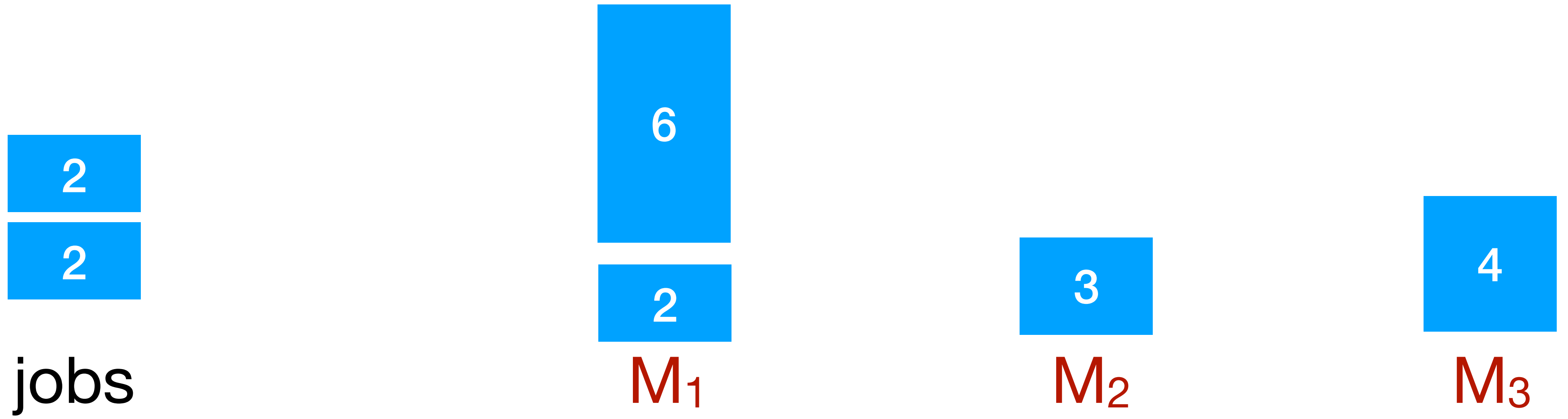

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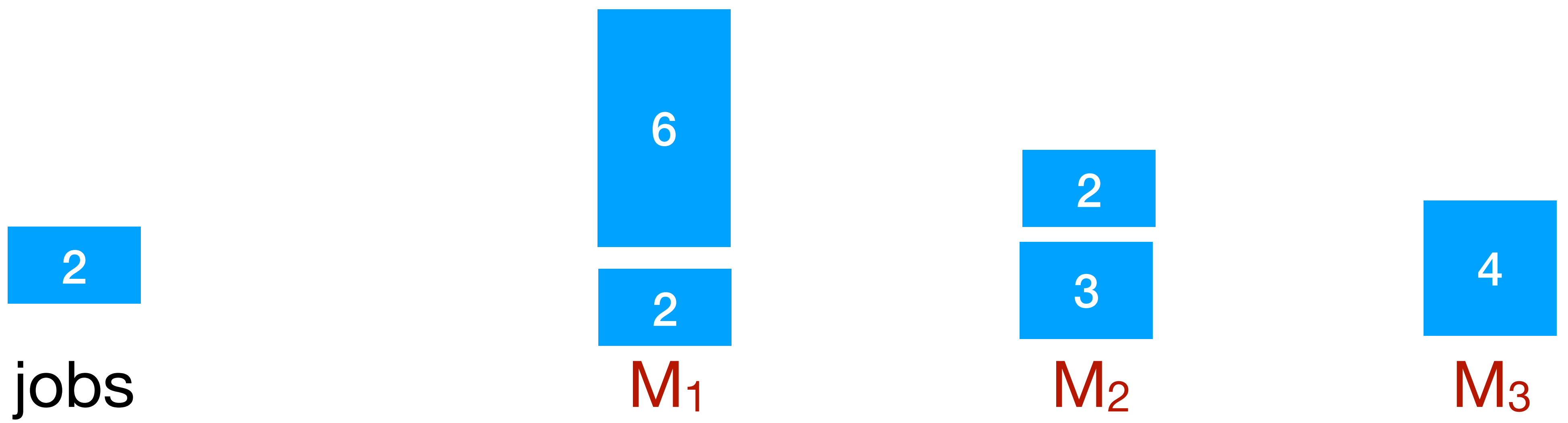

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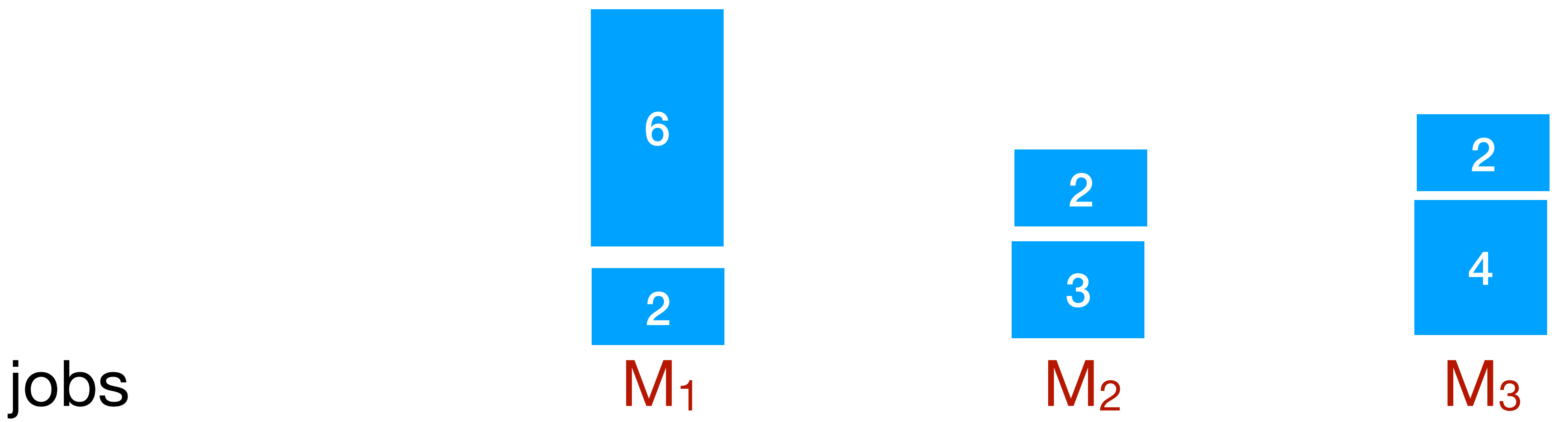

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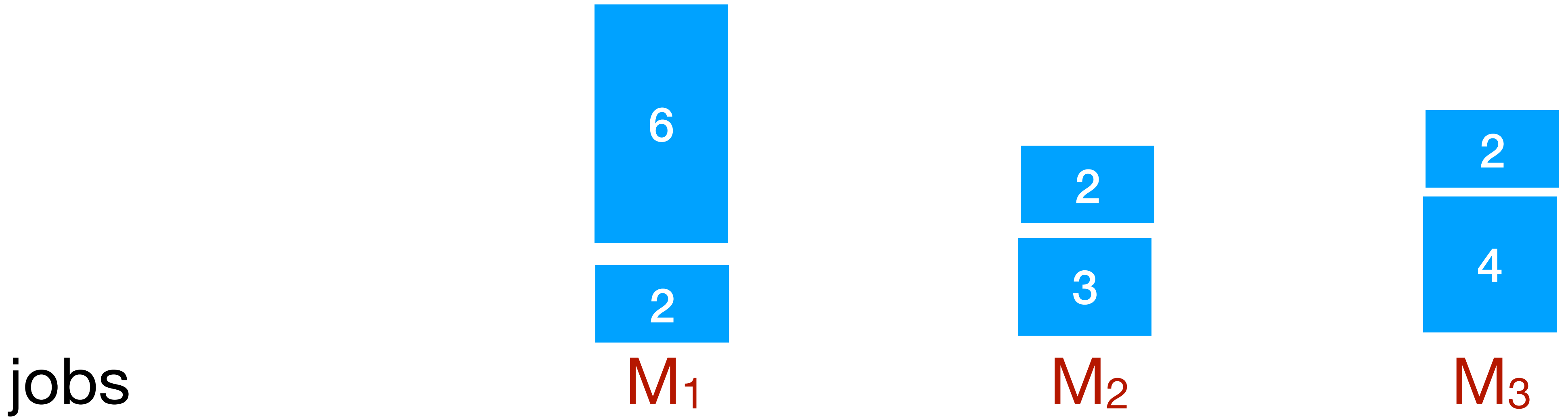

makespan = 8

Load Balancing

- The load balancing problem on identical machines is NP-hard.
- We will design greedy approximation algorithms for it.

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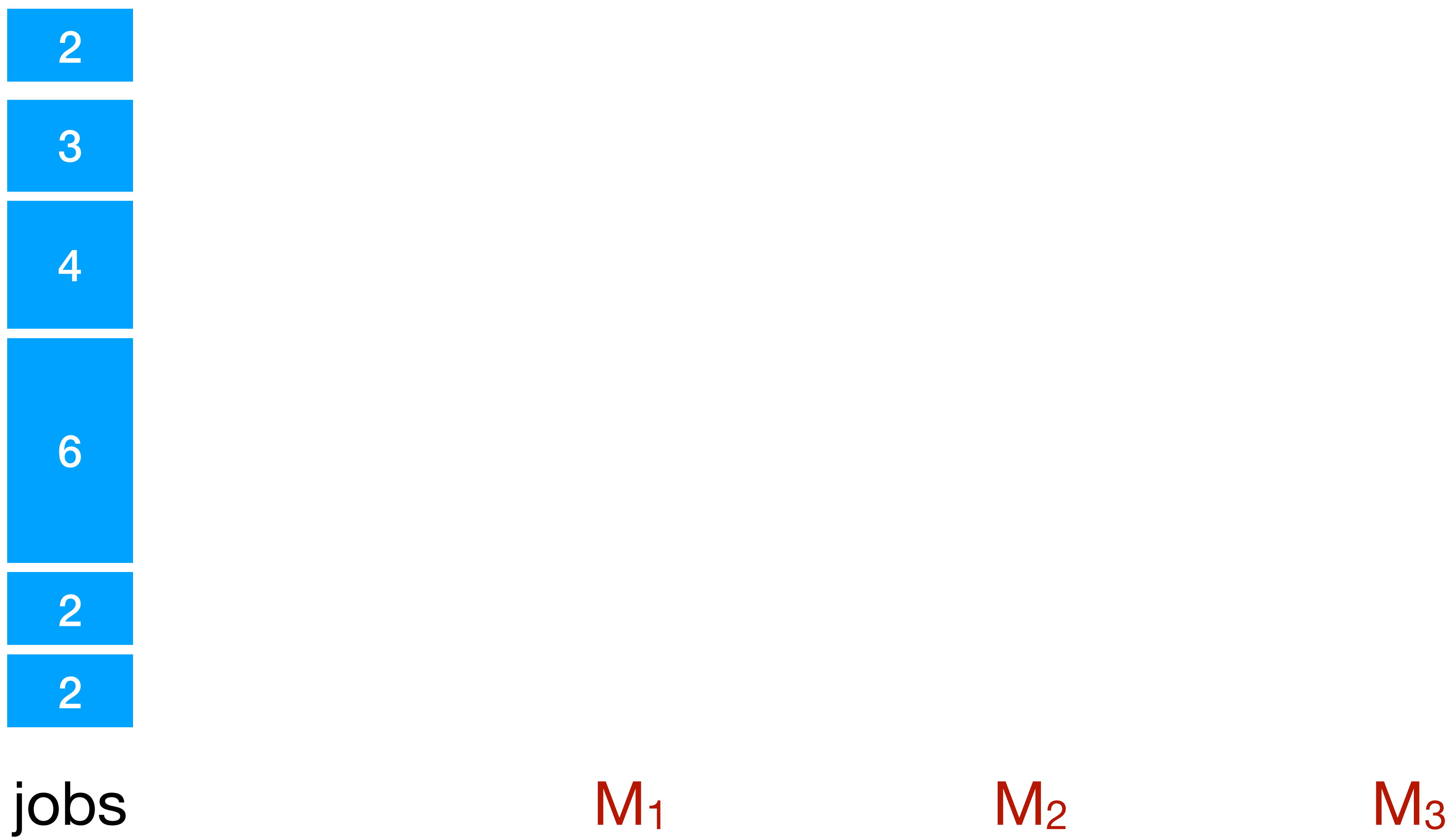
Algorithm **Greedy-Balance**

Start with no jobs assigned
Set $T_i = 0$ and $A(i) = \varnothing$ for all machines M_i
For $j = 1, \ldots, n$
    Let M_i be the machine that achieves the minimum $\min_k T_k$
    Assign job j to machine M_i
    Set $A(i) = A(i) \cup \{ j \}$
    Set $T_i = T_i + t_j$
EndFor

Example



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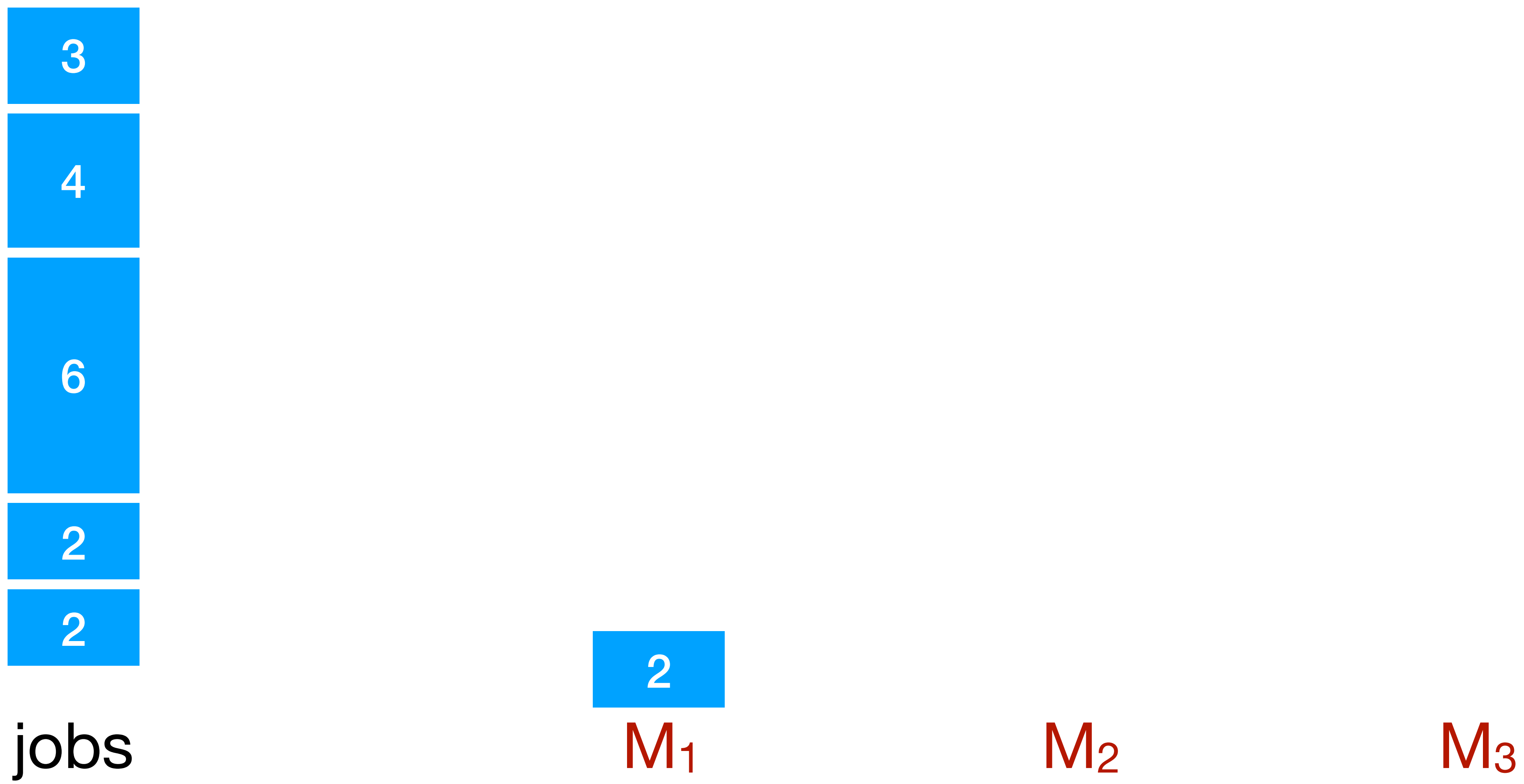

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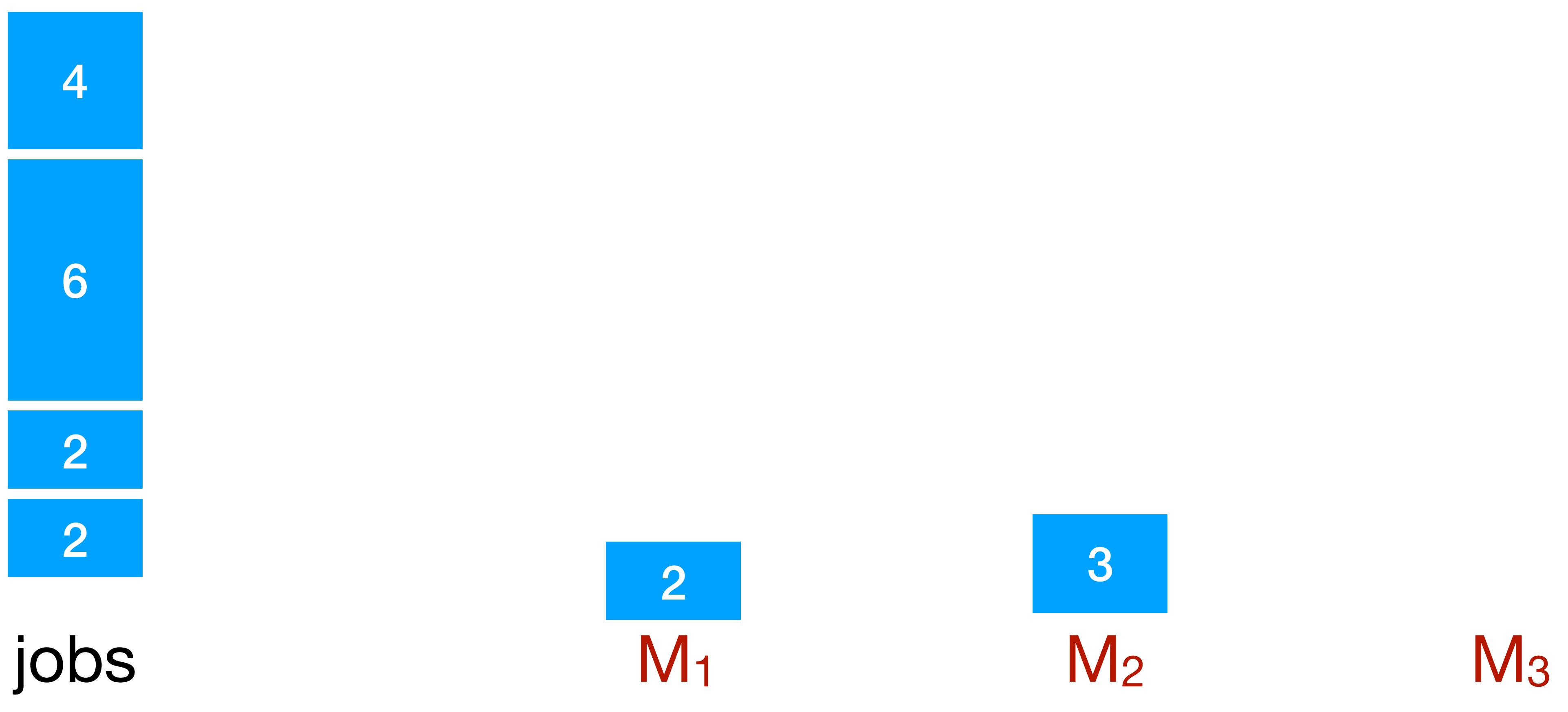

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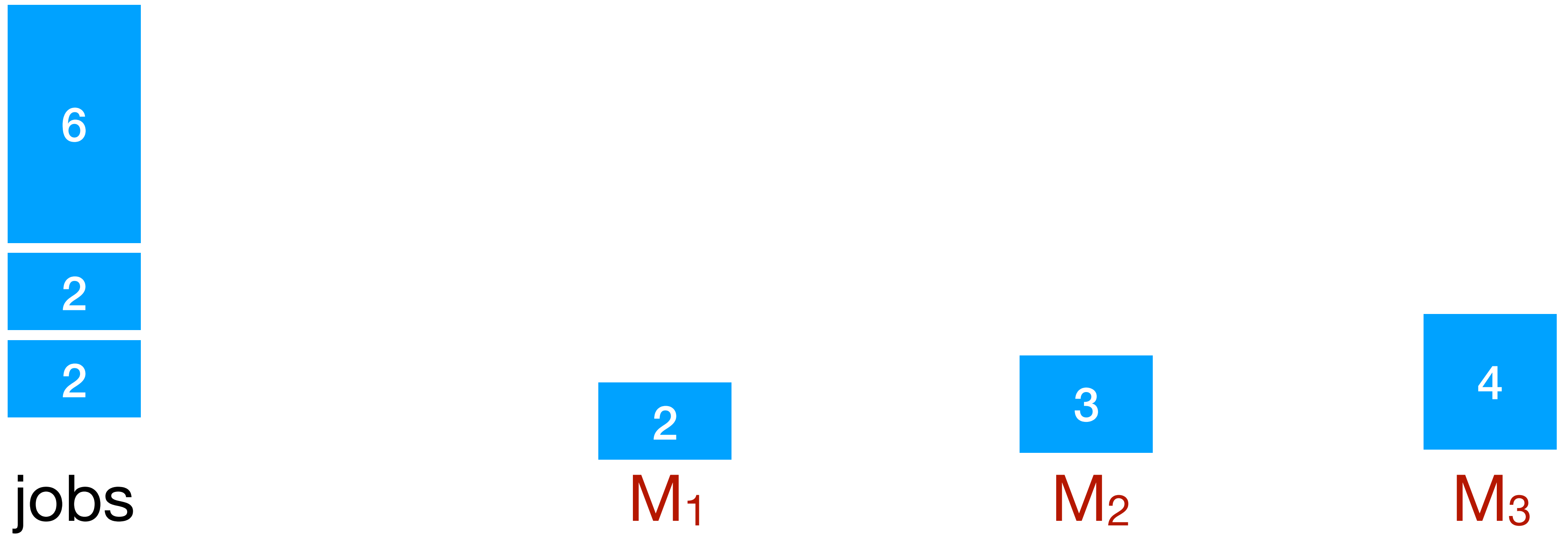

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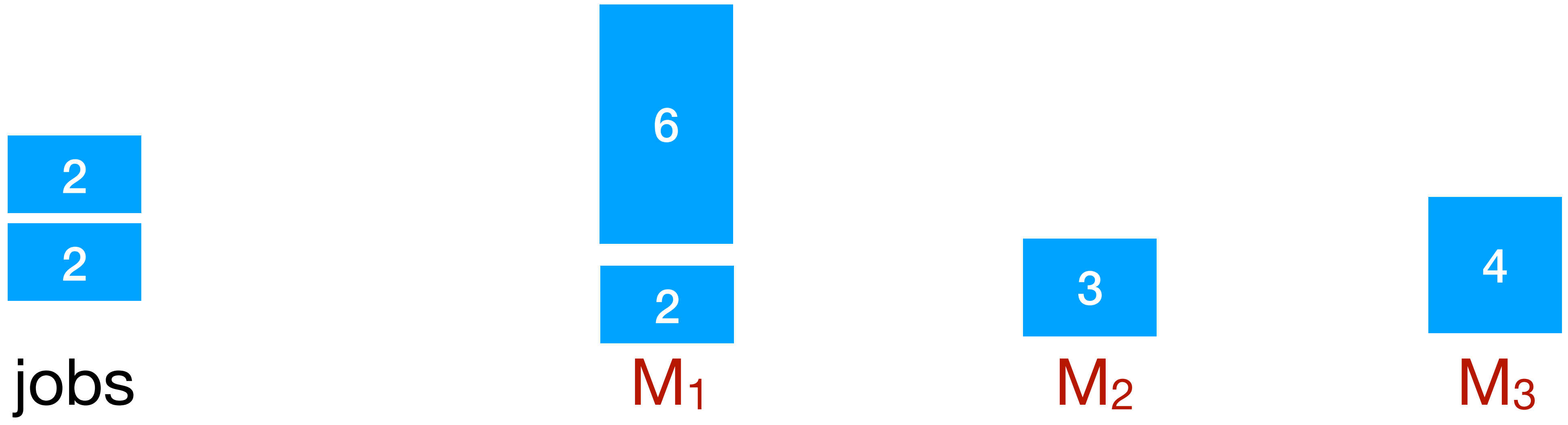

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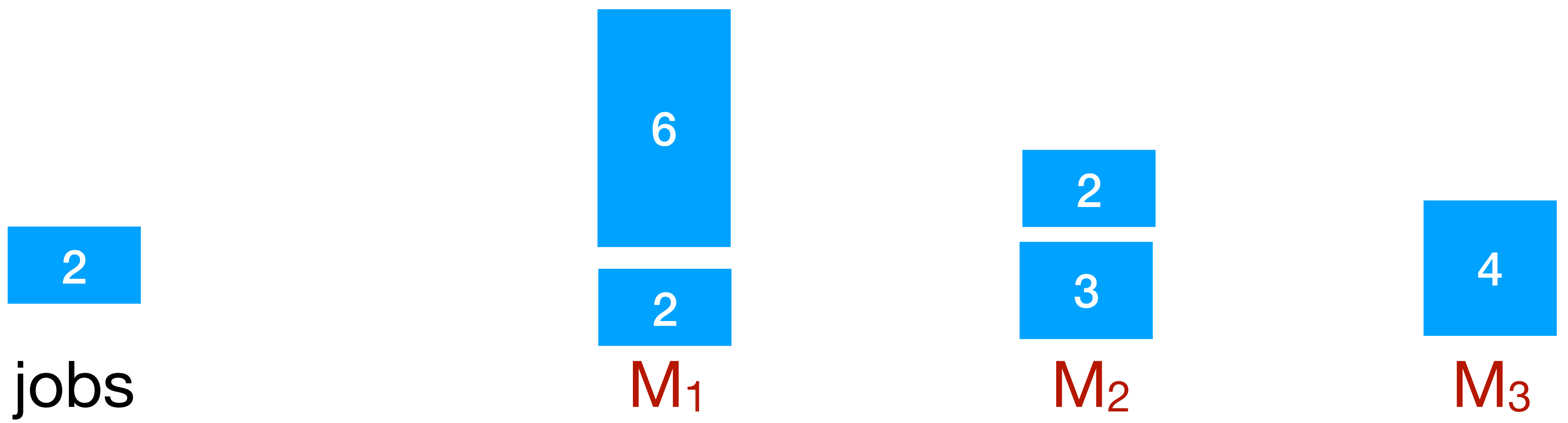

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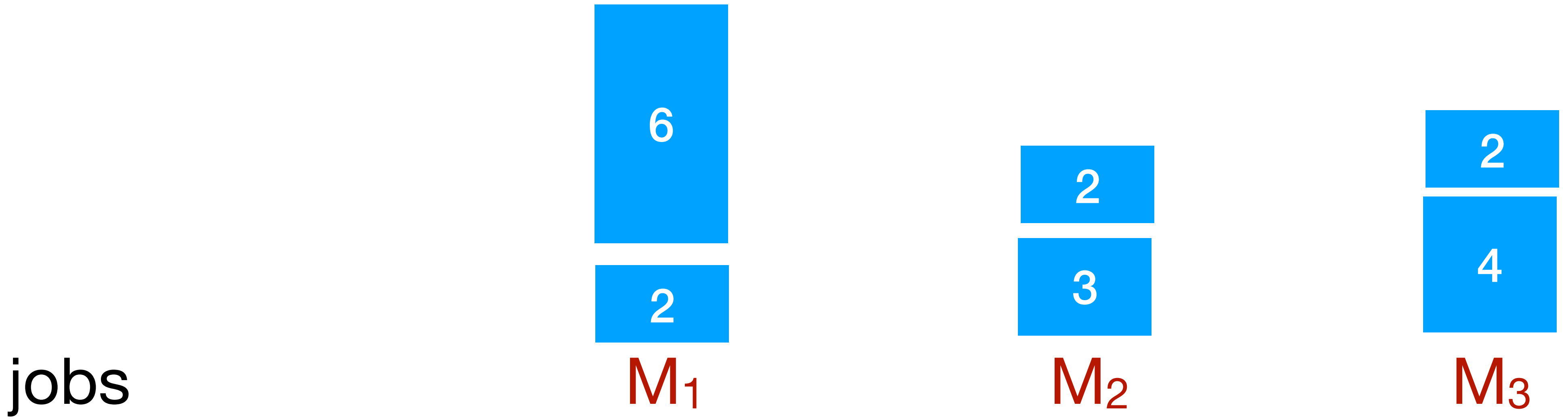

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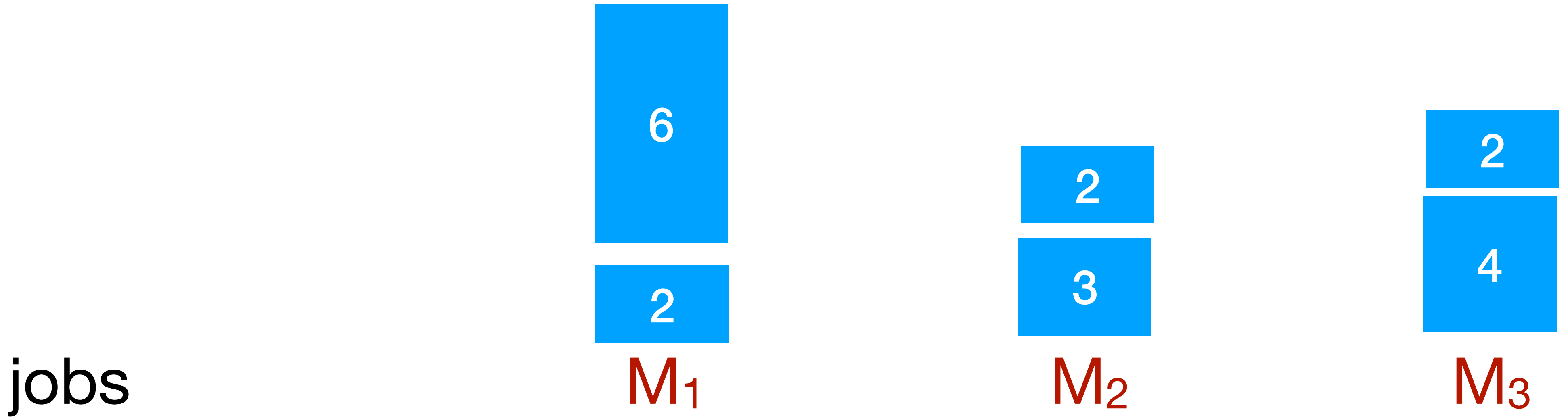

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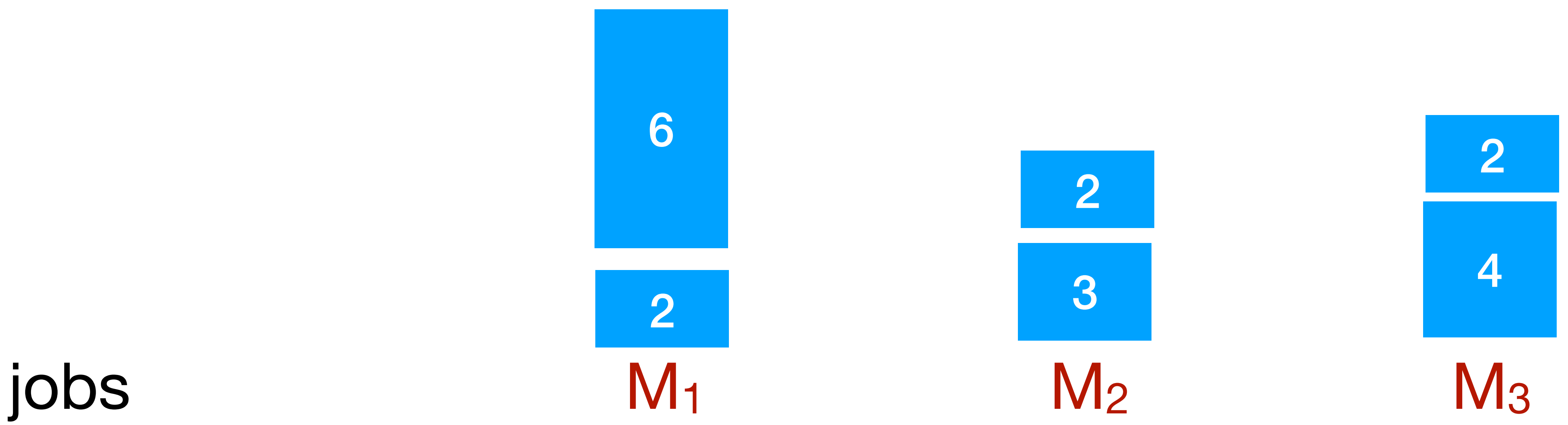

makespan = 8
A makespan of 7 is possible

Notation

- Let T be the makespan achieved by **Greedy-Balance**.
- Let T* be the optimal makespan.

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    - Bounding the optimal from below (for minimisation problems) and from above (for maximisation problems).

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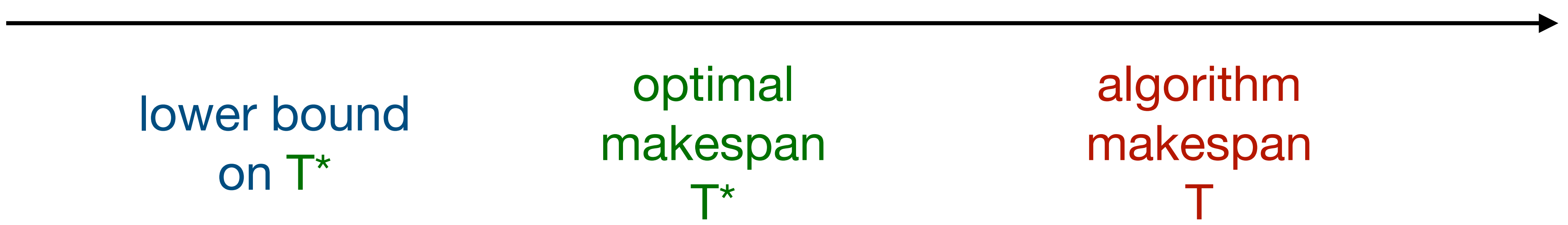

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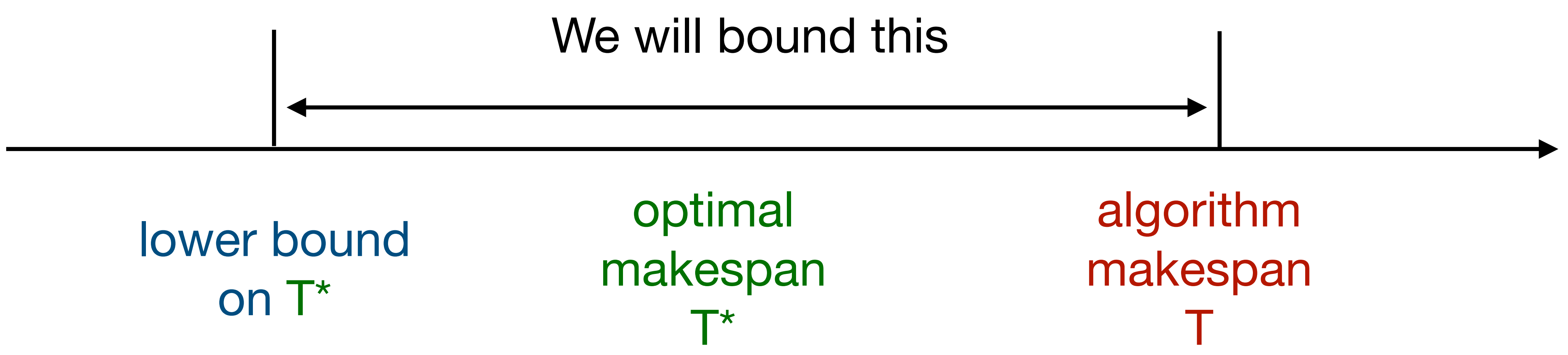

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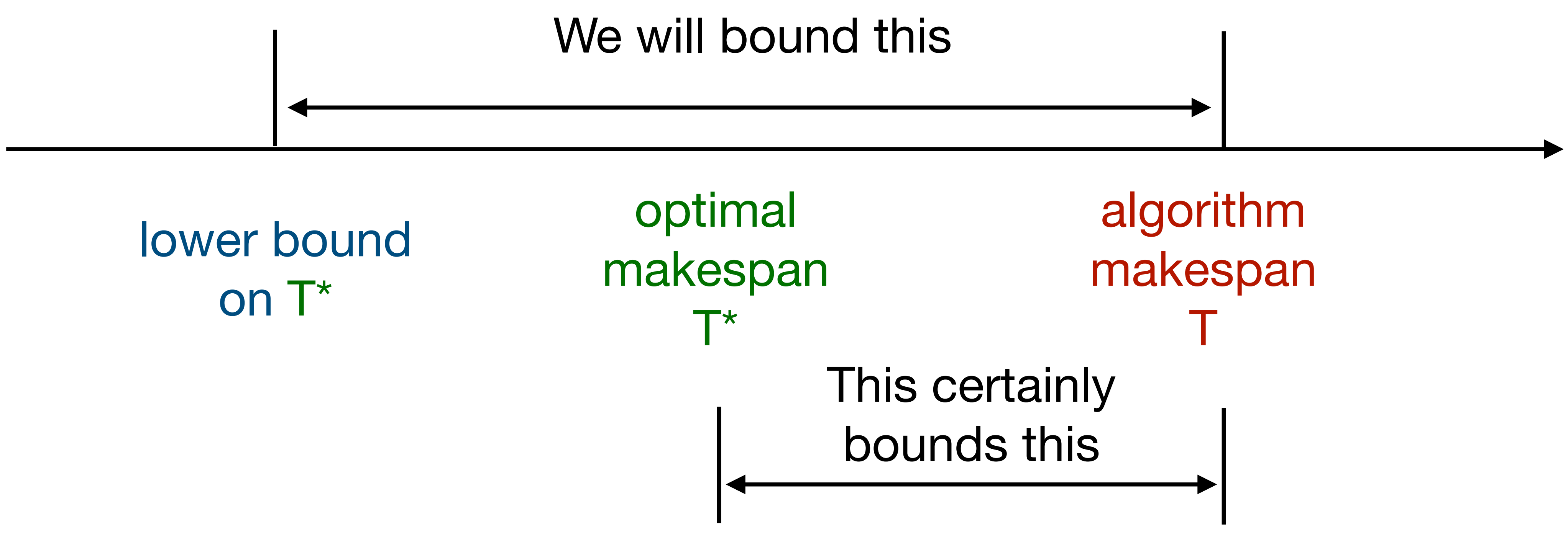

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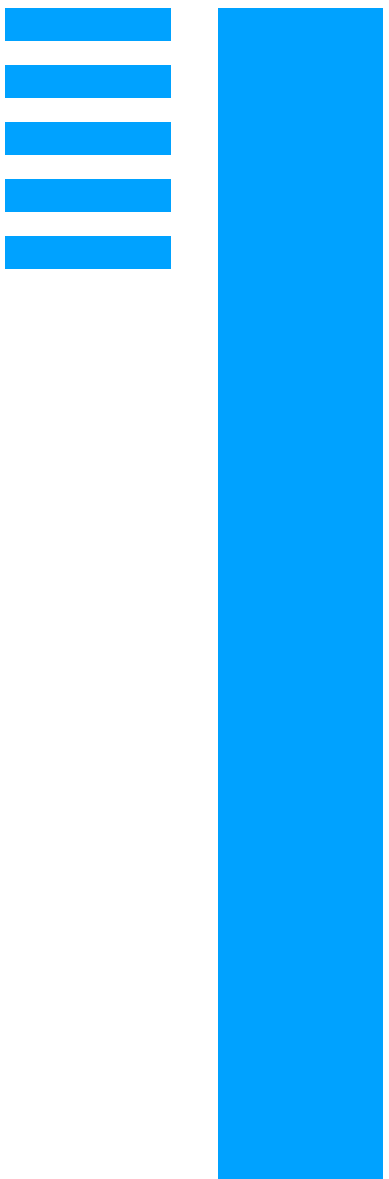

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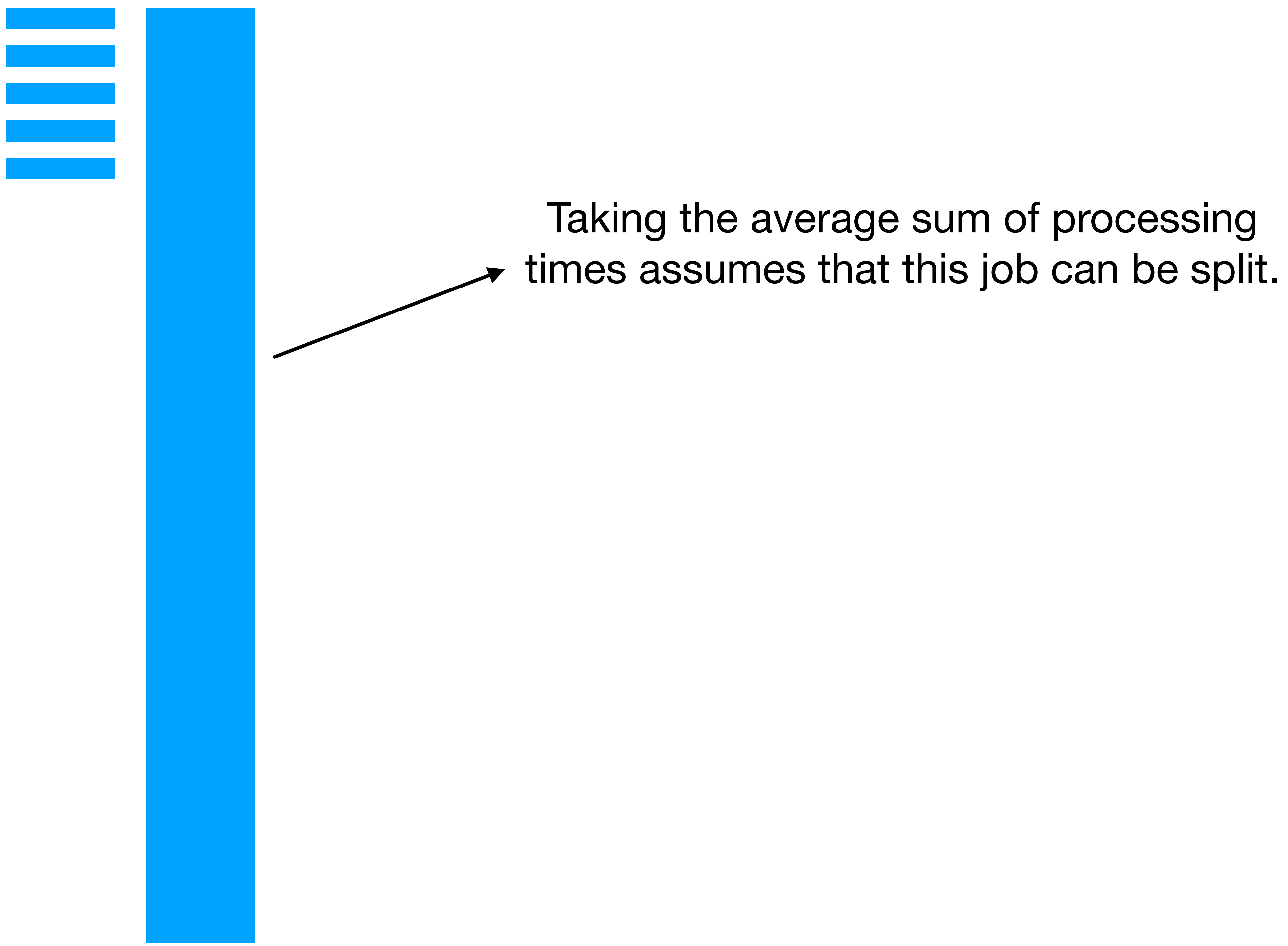

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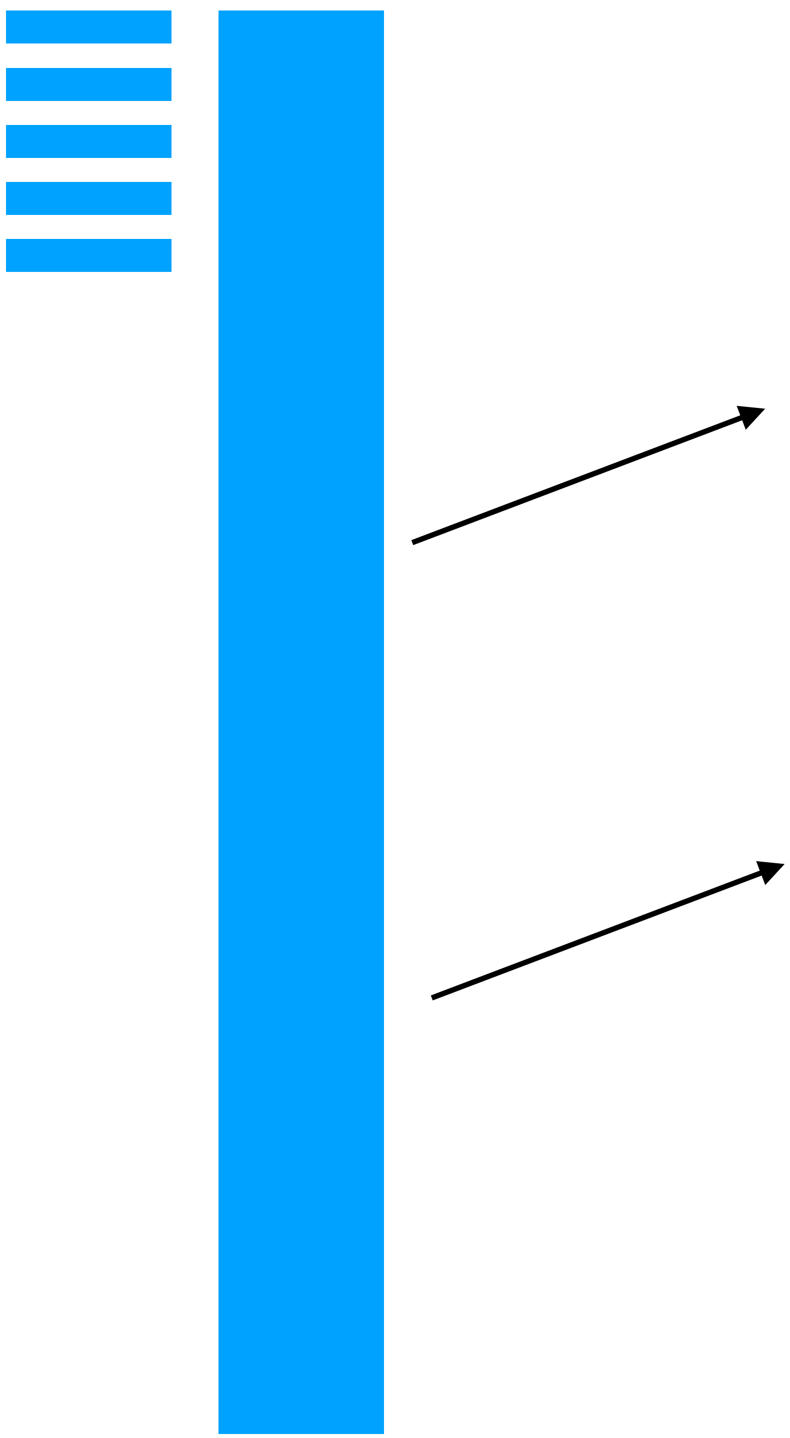

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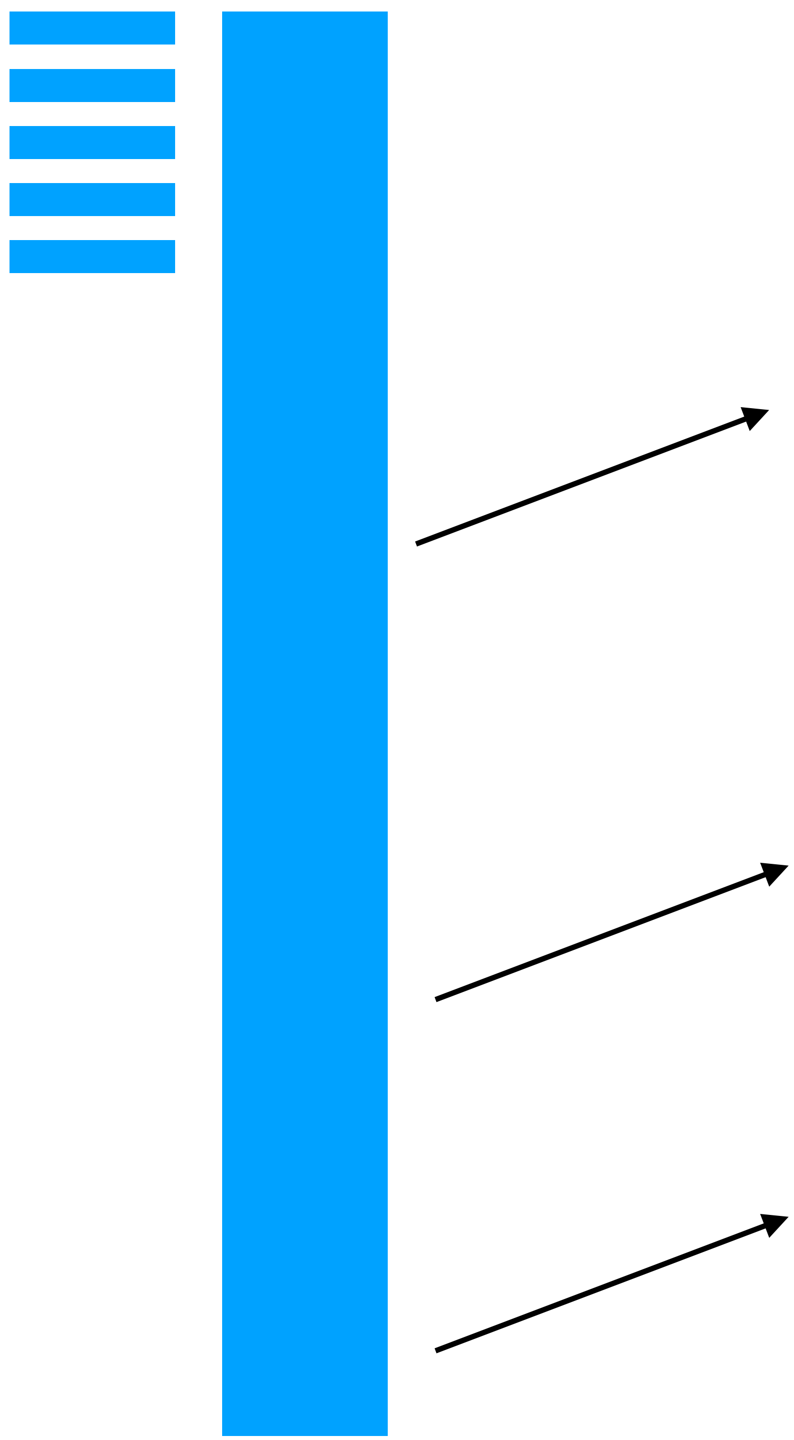

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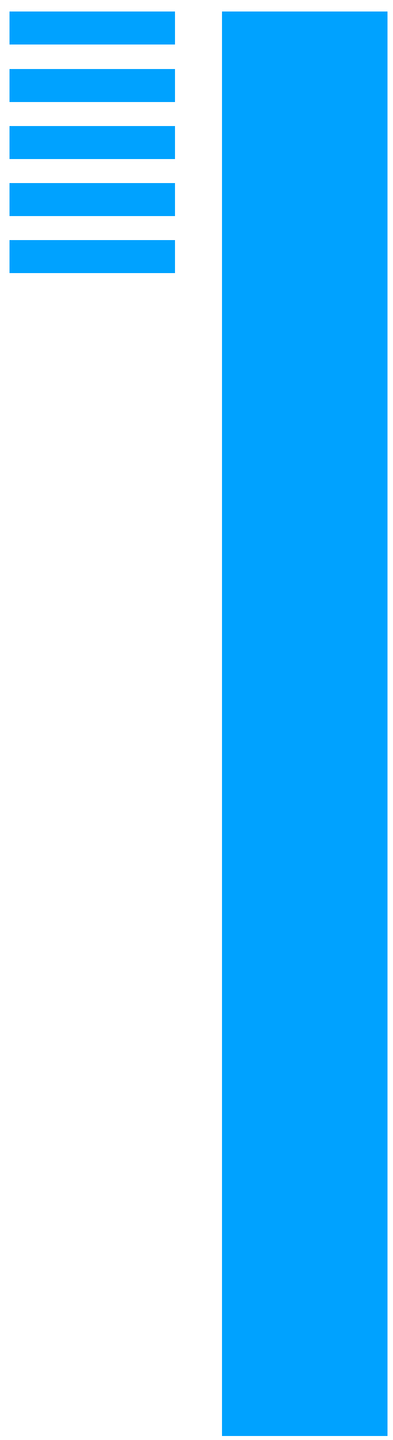

The bound can be good in situations where jobs have fairly similar processing times.

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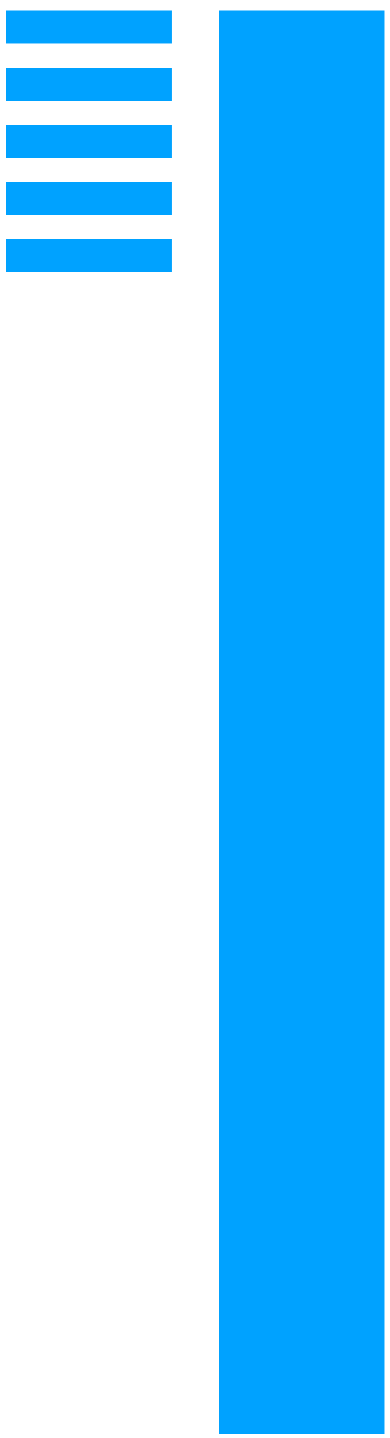

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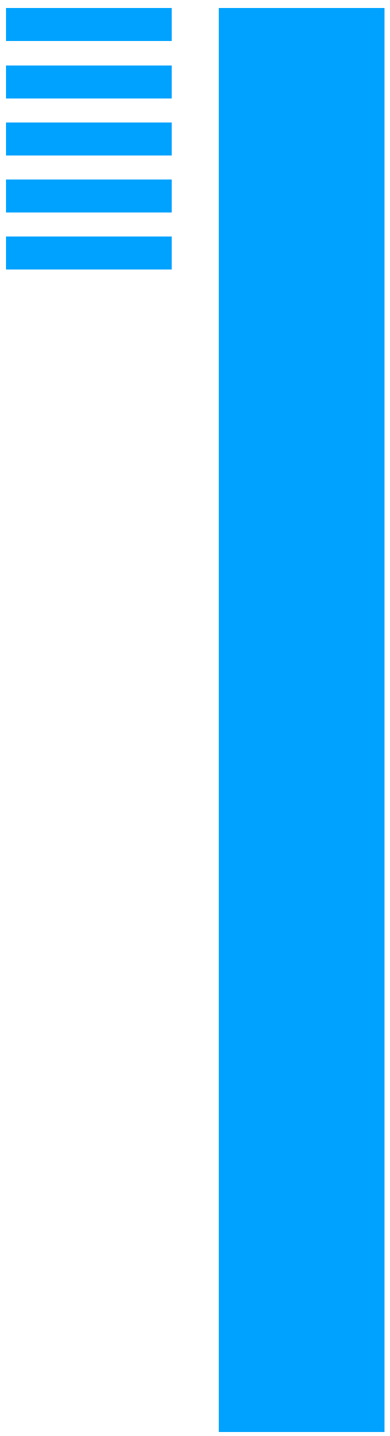

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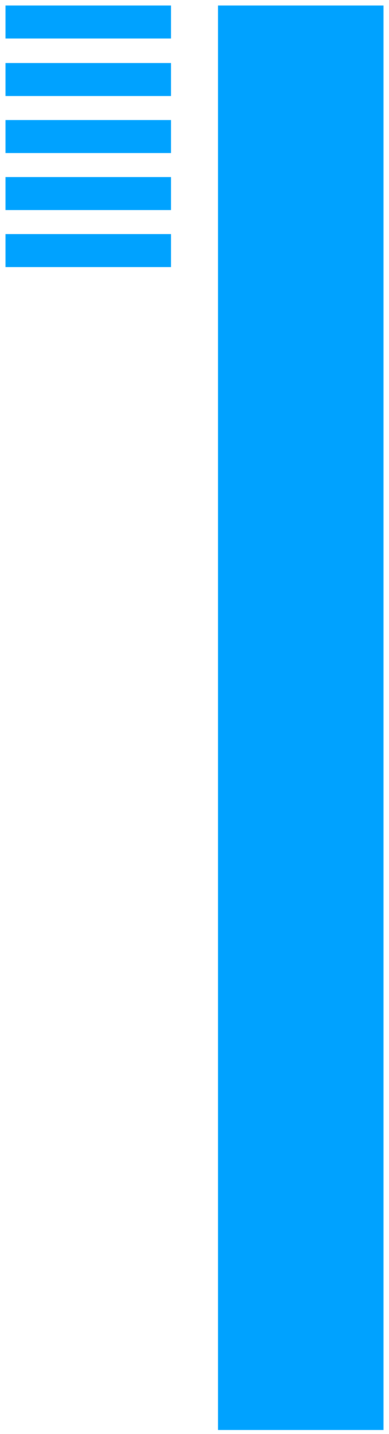

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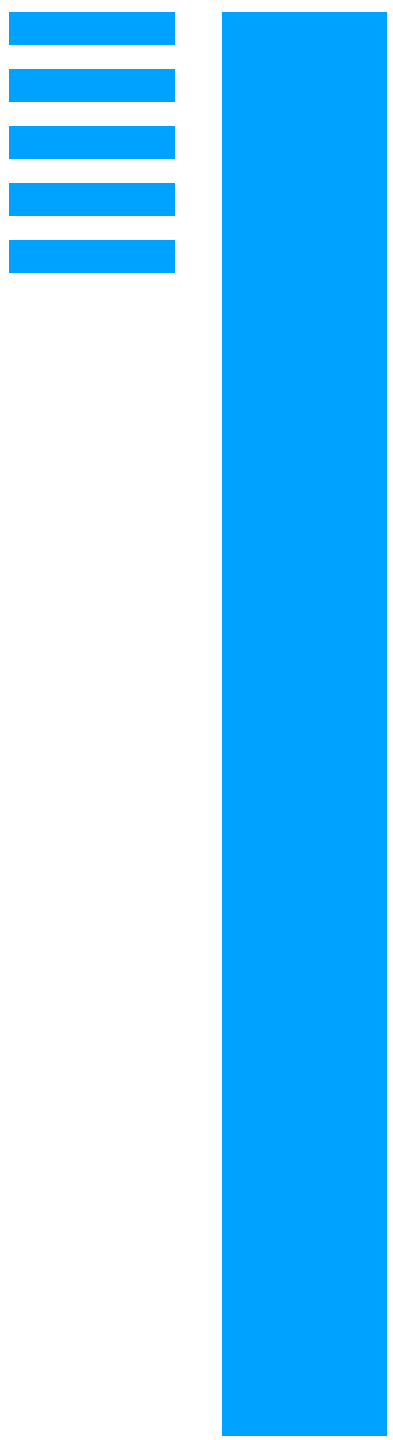

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In other cases, it might not be such a good bound.

But we will actually use both bounds!

Lower bounding the optimal

- Two lower bounds:

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The performance of Greedy-Balance

- Theorem: Algorithm Greedy-Balance produces an assignment of jobs to machines with makespan $T \leq 2T^*$.

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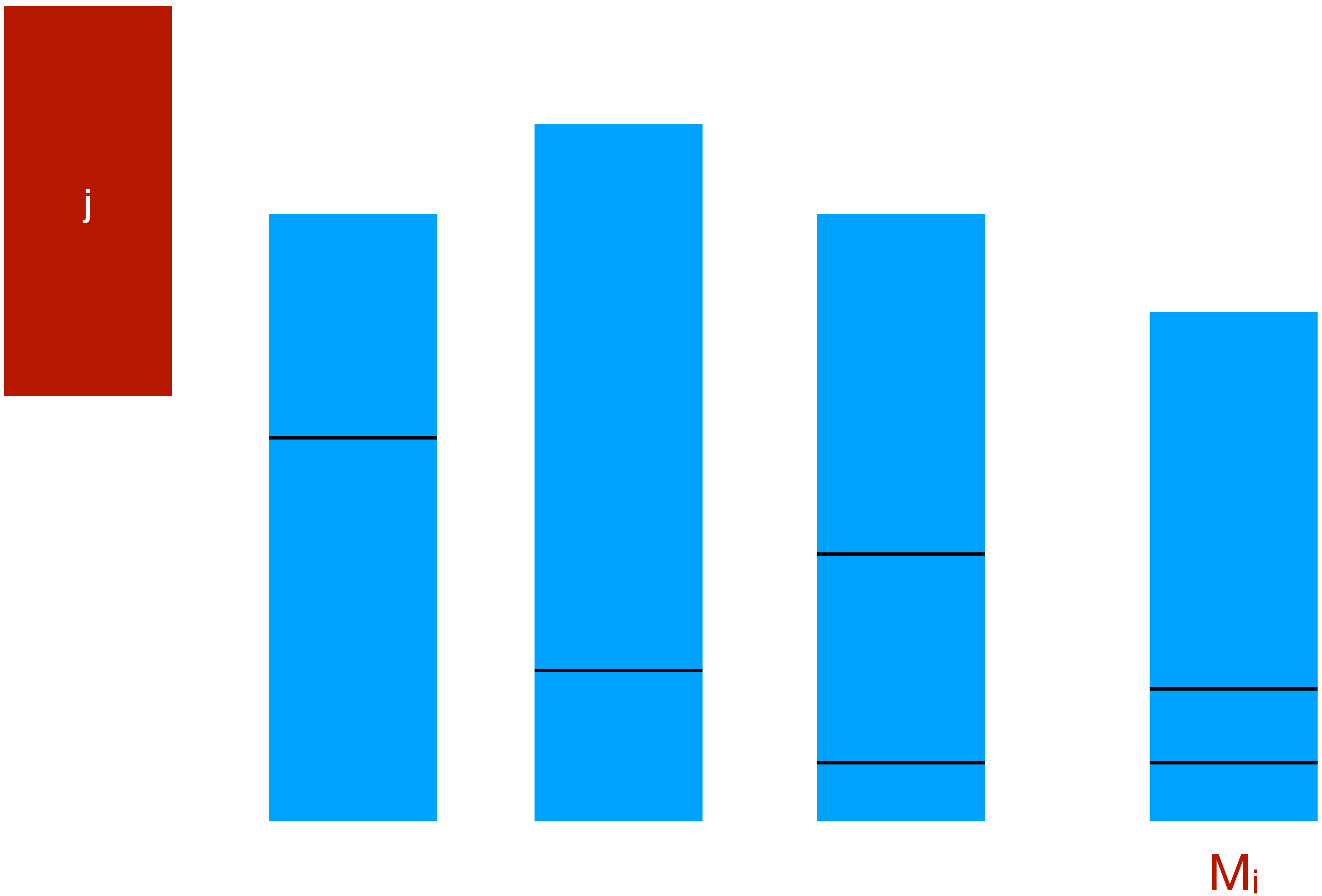

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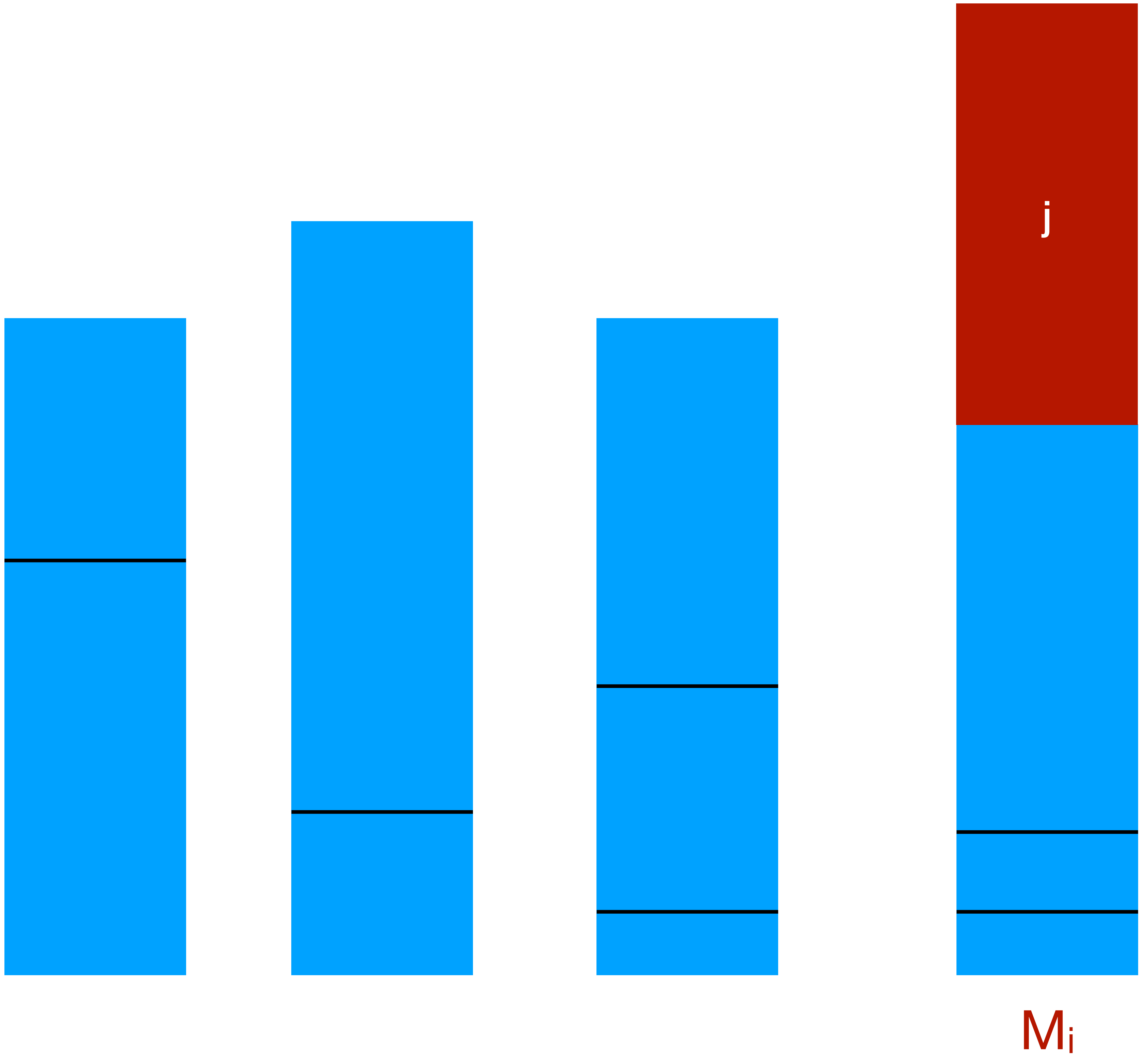

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- Summing up over all machines we get:

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$T_i - t_j \leq T^*$ **(first lower bound)**

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  - The load of machine j was $T_i - t_j$.
  - This was the smallest load among all machines (why?)
  - Every other machine has load at least $T_i - t_j$.

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  - The load of machine j was T_i - t_j.
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  - Obviously $t_j \leq \max_k t_k \leq T^*$

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$T \leq 2T^*$ **(since j was the final job)**

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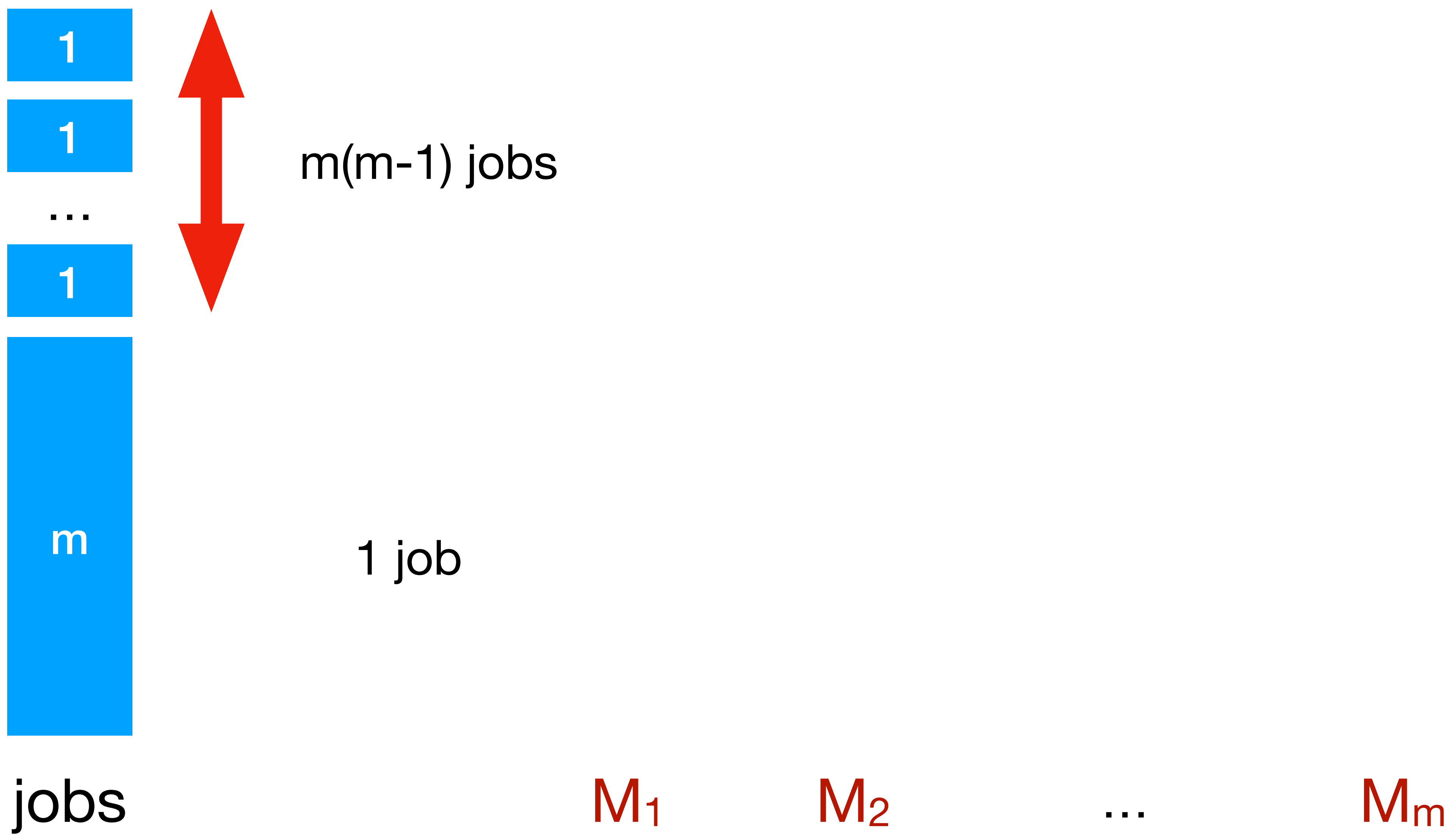
    - In other words, is our analysis of the algorithm *tight*?

Tight example for Greedy-Balance



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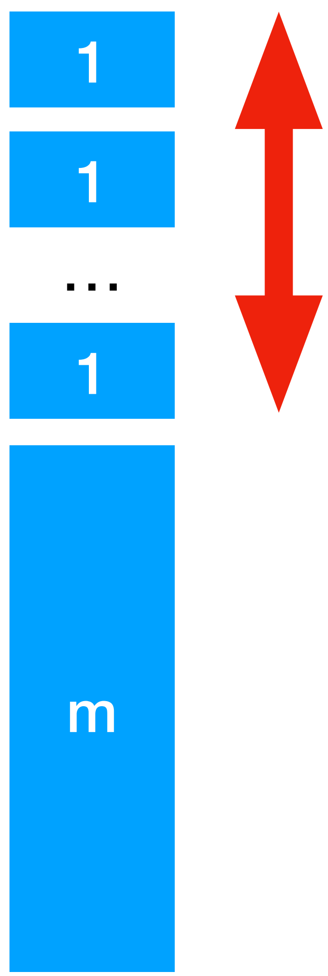

m(m-1) jobs

1 job

jobs

M_1 M_2 ... M_m

Greedy-Balance assigns *m-1* “small” jobs to each machine and then finally assigns the “large” job to one machine.

Makespan: 2m-1

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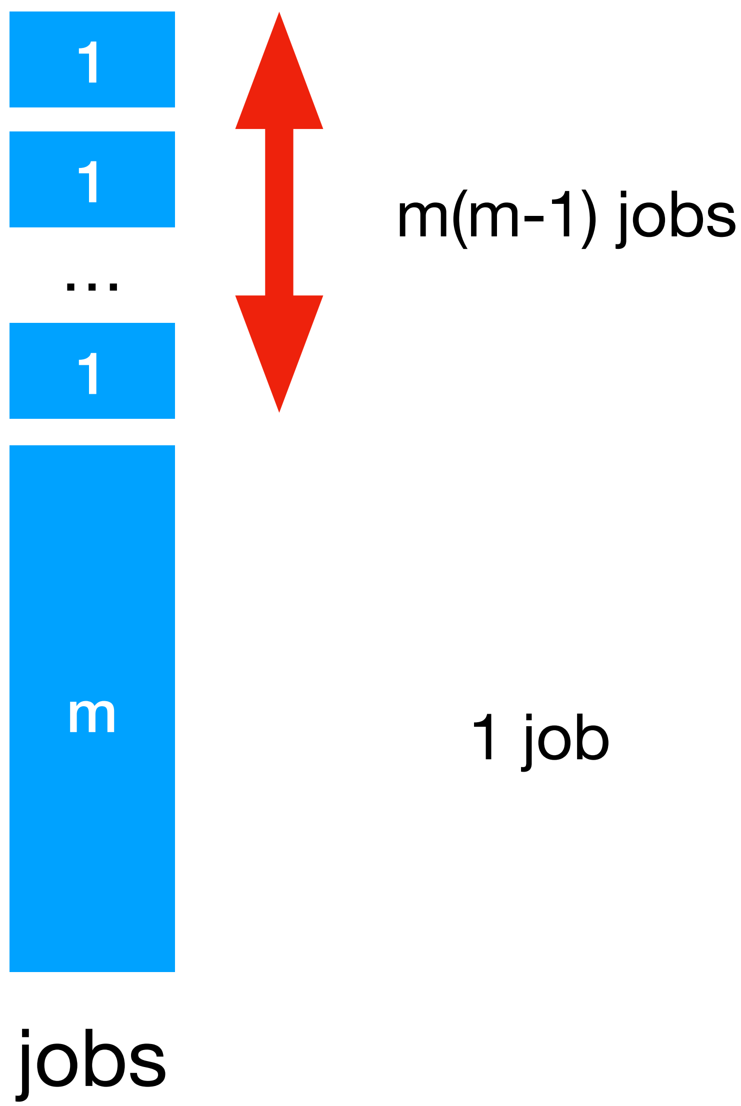

Greedy-Balance assigns *m-1* "small" jobs to each machine and then finally assigns the "large" job to one machine.

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The optimal assigns the "large" job to one machine, and evenly spreads the "small" jobs over the remaining *m-1* machines.

Makespan: m

M_1 M_2 ... M_m

Approximation Ratio

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- Consider a minimisation problem P and an objective obj.

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  - Consider an approximation algorithm A.
  - Consider an input x to the problem P.
  - Let obj(A(x)) be the value of the objective from the solution of A on x.
  - Let opt(x) be the minimum possible value of the objective on x.

Approximation ratio

- The approximation ratio of A is defined as

$$\max_x \text{obj}(A(x)) \,/\, \text{opt}(x)$$

  - i.e., the worst case ratio of the objective achieved by the algorithm over the optimal value of the objective, over all possible inputs to the problem.

Approximation ratio

All inputs to the problem P

Algorithm A

Optimal

Approximation ratio

All inputs to the problem P

Input A

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compute obj on
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Approximation ratio

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Ratio on Input A

Approximation ratio

All inputs to the problem P

Algorithm A

Optimal

Ratio on Input A

Approximation ratio

All inputs to the problem P

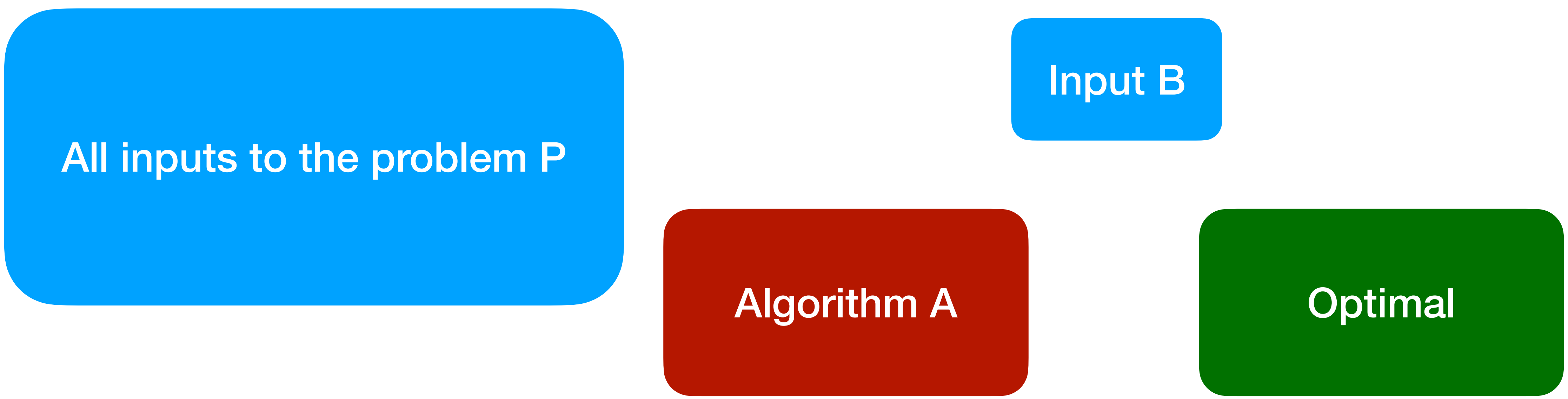
Input B

Algorithm A

Optimal

Ratio on Input A

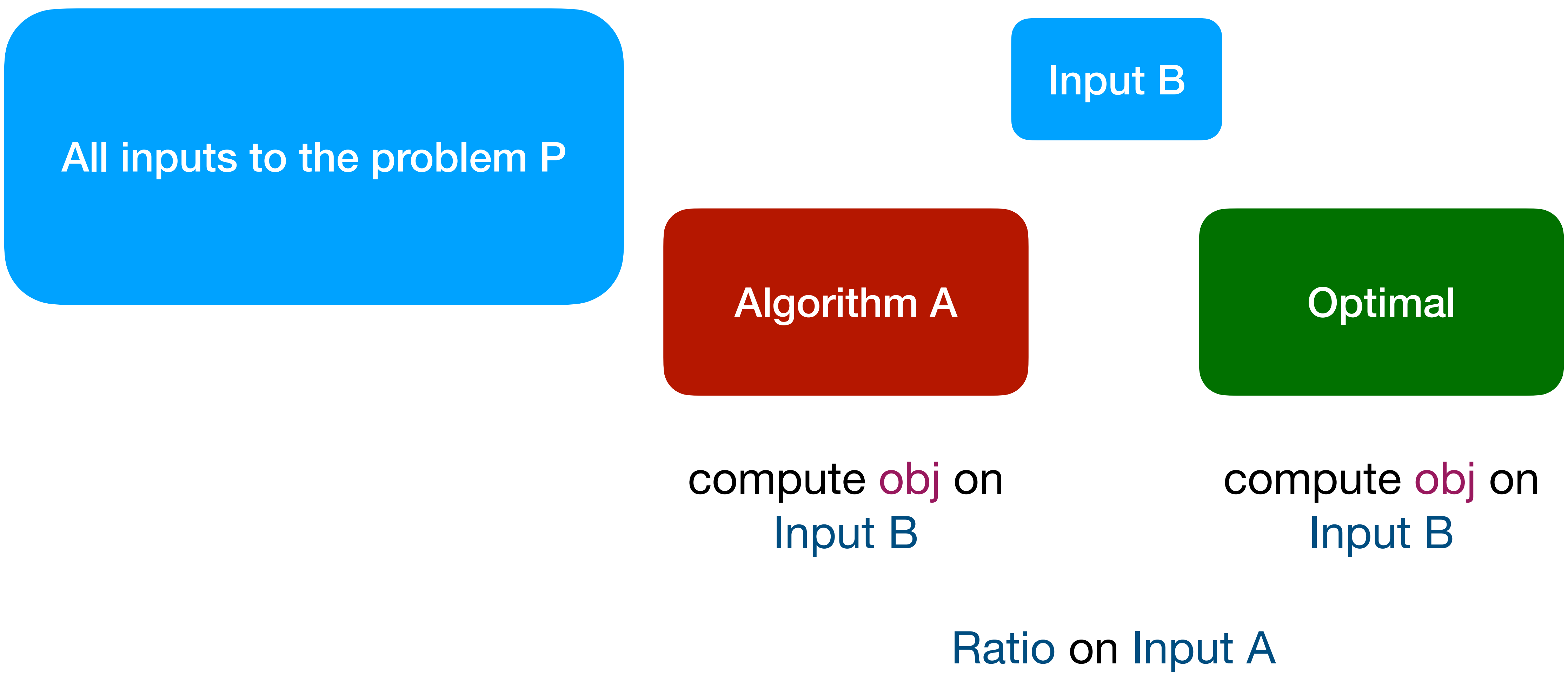
Approximation ratio



compute obj on
Input B

Ratio on Input A

Approximation ratio



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compute obj on Input B

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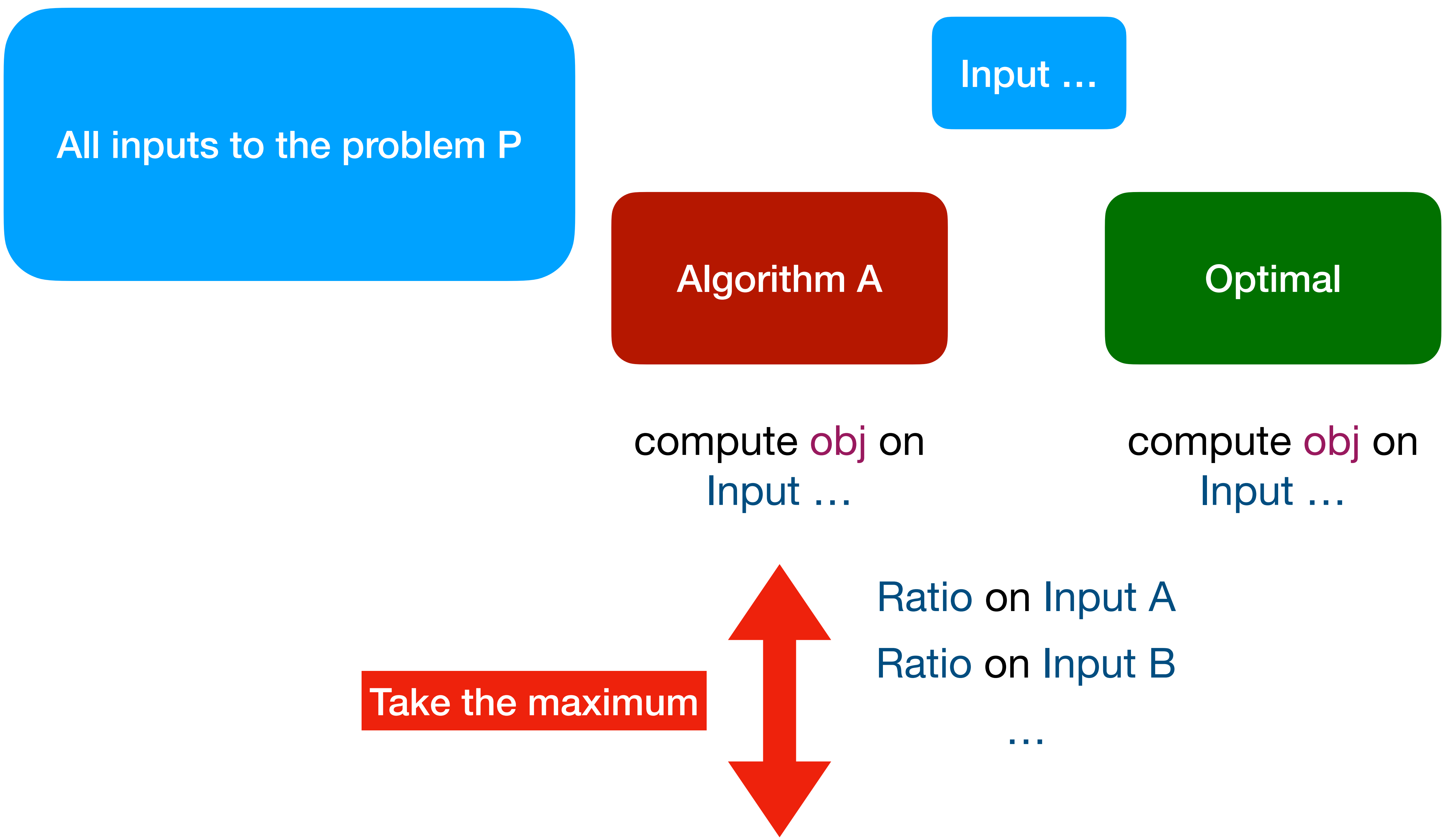

Approximation Ratio

- That means that:
  - In order to prove an upper bound on the approximation ratio, we have to somehow argue about *all* inputs to the problem.
  - In order to prove a lower bound on the approximation ratio, we have to argue about *one* input to the problem.

Approximation ratio

- For maximisation problems, we define

$$\max_x \text{opt}(x) / \text{obj}(A(x))$$

  - i.e., the worst case ratio of the optimal value of the objective over the value of the objective achieved by the algorithm, over all possible inputs to the problem.

  - Convention, to have approximation ratios always be ≥ 1.

Challenges

- What does "*close*" to the optimal mean? How do we measure that?
- How do we make such an argument, if we cannot really find the optimal?
- How do we know if our algorithm is the best possible? Can we get "*closer*" to the optimal?

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A better greedy algorithm for load balancing

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  - Pick any job.
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    - Pick any job.
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We did not really take into account the order in which we consider the jobs.

A better greedy algorithm for load balancing

- **Sorted-Balance**:
  - Sort the jobs in non-increasing order of processing times.
  - Pick a job according to this order.
  - Assign it to the machine with the smallest load so far.
  - Remove it from the pile of jobs.

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  - Each job goes to a different machine.
  - **Sorted-Balance** produces an optimal allocation.
  - The same was actually true for **Greedy-Balance**.

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    - Consider the first $m+1$ jobs in sorted order.
    - Each one of them takes at least t_{m+1} time.
    - Since there are m machines, there must be one machine that receives at least two of these jobs.
    - The load on this machine will be at least $2t_{m+1}$.

Lower bounding the optimal

- Two lower bounds:

$$T^* \geq \frac{1}{m} \sum_{j=1}^{n} t_j$$

$$T^* \geq \max_j t_j$$

Lower bounding the optimal

- Three lower bounds:

$$T^* \geq \frac{1}{m}\sum_{j=1}^{n} t_j$$

$$T^* \geq \max_j t_j$$

$$T^* \geq 2t_{m+1}$$

The performance of Sorted-Balance

- Theorem: Algorithm Sorted-Balance produces an assignment of jobs to machines with makespan $T \leq (3/2)T^*$.

The proof

- Let M_i be the machine with the maximum load according to the assignment of **Sorted-Balance**.

- If M_i is assigned a single job, the outcome is optimal.

- Assume M_i that is assigned at least two jobs and let j be the last job assigned to the machine.

  - Note that $j \geq m+1$

  - Therefore, $t_j \leq t_{m+1} \leq (1/2)T^*$

The proof (still the same argument)

- Every other machine has load at least T_i - t_j.
- Summing up over all machines we get:

$$\sum_k T_k \geq m(T_i - t_j) \Rightarrow T_i - t_j \leq \frac{1}{m} \sum_k T_k$$

$T_i - t_j \leq T^*$ **(first lower bound)**

The proof (previous argument)

- Consider the last job j that was assigned to any machine (assume machine M_i) by **Greedy-Balance**.

- Consider the time when this assignment took place.

  - The load of machine j was T_i - t_j.

  - This was before we added the job.

  - After we add the job, the load is $T_i - t_j + t_j$.

- Obviously $t_j \leq \max_k t_k \leq T^*$

The proof (new argument)

- Consider the last job j that was assigned to some machine M_i by Greedy-Balance.

- Consider the time when this assignment took place.

  - The load of machine j was T_i - t_j.

  - This was before we added the job.

  - After we add the job, the load is T_i - t_j + t_j.

  - We established that $t_j \leq t_{m+1} \leq (1/2)T^*$

The proof

$T_i - t_j \leq T^*$ **(first lower bound)**

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$$T_i \leq \frac{3}{2}T^*$$

$$T \leq \frac{3}{2}T^*$$

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- The Sorted-Balance algorithm actually gives a 4/3 approximation ratio, with a better analysis.
- For the load balancing problem on identical machines, there is a Polynomial Time Approximation Scheme (PTAS).
  - An algorithm which, given an input and a constant parameter ε, runs in polynomial time and produces an outcome which is (1+ε) far from the optimal.

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  - A PTAS (or an FPTAS, more about that later) is the best approximation we can hope for, for an NP-hard problem.

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- Generally:
  - A PTAS (or an FPTAS, more about that later) is the best approximation we can hope for, for an NP-hard problem.
  - Sometimes it is impossible to get that close.
  - Inapproximability **α** of problem P:
    - There is no polynomial time algorithm that achieves an approximation ratio better than **α**.

Reading

- Kleinberg and Tardos 11.1
- Roughgarden 20.1
- Williamson and Shmoys - The Design of Approximation Algorithms 1.1, 2.3
  - Available via the library, also for free on https://www.designofapproxalgs.com/
  - (This is probably my favourite book!)