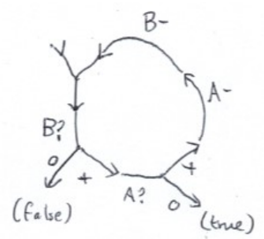
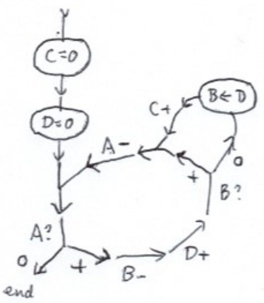
Informatics 2 – Introduction to Algorithms and Data Structures

Tutorial 10: Register machines and computability (SOLUTIONS)

1. (a) *Design a flowchart for a register machine that tests whether 'A < B'.*



(b) *Design a machine that computes 'A div B' and 'A mod B' (assuming B is non-zero), storing the results in C and D respectively.*



Here, for clarity, we have assumed given some very simple components. 'C=0' and 'D=0' do what they say; 'B ← D' copies the value of D to B, setting D to 0 in the process.

(c) *Show that if both* $f : \mathbb{N} \to \mathbb{N}$ *and* $g : \mathbb{N} \to \mathbb{N}$ *are RM-computable, then so is their* composition h *defined by* $h(n) = g(f(n))$.

Saying f is RM-computable means that there's a register machine F such that, for any $n \in \mathbb{N}$, if F is run on an initial state with A $= n$, it will terminate in a final state with A $= f(n)$. Likewise, g is RM-computable if there's a machine G doing the same job for G. Given such machines, we may simply plug them together by connecting the exit point of F to the entry point of G. (Strictly speaking, we first need to ensure F and G have the same number of registers, which we may do by adding extra (unused) registers to F or G as required.)

the resulting machine will compute the composition h as required.

(d) *Show that if $e, f, g : \mathbb{N} \to \mathbb{N}$ are all RM-computable, then so is the function k defined by*

$$k(n) \;=\; \text{if } e(n) = 0 \text{ then } f(n) \text{ else } g(n)$$

Suppose e, f, g are computed by register machines E, F, G respectively. Let r be two more than the maximum number of registers of E, F, G, and expand E, F, G to equivalent machines E', F', G' with r registers.

Our machine for computing k will work as follows, given an initial state with A $= n$.

- Copy n from A into the two spare registers, then copy one of them back to A.
- Use E to compute $e(n)$ (in A), then use 'A?' to branch on whether $e(n) = 0$.
- On the 0 branch, copy the value of n back into A, then apply F.
- On the + branch, copy n back into A and apply G.
- Merge the two exit points into one.

2. (a) *What about the predicate 'the machine coded by m, when applied to the inputs coded by n, halts within k steps'? Would you expect this to be RM-decidable? Informally justify your answer.*

This is certainly decidable. Given m, n and k, it is a purely mechanical task to simulate the execution of machine m on input n for up to k steps. This simulation will complete within finite time, and by then we'll know if the computation in question halts within k steps.

So the given predicate is decidable by a mechanical procedure. By an informal appeal to Church's thesis, then, we expect it to be decidable by a register machine. (Alternatively, one could explicitly construct such a machine and show it did this, but life is too short.)

(b) *Let T be the set of all codes for register machines that compute some* total *function $\mathbb{N} \to \mathbb{N}$. It would be nice if there were some register machine that could tell us, given any $m, m' \in T$, whether the machines represented by m and m' gave rise to the* same *total function. Show however that no such machine is possible.*

Suppose such a machine D existed. Here's how we could use it to solve the halting problem.

- Given any m (coding a register machine) and n (coding a memory state), we can use our solution to (a) to construct a machine $P_{m,n}$ that computes the function

$$k \;\mapsto\; (\text{loop if machine } m \text{ on input } n \text{ halts within} \leq k \text{ steps},\ 0 \text{ otherwise})$$

- This machine $P_{m,n}$ will have a certain numerical code $p_{m,n}$. What's more, since the construction of $P_{m,n}$ is uniform in m and n, it will be possible to compute $p_{m,n}$ given m and n.
- The trick is to note that the computation of machine m on input n continues forever if and only if the function computed by $P_{m,n}$ is total. (If so, it will be the constant 0 function.) So we could solve the halting problem as follows: given m, n, compute $p_{m,n}$, then run the supposed machine D on $p_{m,n}$.