Algorithms and Data Structures

Content and Basic Notions

Algorithms and Data Structures

- A continuation of Introduction to Algorithms and Data Structures (INF2 - IADS).
- Mostly same techniques, more advanced applications.
  - Divide-and-Conquer, Greedy, Dynamic Programming
- More emphasis on “Algorithms” rather than “Data Structures”.
- More theorem proving.

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- Review of algorithm design basics, including *time and space complexity*, and *asymptotic notation*.
- The *divide-and-conquer* paradigm:
  - Sorting, matrix multiplication, Fourier transform.
  - Upper and lower bound proofs.
  - Solving recurrence relations.

The course content (cont)

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- The *greedy* paradigm:

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- The *greedy* paradigm:
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  - Linear programming.
- The *dynamic programming* paradigm:
  - Matrix-chain multiplication and other examples.

What is an algorithm?

- ***A set of instructions for solving a problem or performing a computation.***

- **Origin of the name:** Latinisation of the name given by Persian scholar Muhammad ibn Musa al-Khwarizmi.

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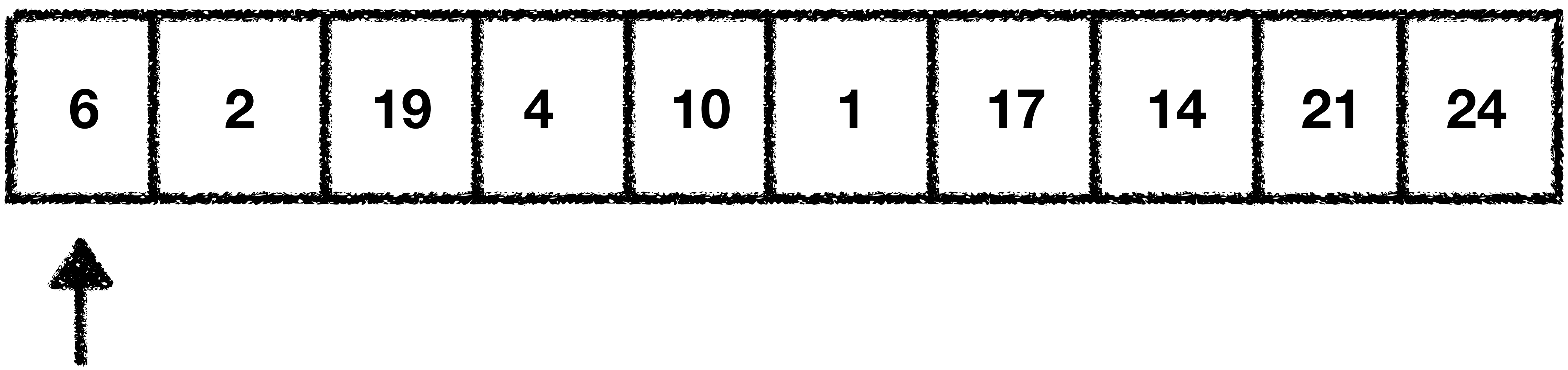

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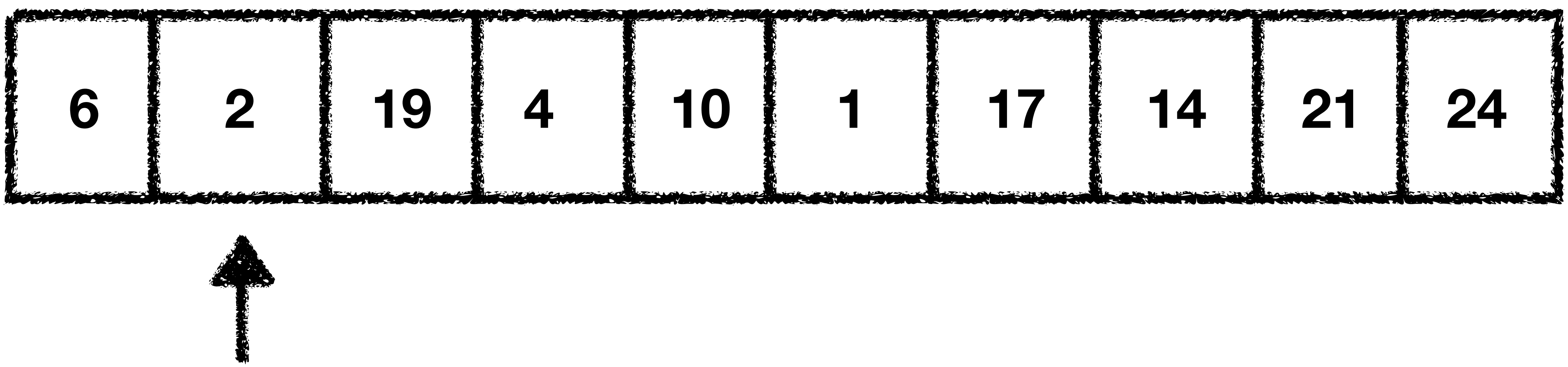

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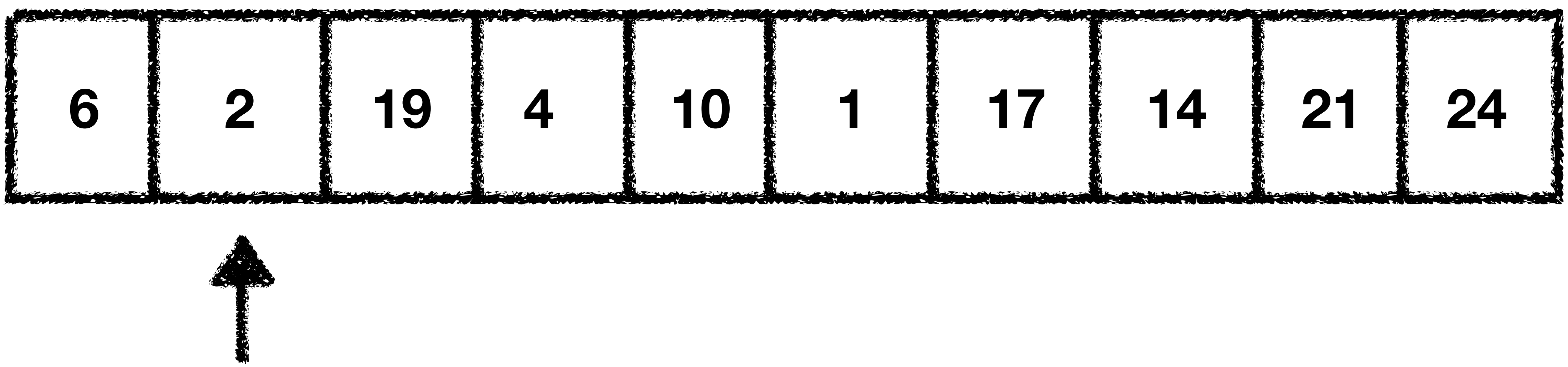

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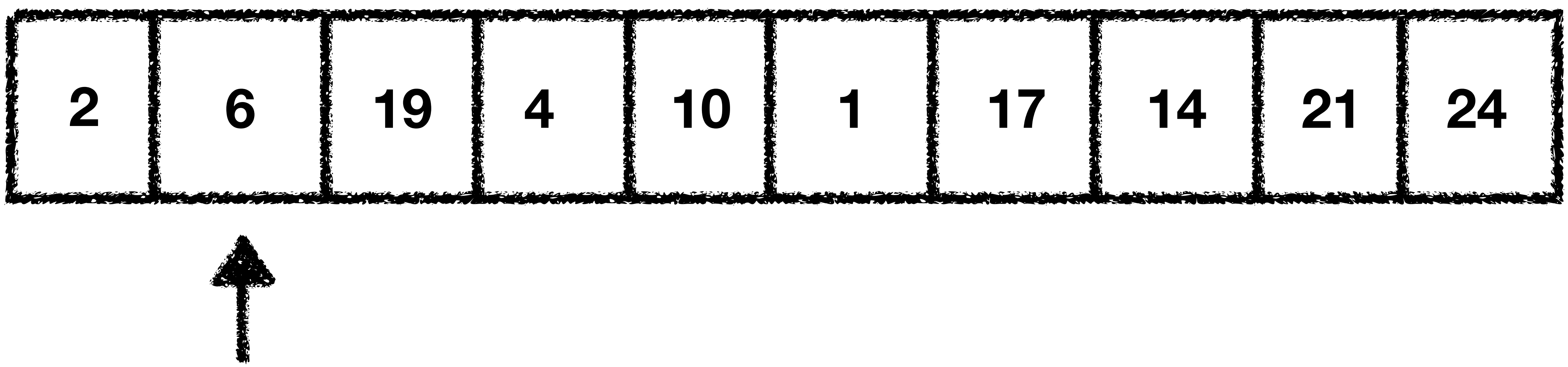

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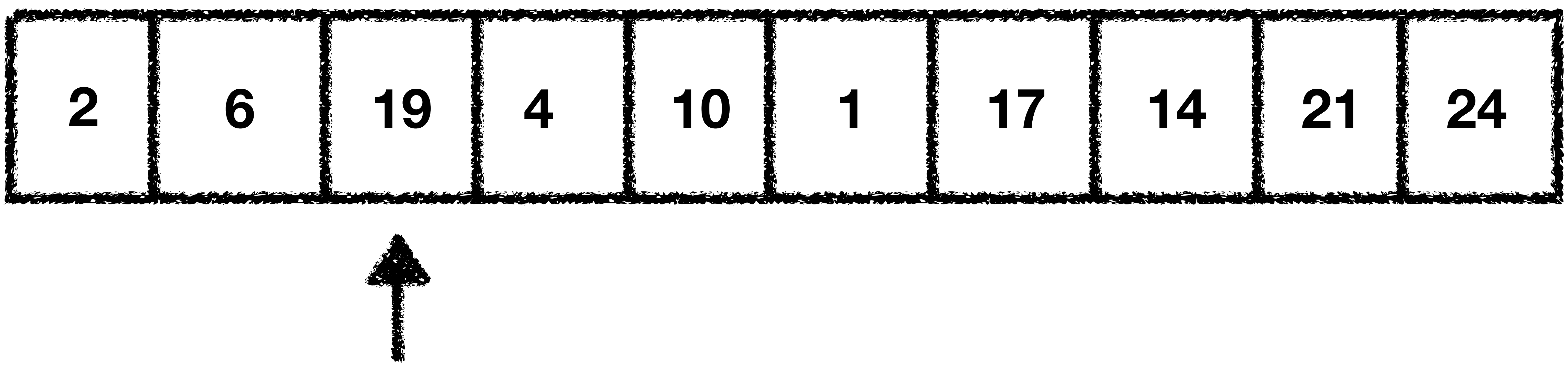

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Is 19 < 6?

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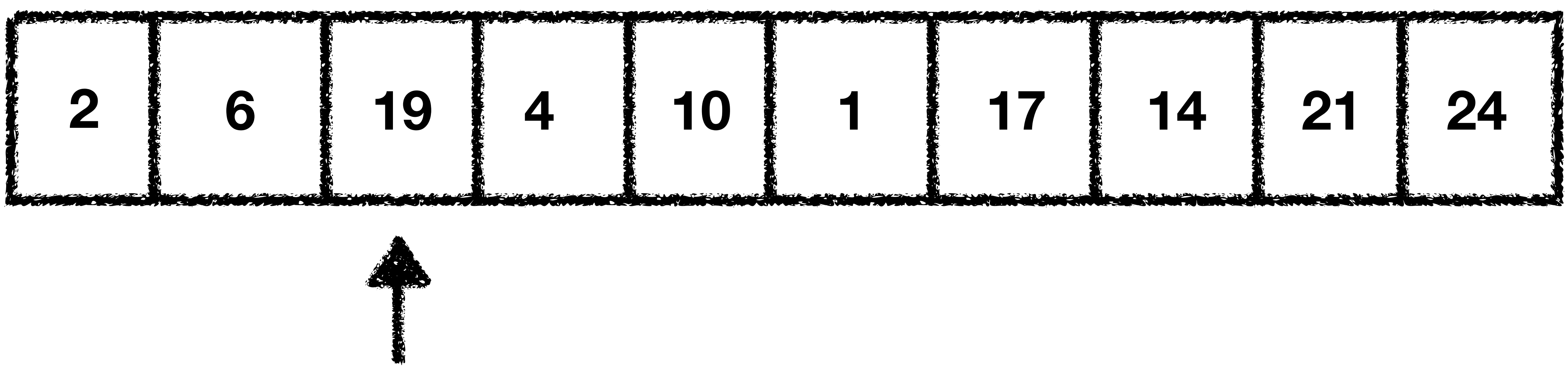

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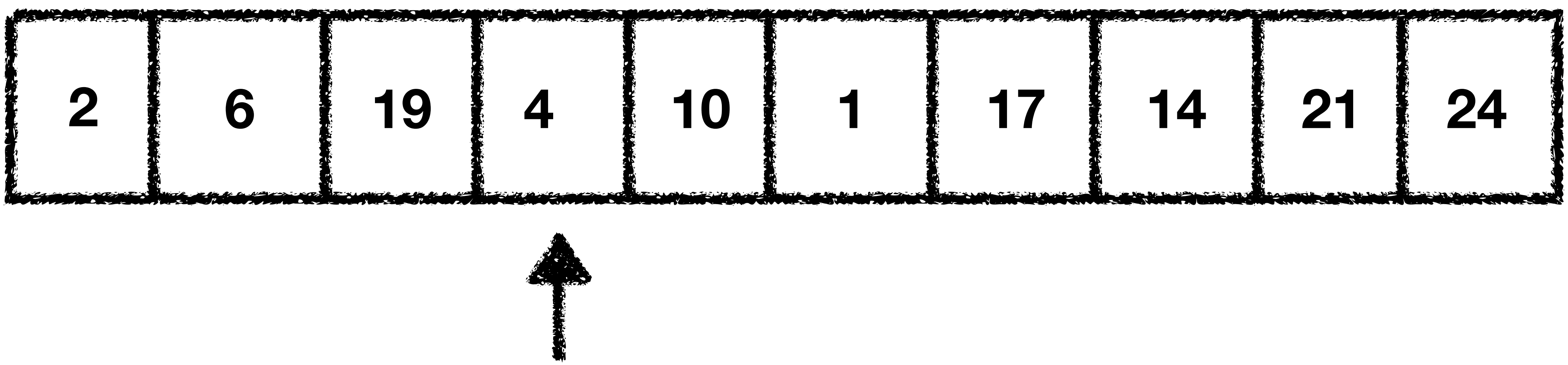

Is 4 < 19?

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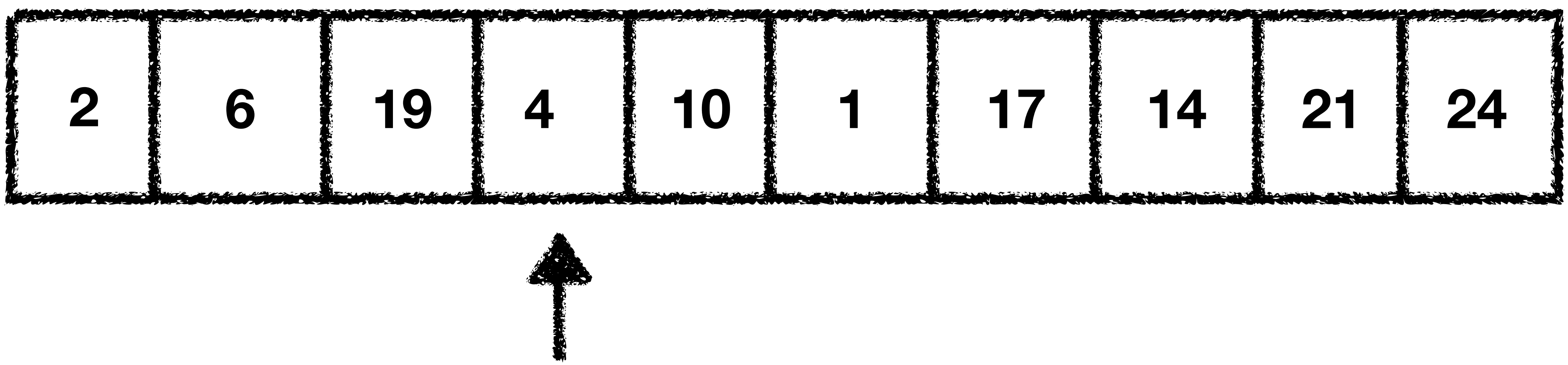

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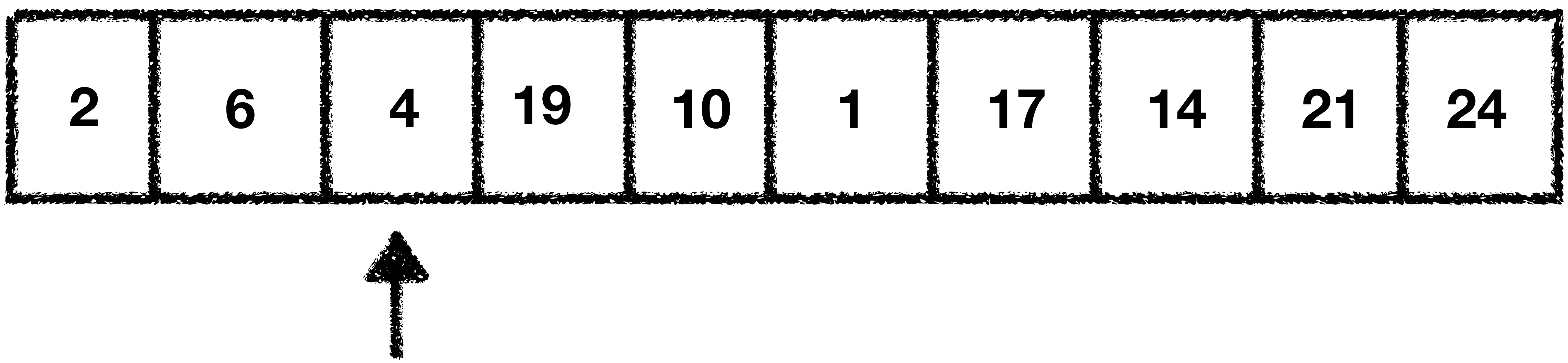

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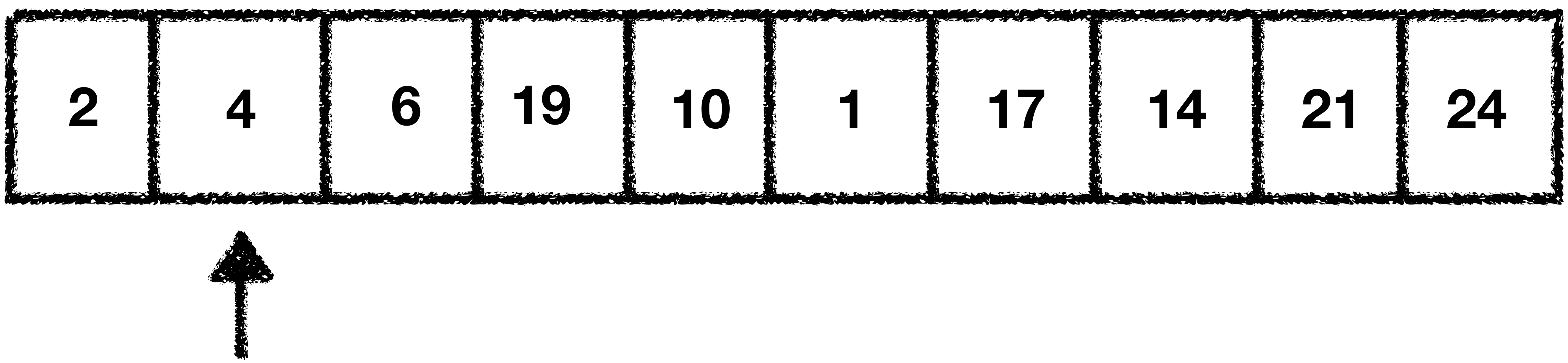

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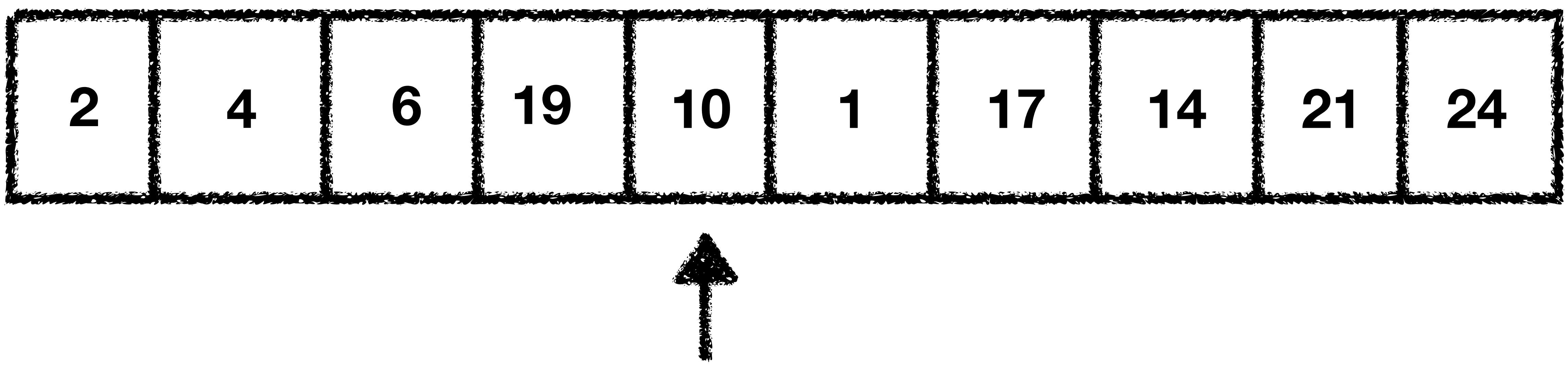

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FOR j ← 2 TO length[A]
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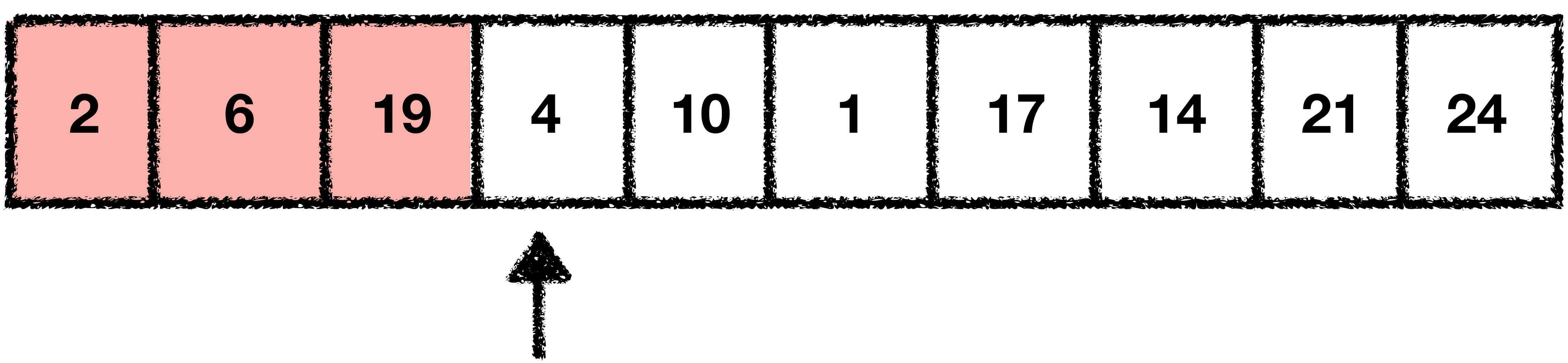

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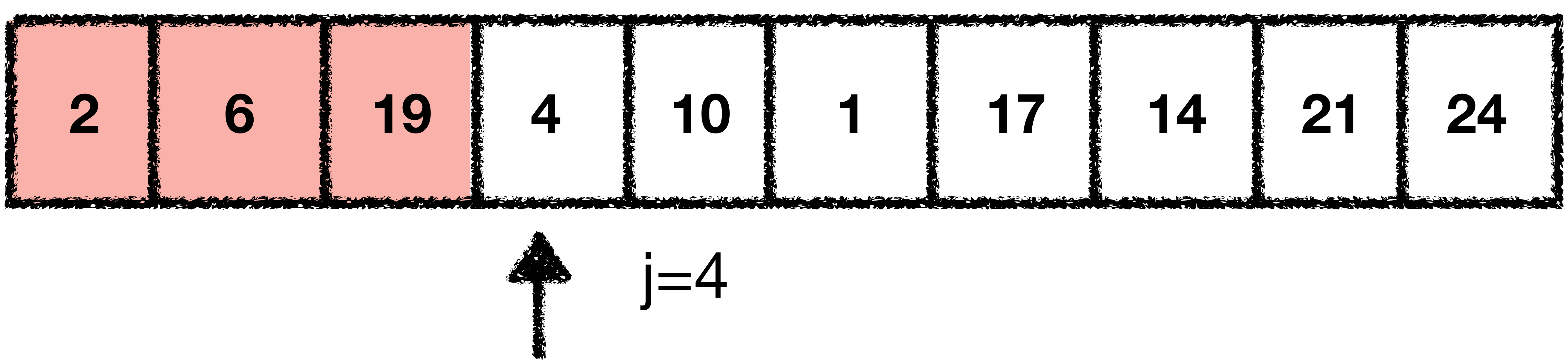

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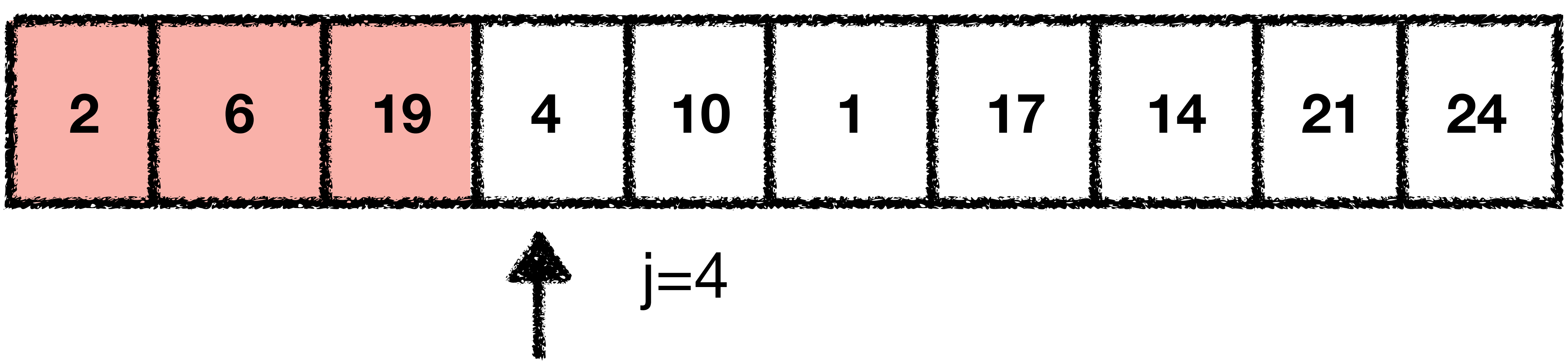

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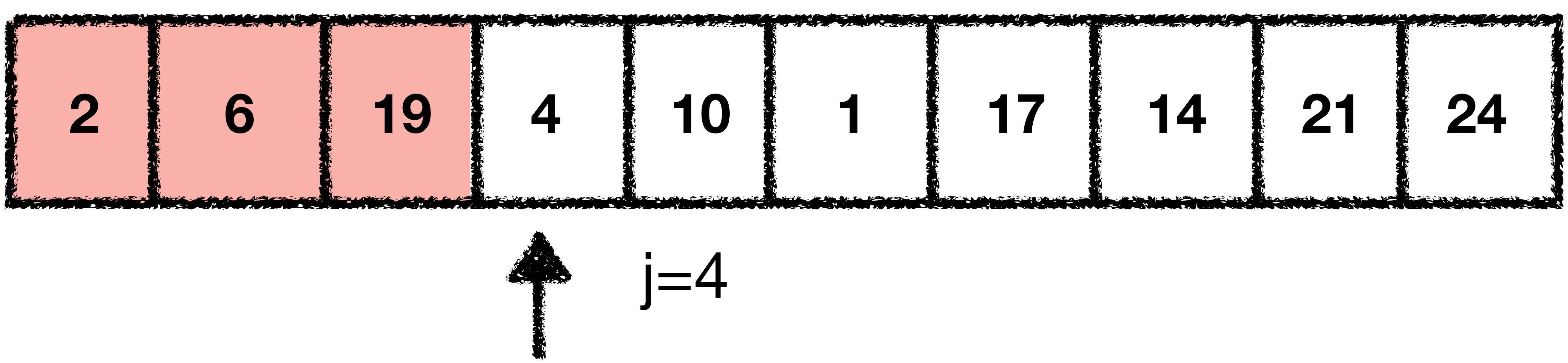

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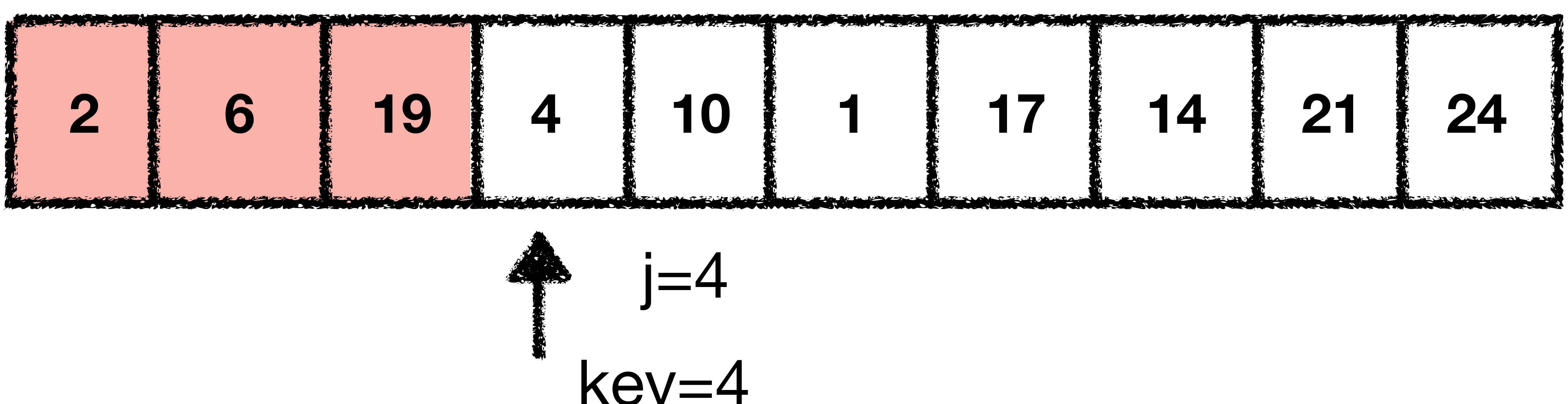

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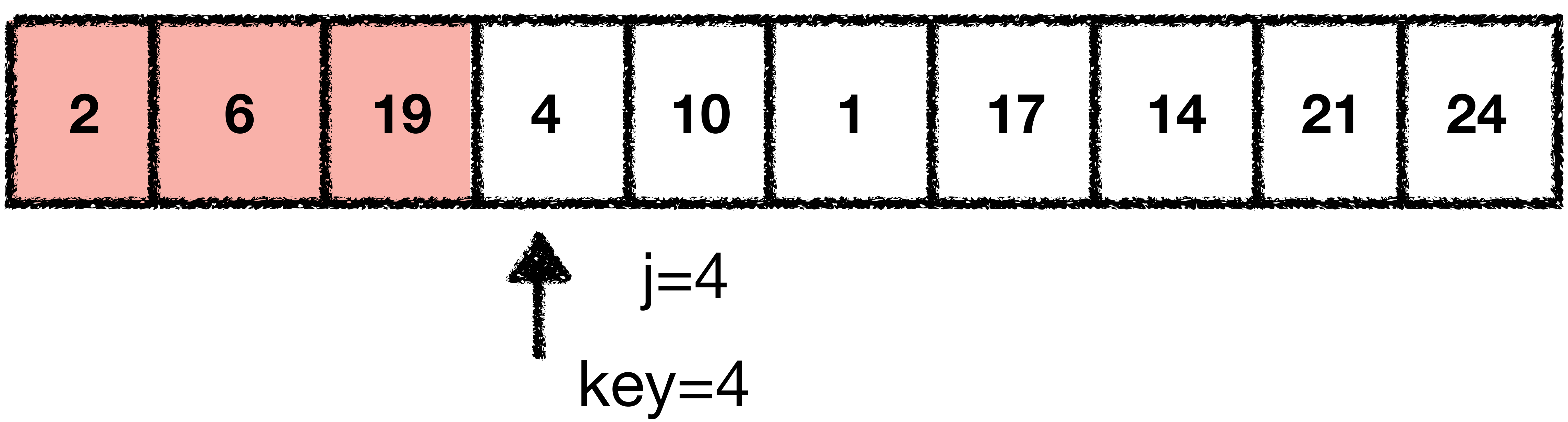

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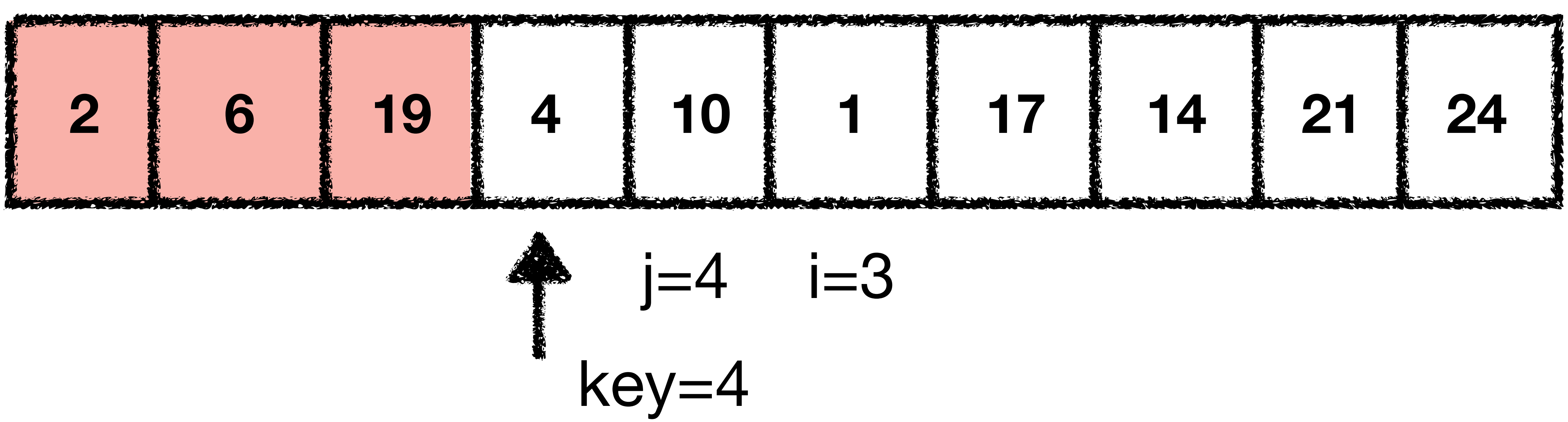

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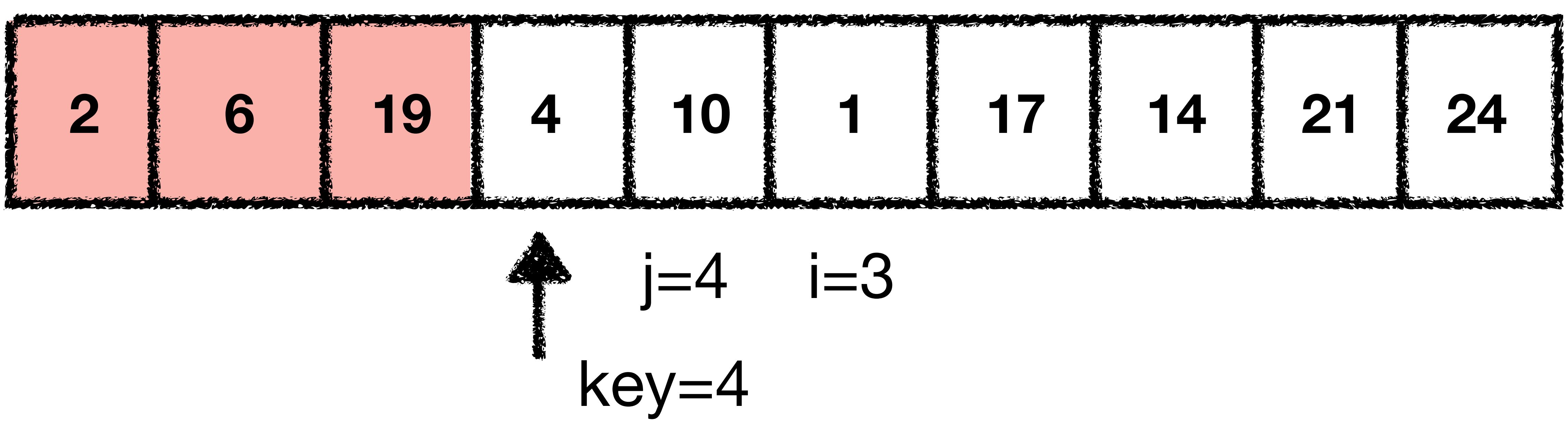

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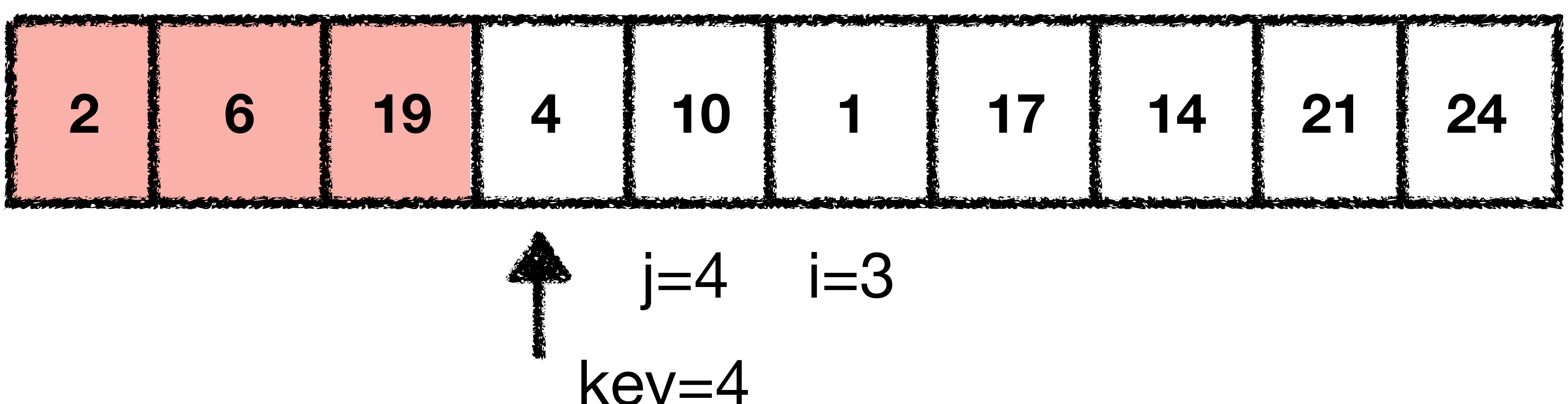

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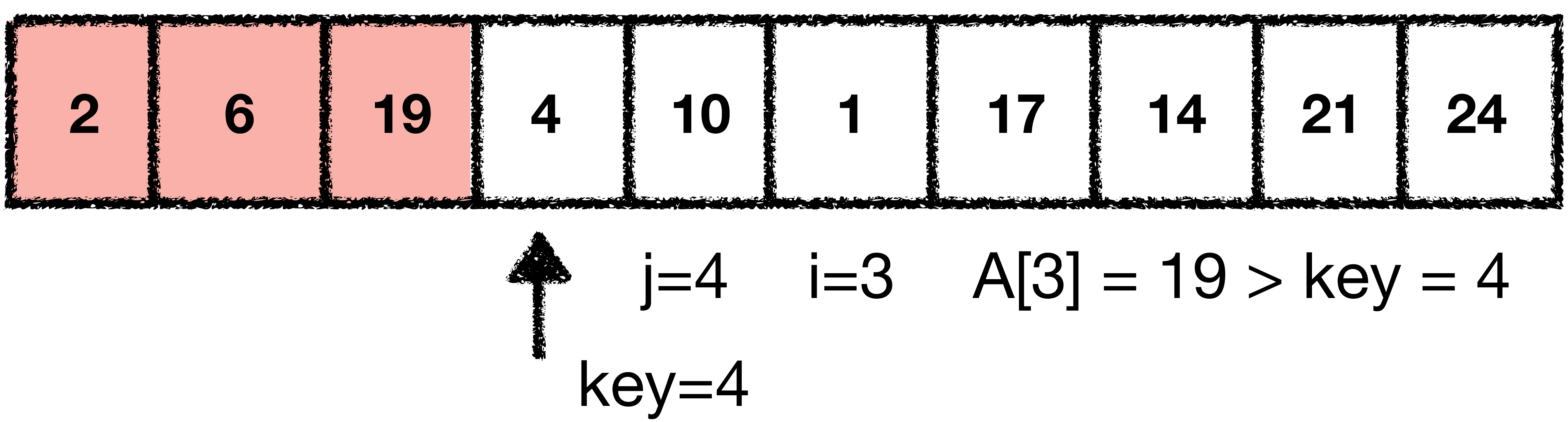

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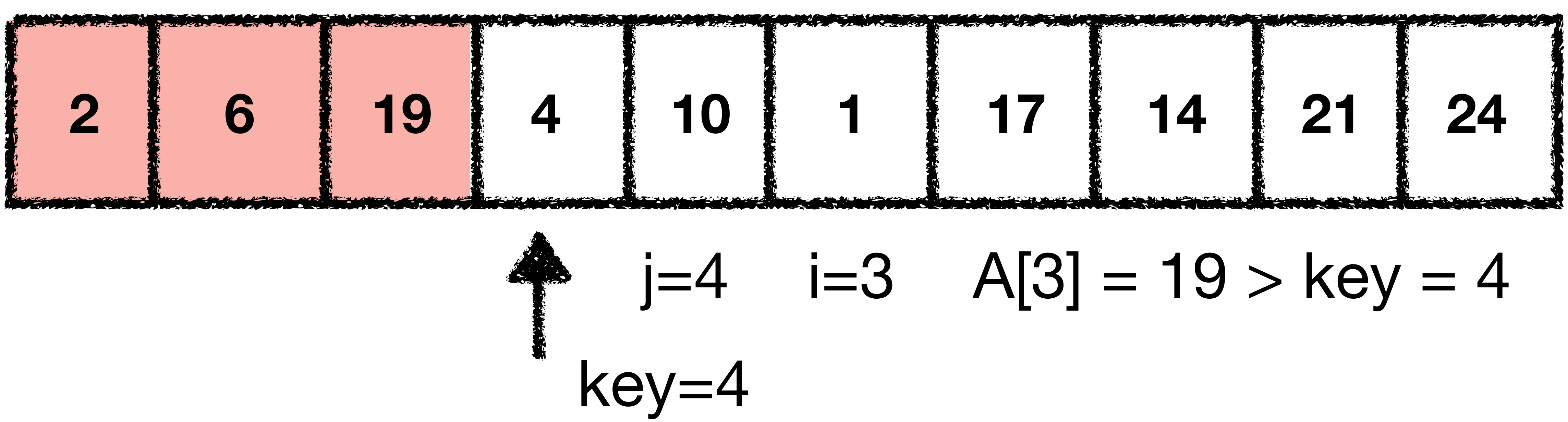

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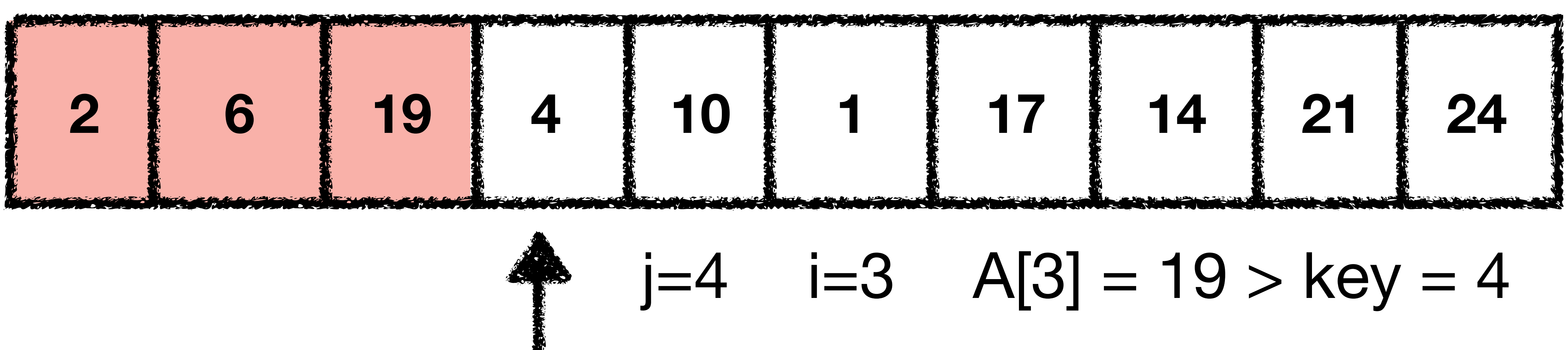

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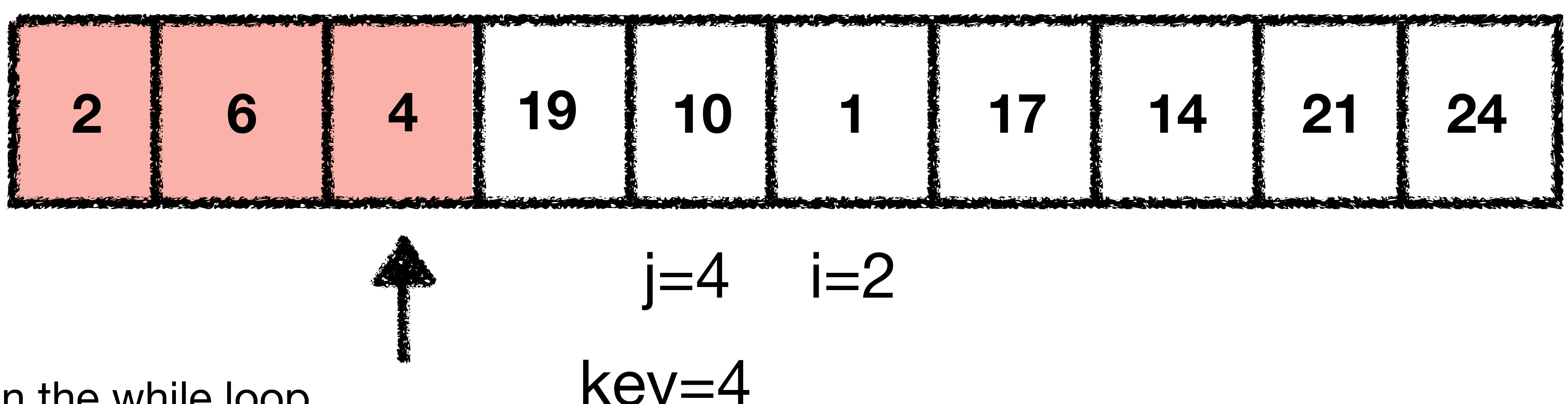

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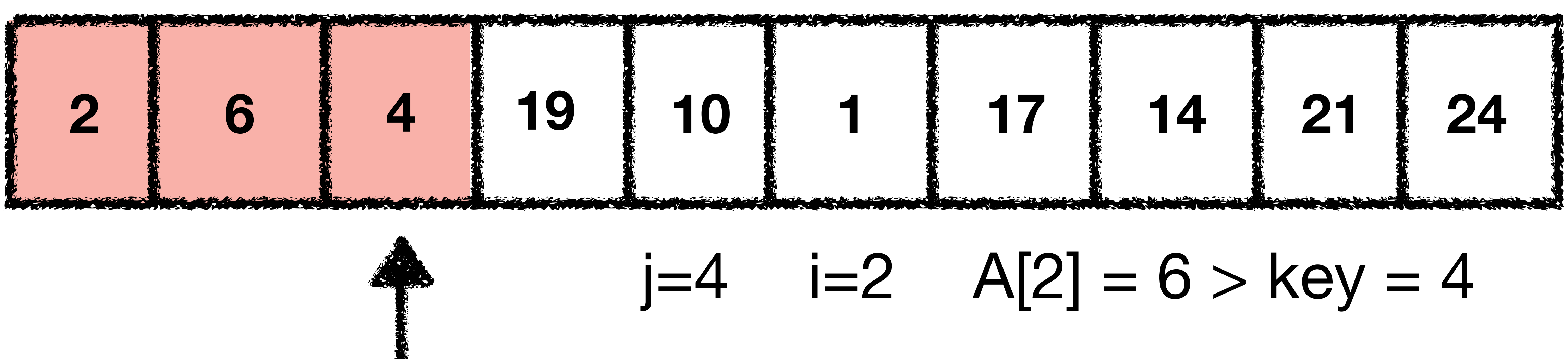

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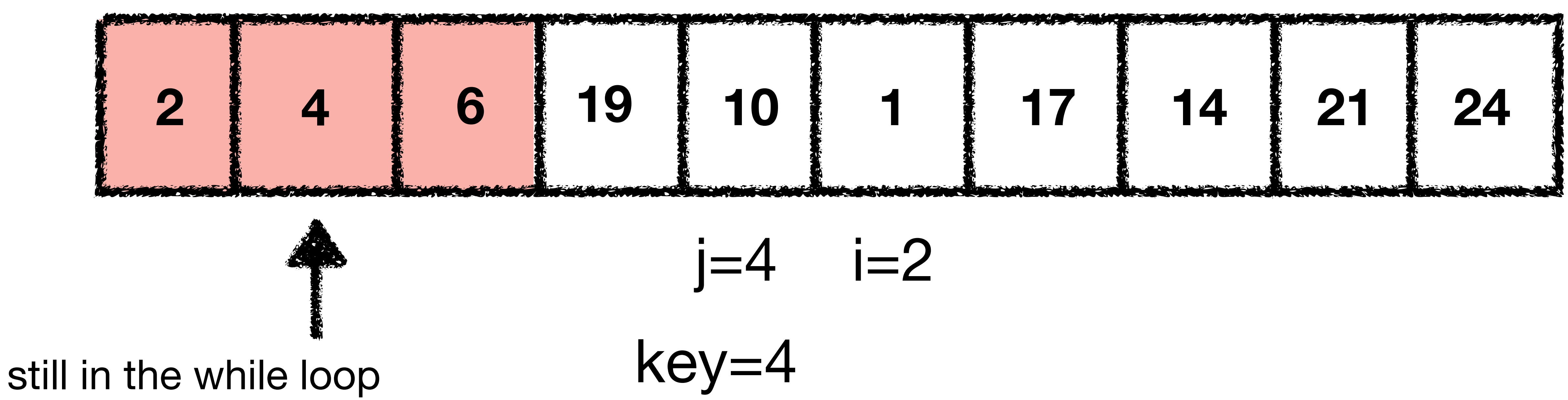

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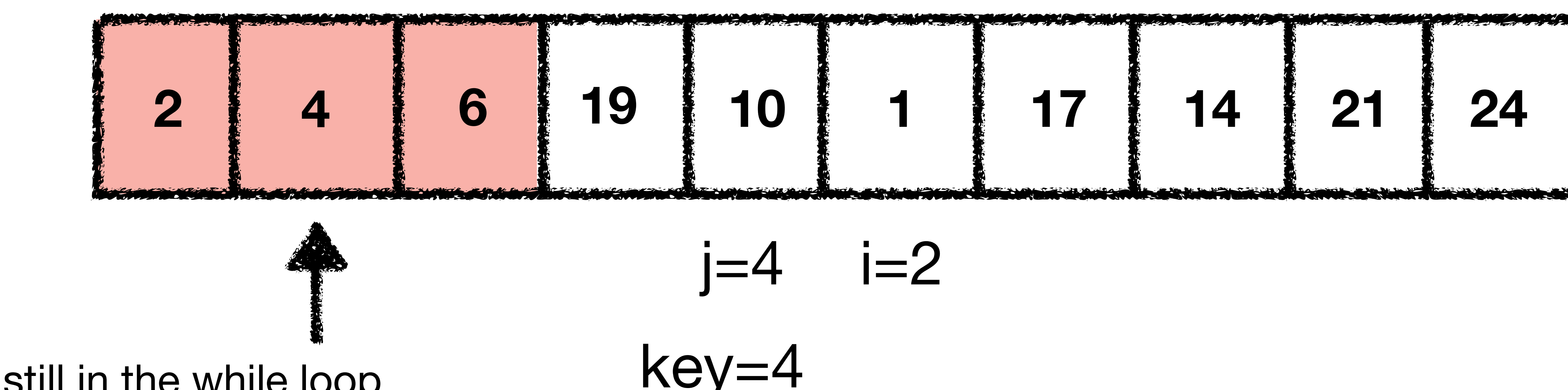
- The algorithm maintains a sorted array in each iteration (each time the for loop is executed).



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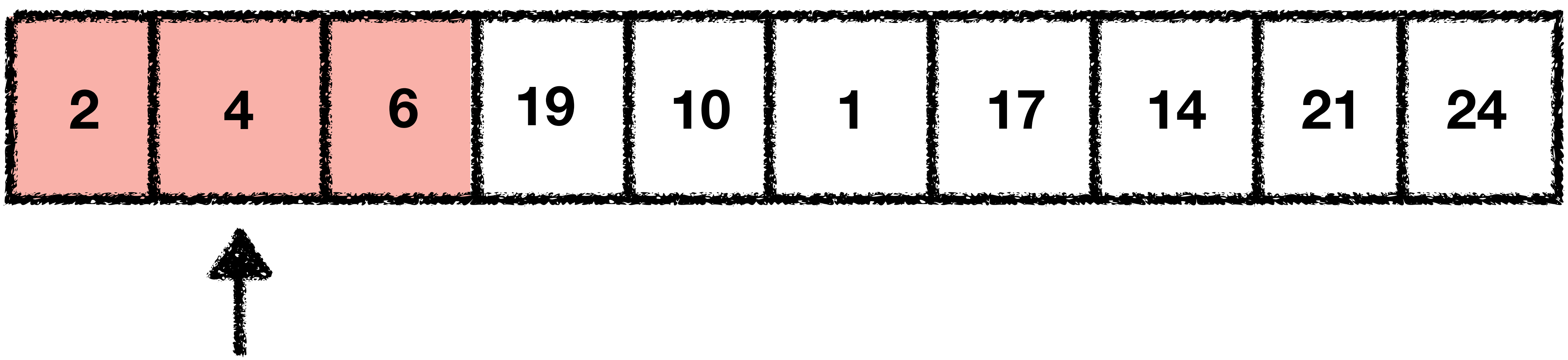

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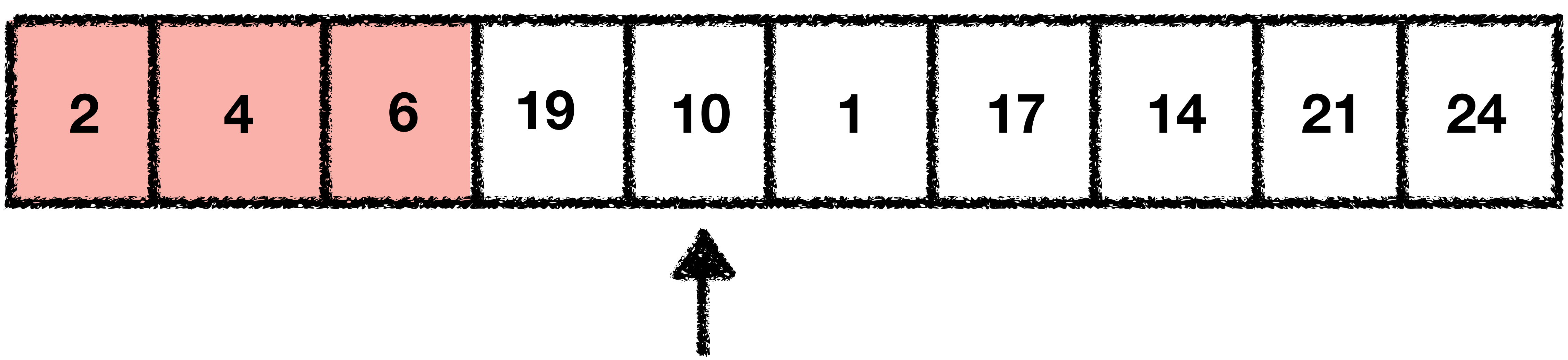

j=5

still in the while loop

What should we expect from algorithms?

- Correctness: It computes the desired output.
- Termination: Eventually terminates (or with high probability).
- Efficiency:
  - The algorithm runs *fast* and/or uses *limited memory.*
  - The algorithm produces a “good enough” outcome.

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  - Proof techniques (induction, proof-by-contradiction etc).

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  - Proof techniques (induction, proof-by-contradiction etc).
- For those of you that took INF2-IADS: We did this a lot there!

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Time Complexity

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  - Control (branch, subroutine call, return, etc).
- Each instruction is carried out in constant time.
- We can count the number of instructions, or the number of steps.

Example: Running Time of InsertionSort

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INSERTION_SORT (A)

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- Q: What is the total and the auxiliary memory usage of InsertionSort?

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- **Convention:** When we say "the running time of Algorithm A", we mean the worst-case running time, over all possible inputs to the algorithm.
- We can also measure the best-case running time, over all possible inputs to the problem.
- In between: average-case running time.
    - Running time of the algorithm on inputs which are chosen at random from some distribution.
    - The appropriate distribution depends on the application (usually the uniform distribution - all inputs equally likely).

Example: Average Running Time of InsertionSort

INSERTION_SORT (A)

FOR j $\leftarrow$ 2 **TO** length[A] **n times**
    DO key $\leftarrow A[j]$ **n-1 times**
        {Put $A[j]$ into the sorted sequence $A[1\,..\,j-1]$}
        $i \leftarrow j-1$ **n-1 times**
        WHILE $i > 0$ and $A[i] >$ key $\sum_{j=2}^{n} t_j$ **times**
            DO $A[i+1] \leftarrow A[i]$
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- Select an input uniformly at random from all possible sequences with n numbers.

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INSERTION_SORT (A)

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2.        DO key ← A[j]                              n-1 times
3.             {Put A[j] into the sorted sequence A[1 . . j − 1]}
4.             i ← j − 1                              n-1 times
5.             WHILE i > 0 and A[i] > key             Σ_{j=2}^{n} t_j times
6.                  DO A[i +1] ← A[i]                 Σ_{j=2}^{n} (t_j − 1) times
7.                     i ← i − 1
8.             A[i + 1] ← key                         n-1 times
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for loops, the tests are executed one more time than the loop body

- Select an input uniformly at random from all possible sequences with n numbers.
- On average, key will be smaller than half of the elements in A[1,…,j].

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Bounded by some cn^2 for some constant c

Asymptotic Notation

- When n becomes large, it makes less of a difference if an algorithm takes 2n or 3n steps to finish.
- In particular, 3logn steps are fewer than 2n steps.
- We would like to avoid having to calculate the precise constants.
- We use asymptotic notation (next lecture).