Algorithms and Data Structures

Upper and Lower Bounds for Sorting, Matrix Multiplication

Worst-case Running Times, Upper Bounds

Algorithm **Mergesort**(**A**[$i,\ldots,j$])

If $i=j$, return i

$q=(i+j)/2$

$\mathbf{A}_{left}$=**Mergesort**(**A**[$i,\ldots,q$])
$\mathbf{A}_{right}$=**Mergesort**(**A**[$q+1,\ldots,n$])
return **Merge**($\mathbf{A}_{left}$, $\mathbf{A}_{right}$)

Recurrence relation:

$$T(n) = 2T(n/2) + f(n)$$

where $f(n) = O(n)$

If we solve the recurrence relation we obtain
$T(n) = O(n \lg n)$

To be exactly precise

Recurrence relation: For some constant c,

$$T(n) = \begin{cases} T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn, & \text{when } n > 2. \\ T(2) \leq c, & \text{otherwise.} \end{cases}$$

How do we solve this recurrence relation?

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The Master Theorem

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Suppose $T(n) \leq \alpha T\left(\lceil n/b \rceil\right) + O(n^d)$
for some constants $\alpha > 0,\ b > 1$ and $d \geq 0$.

$$\text{Then,}\ \ T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b \alpha \\ O(n^d \log_b n), & \text{if } d = \log_b \alpha \\ O(n^{\log_b \alpha}), & \text{if } d < \log_b \alpha \end{cases}$$

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Example: For **MergeSort**, $\alpha = b = 2$ and $d = 1$, we get $O(n \log n)$

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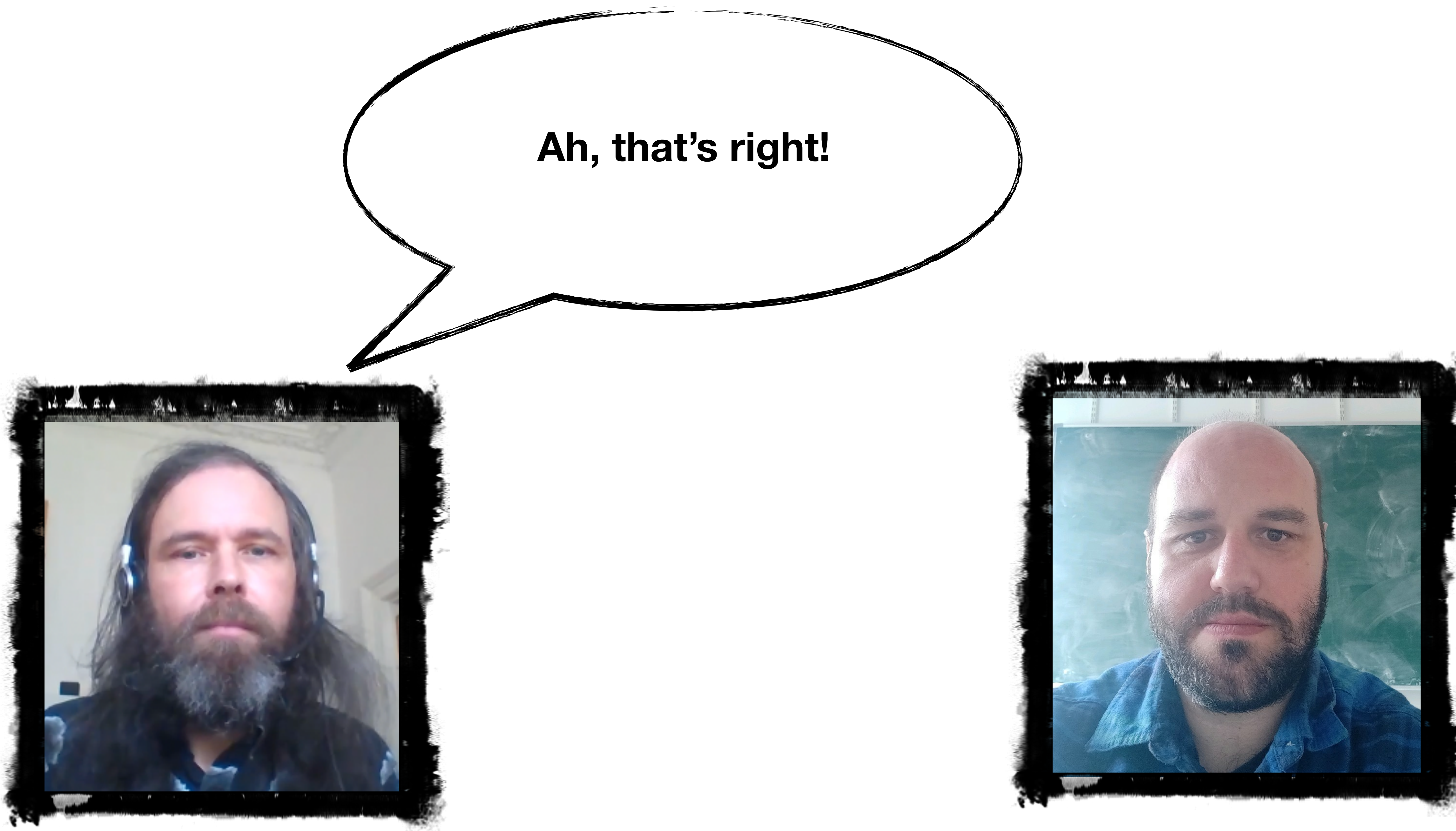

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"Guess and verify": We guess the solution, and verify that it works after substituting in the recurrence relation. Often the argument is by *induction on n.*

"Unrolling the recursion": Figure out the solution for the first few *levels*, and then identifying a pattern. In the end we sum the solutions over all the levels. Often also called the method of "recursion trees".

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For $n = 8$, we have $T(8) = 2T(4) + 8c \leq 12c + 8c = 20c$.
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&= c \cdot 2^m \log_2 2^m - c \cdot 2^m \log_2 2 + c \cdot 2^m = c \cdot 2^m \log_2 m - c \cdot 2^m + c \cdot 2^m \\
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logarithm property

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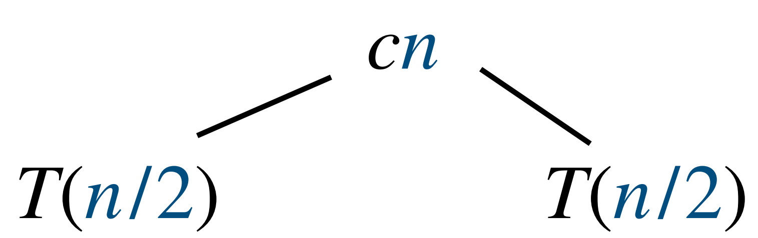

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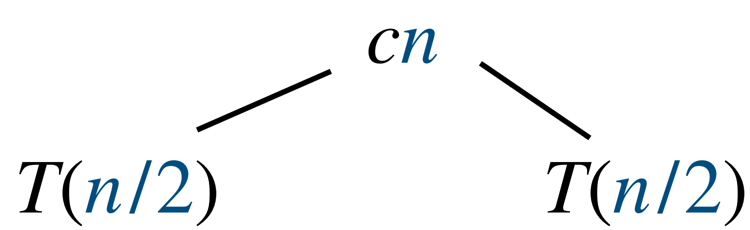
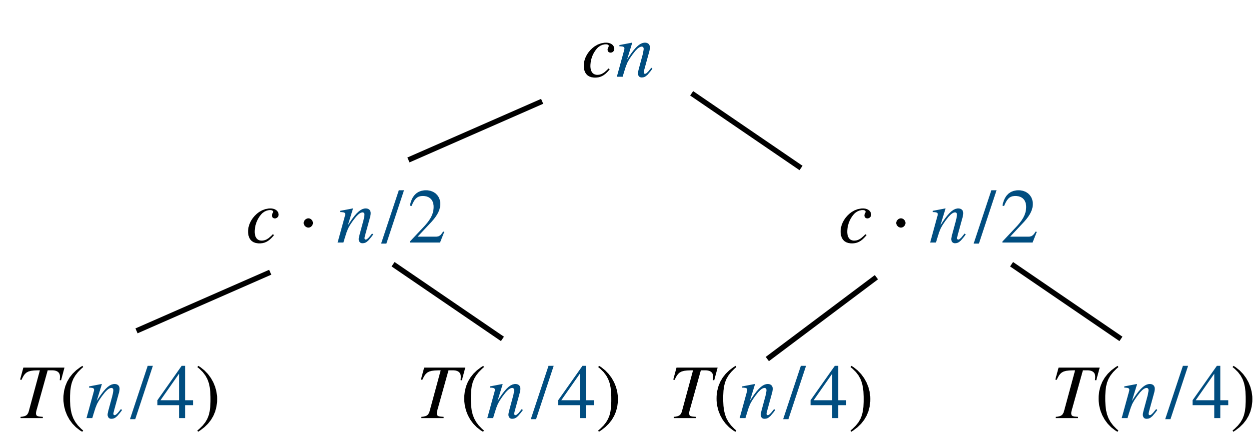


Second iteration: Price of $c \cdot n/2$ for each subproblem,
plus the cost of two subproblems of size $n/4$



Solving the Mergesort recursion

In total, there will be $\log n + 1$ levels (input halved every time).

Level 0 has cost $C_0(n) = cn$

Level 1 has cost $C_1(n) = 2c \cdot n/2 = cn$

Level 2 has cost $C_2(n) = 4c \cdot n/4 = cn$

Level j has cost $C_j(n) = 2^j c \cdot n/2^j = cn$

The last level has cost cn

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Solving the **Mergesort** recursion

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Recurrence relation:

$$T(n) = \sum_{j=1}^{\log n+1} C_j(n) + cn = \sum_{j=1}^{\log n+1} cn + cn$$

Solving the Mergesort recursion

Recurrence relation:

Sum the solutions

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The overall running time is $O(n \log n)$.

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For a formal proof, those techniques could be used together.

The Quicksort algorithm

Algorithm **Quicksort**(**A**[$i,\ldots,j$])

y = **Partition**(**A**[$i,\ldots,j$])

Quicksort(**A**[$i,\ldots,y\text{-}1$])

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What is the worst possible running time?

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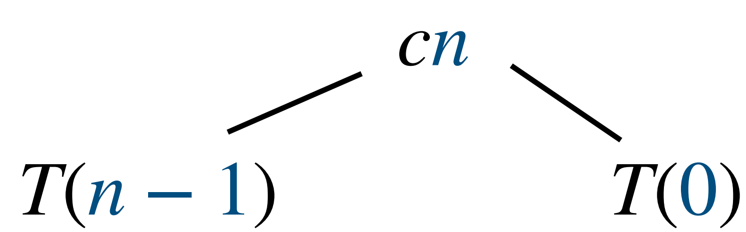
"Unrolling"

First iteration: Price of cn plus the cost of two subproblems of size $n - 1$ and 0.



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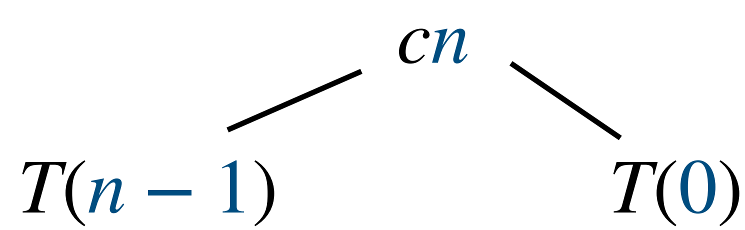
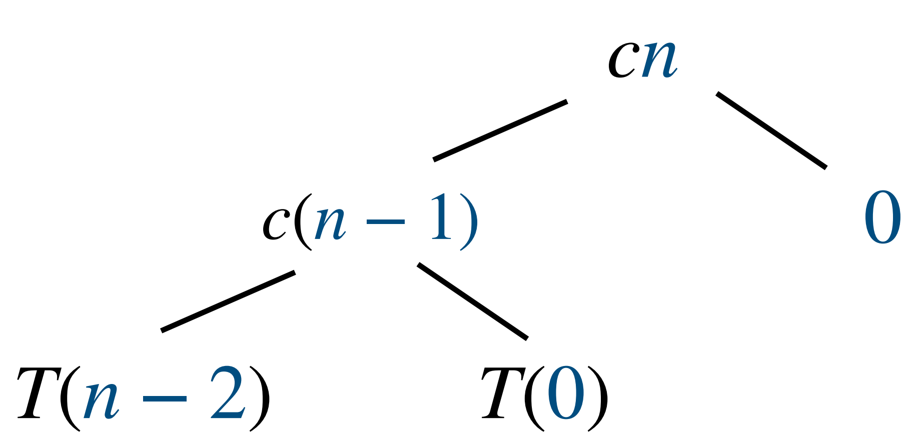


Second iteration: Price of $c(n-1)$, plus the cost of two subproblems of size $n - 2$ and 0.



Solving the Quicksort recursion

How many levels do we have in total?

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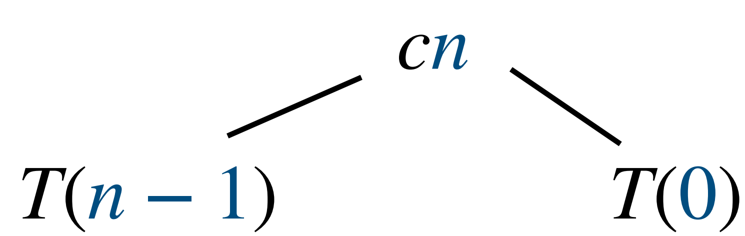

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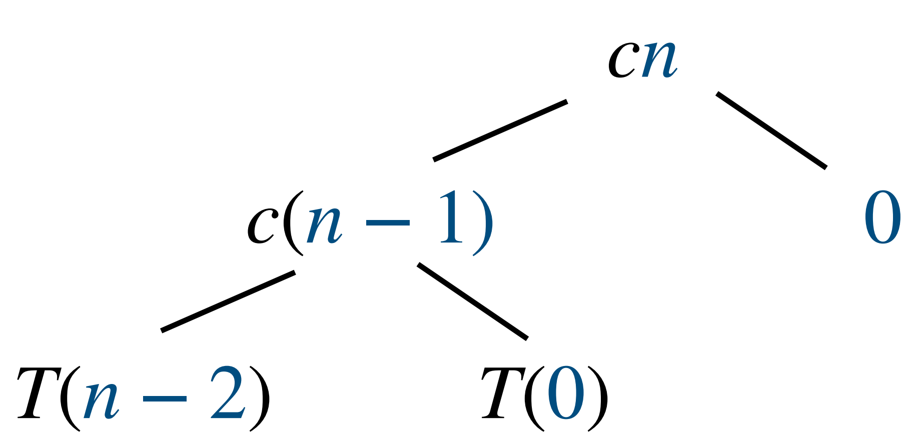

Solving the Quicksort recursion

How many levels do we have in total? n levels.

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Level 1 has cost $C_1(n) = c(n-1)$

Level 2 has cost $C_2(n) = c(n-2)$

Level j has cost $C_j(n) = c(n-j)$

The last level has cost c

Solving the Quicksort recursion

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Recurrence relation:

$$T(n) = \sum_{j=1}^{n-1} C_j(n) = c(n + n - 1 + n - 2 + \ldots + 1)$$

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The overall running time is $O(n^2)$.

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Can we show a lower bound of $\Omega(n^2)$ on the running time of Quicksort?

Upper and Lower (Worst-Case) Bounds

Upper Bound $O(g_1(n))$: On *any possible input* to the problem, our algorithm will take time (at most) $O(g_1(n))$.

Lower Bound $\Omega(g_2(n))$: There *exists at least one input* to the problem, on which our algorithm will take time (at least) $\Omega(g_2(n))$.

When $g_1(n) = g_2(n)$, we say that our running time analysis is *tight*, and we have fully understood the (asymptotic, worst-case) running time of the algorithm.

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Q: Can you think of an input where **Quicksort** takes time $\Omega(n^2)$?

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For us here concretely

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Upper Bound $O(n \log n)$: We have identified an algorithm **(Mergesort)** that has worst-case running time $O(n \log n)$.

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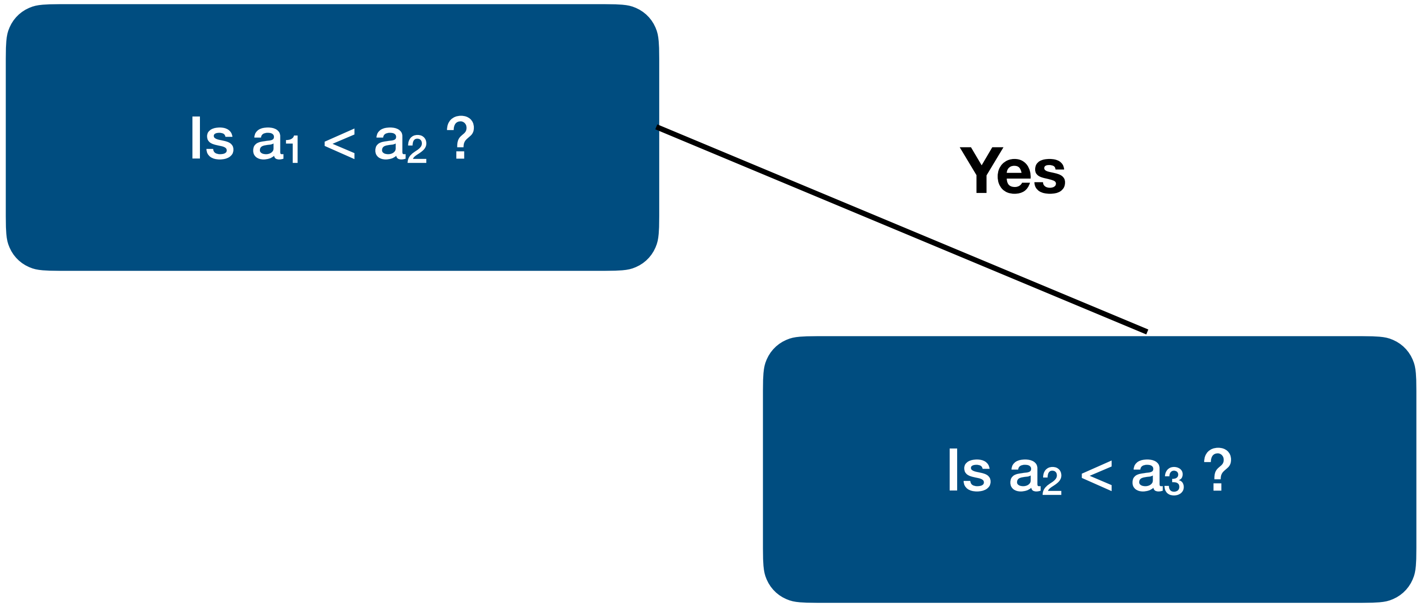
In other words, we will prove that there is no algorithm that is *asymptotically better* than **Mergesort**.

Lower bound for sorting

Lower bound for sorting

Is $a_1 < a_2$?

Lower bound for sorting



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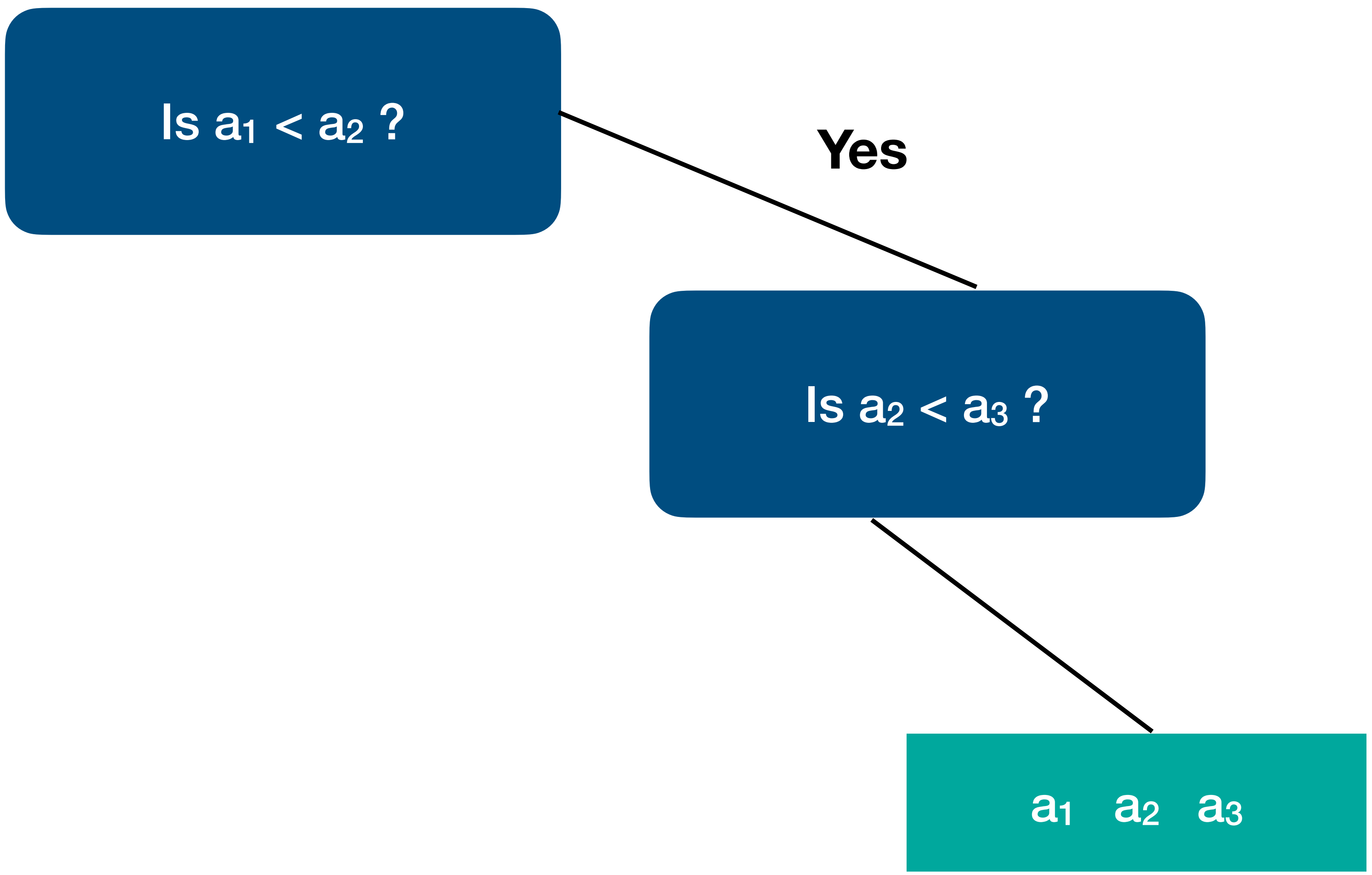

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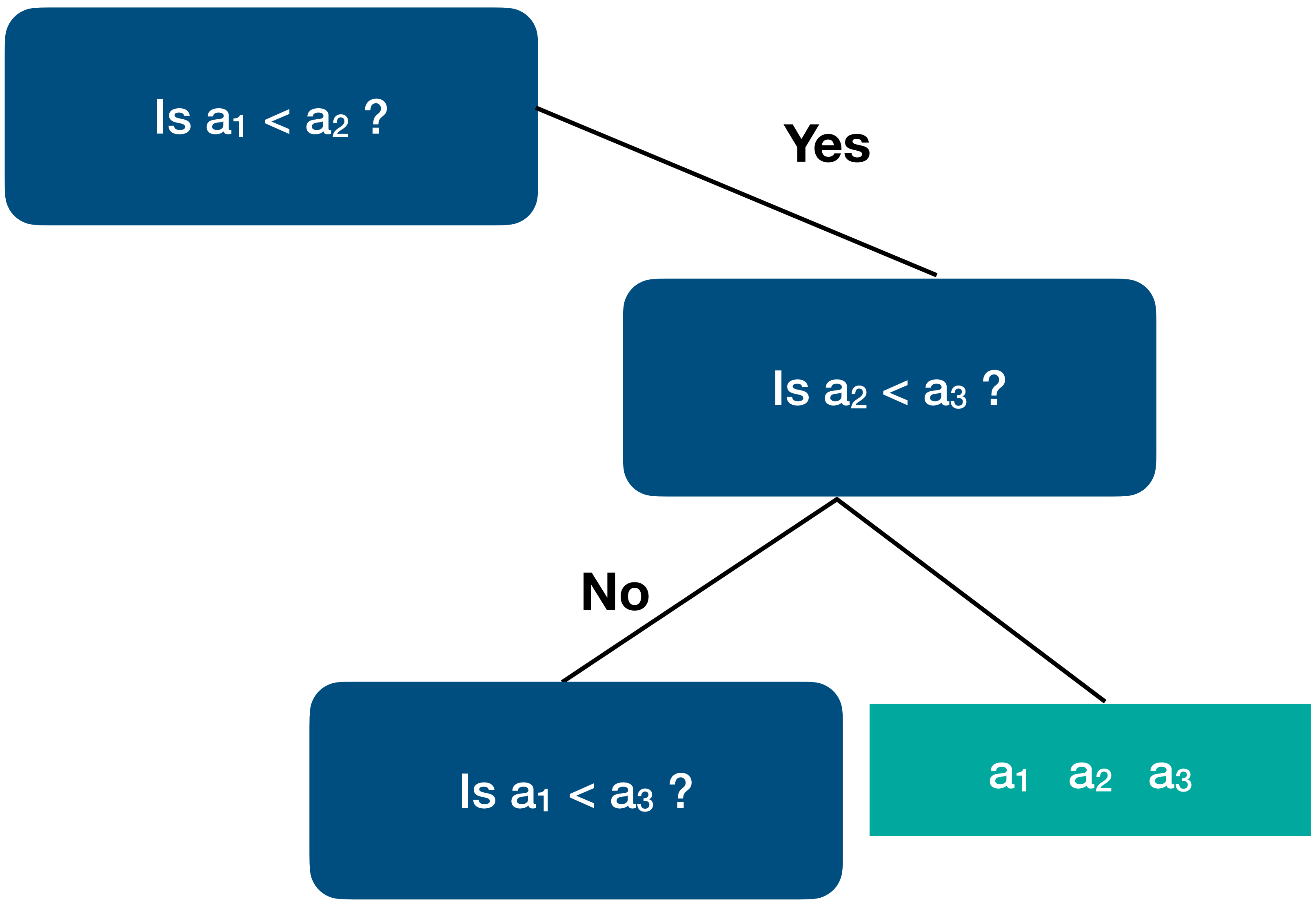

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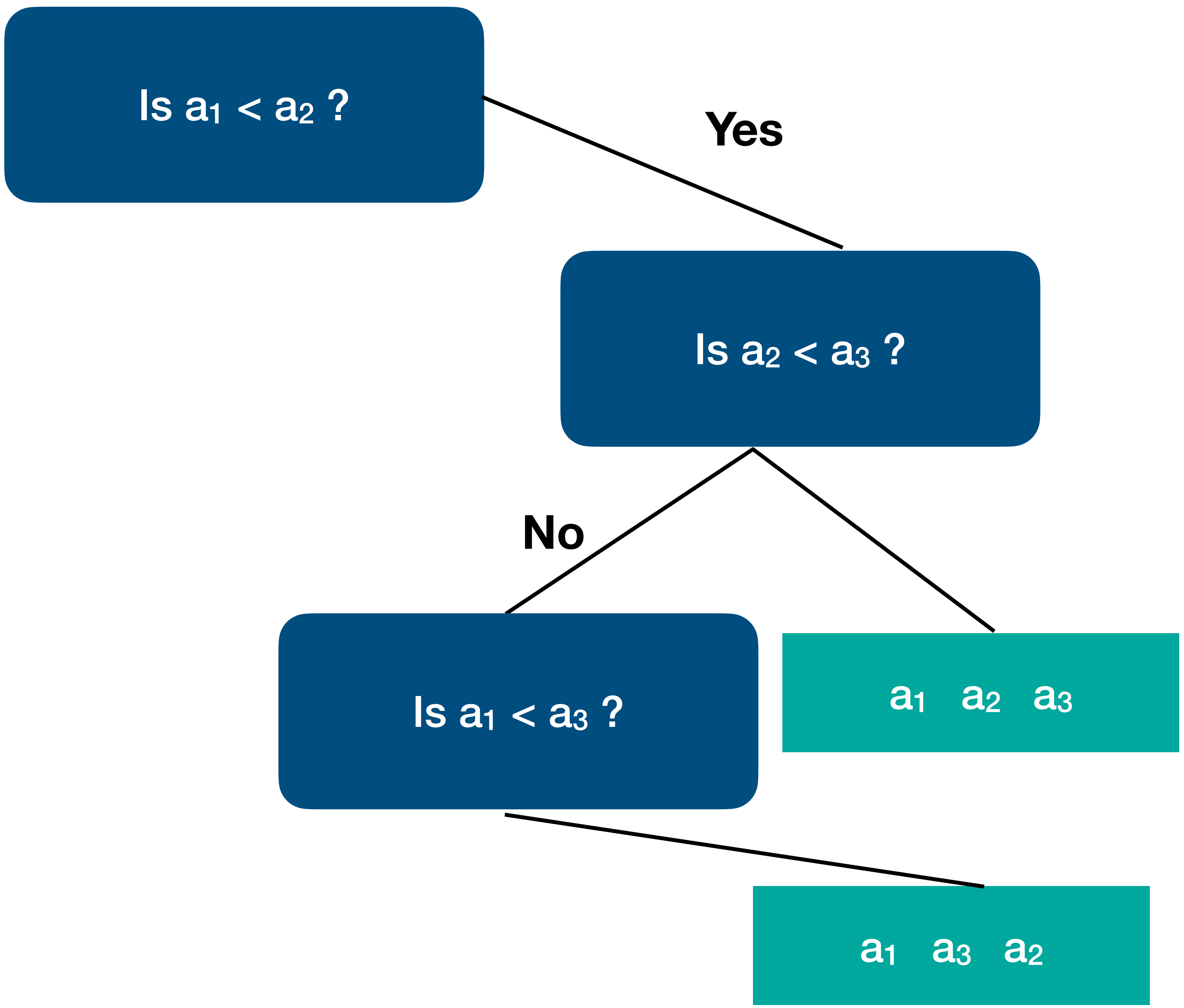

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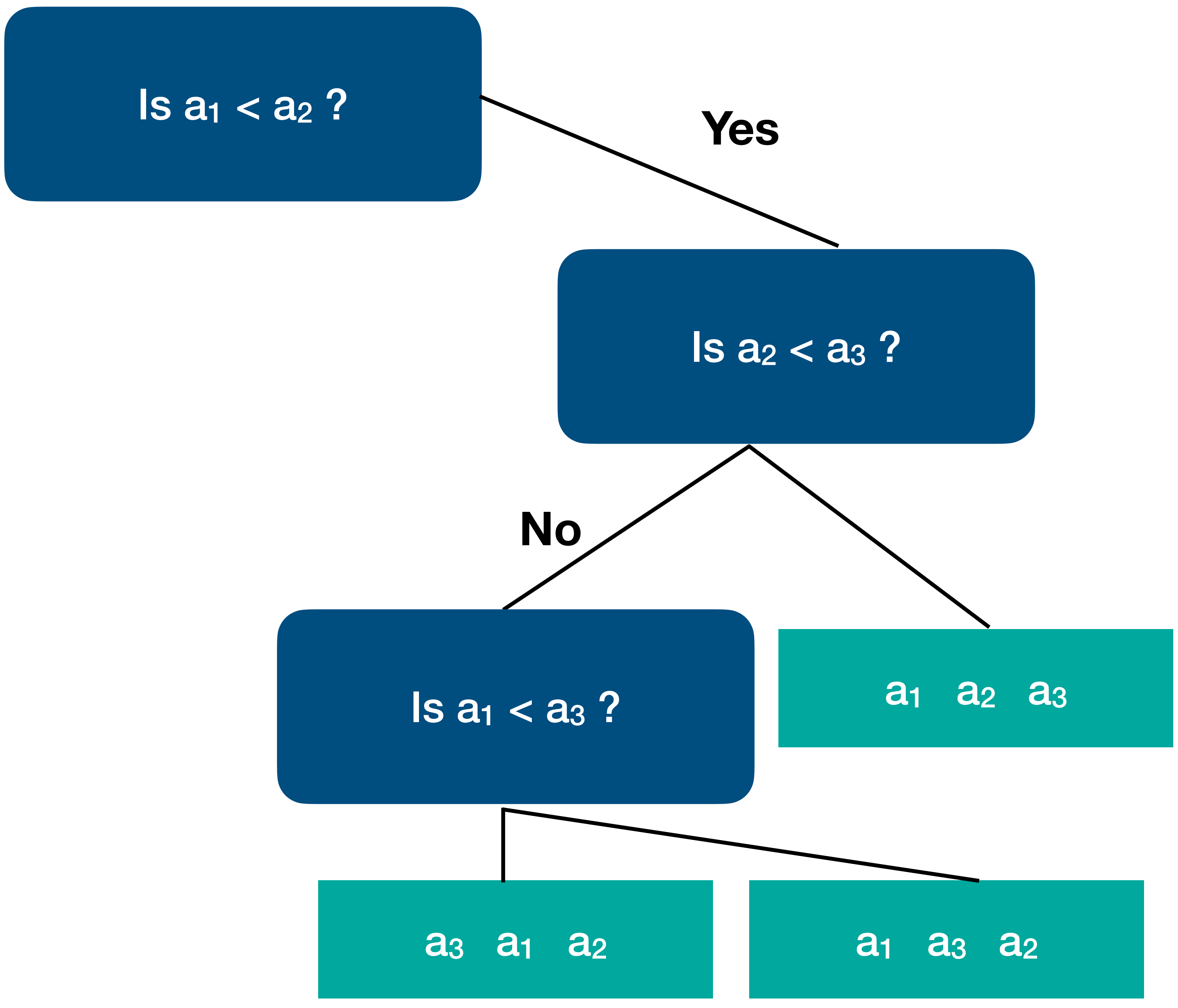

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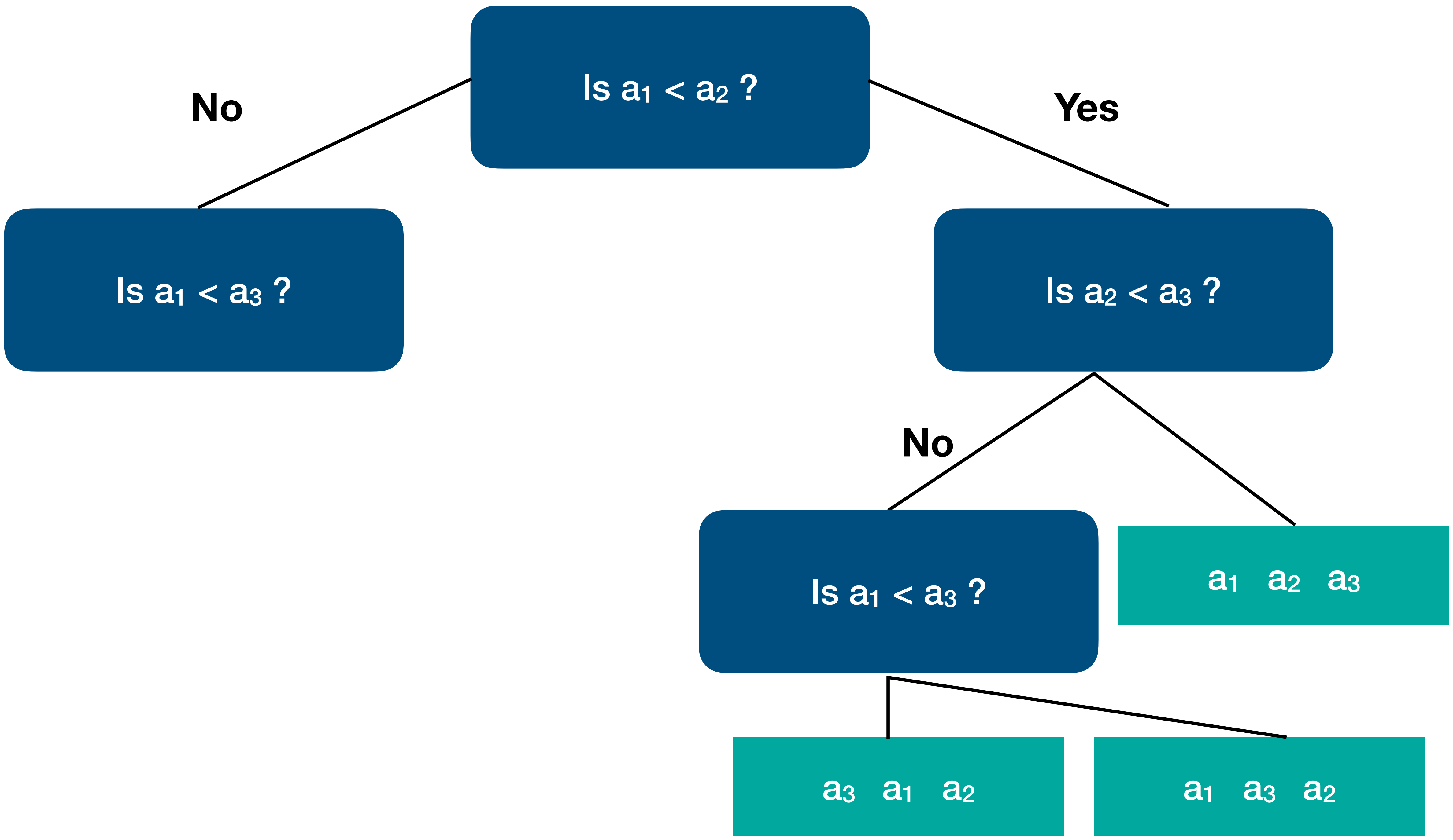

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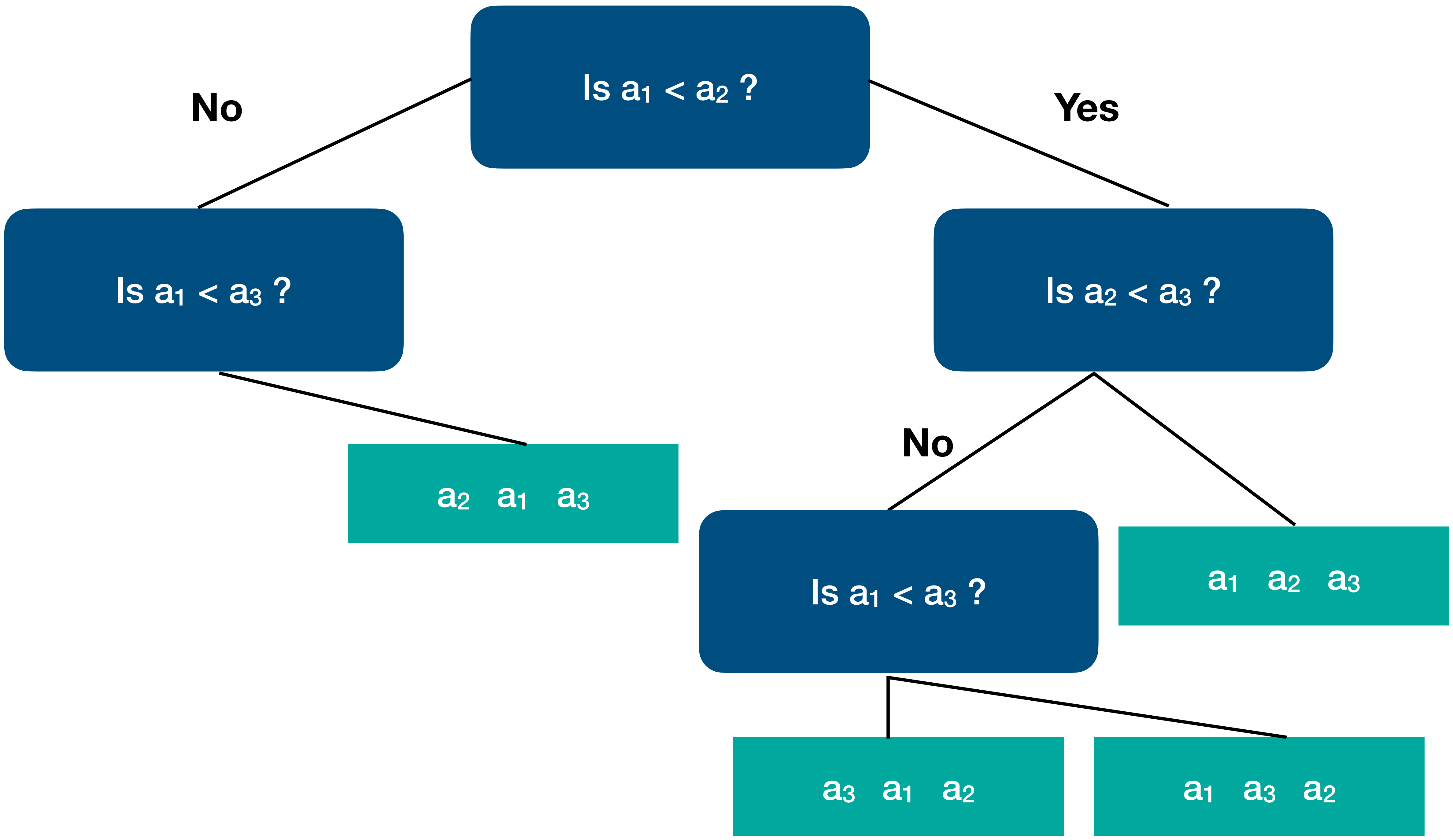

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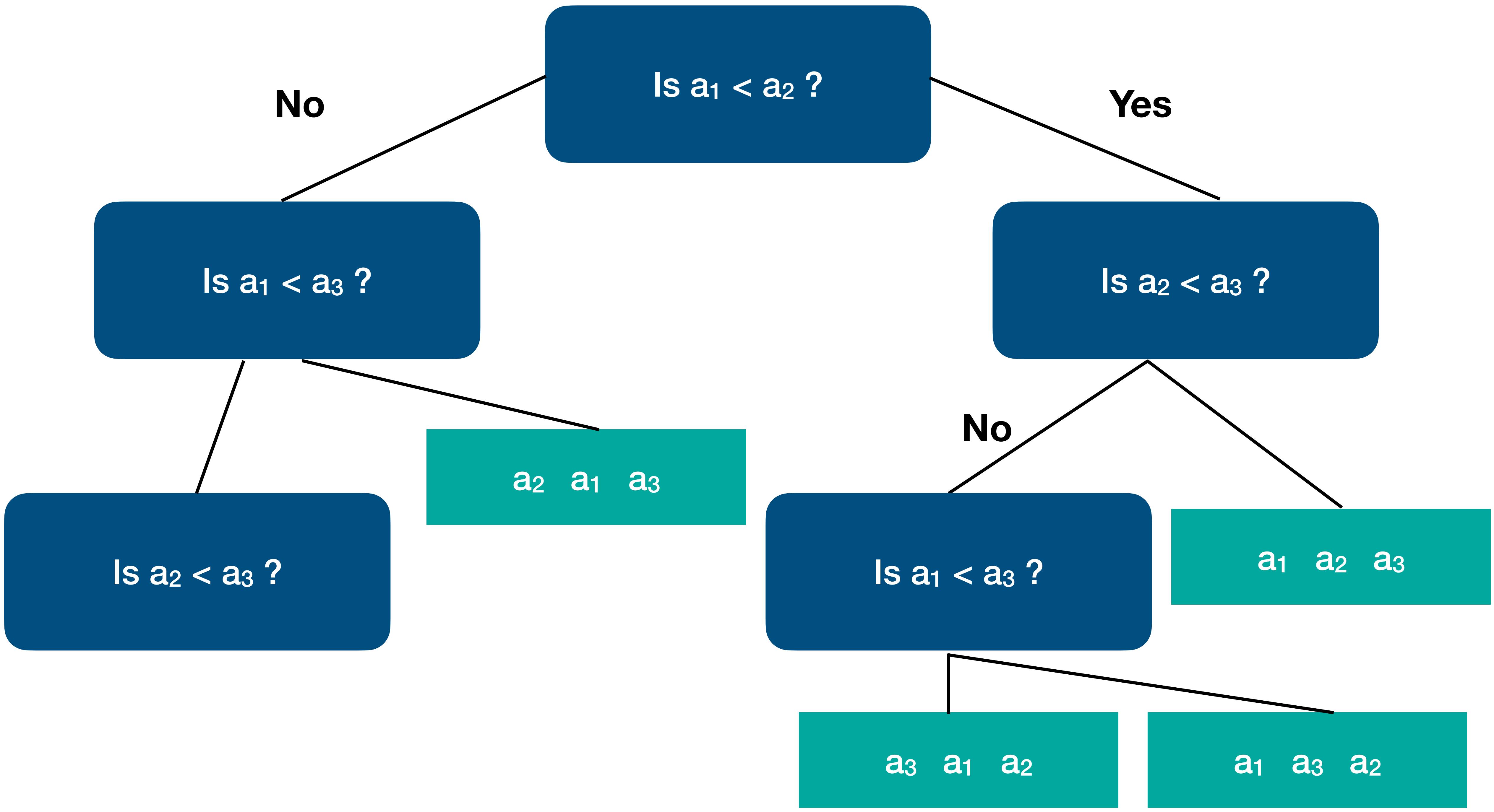

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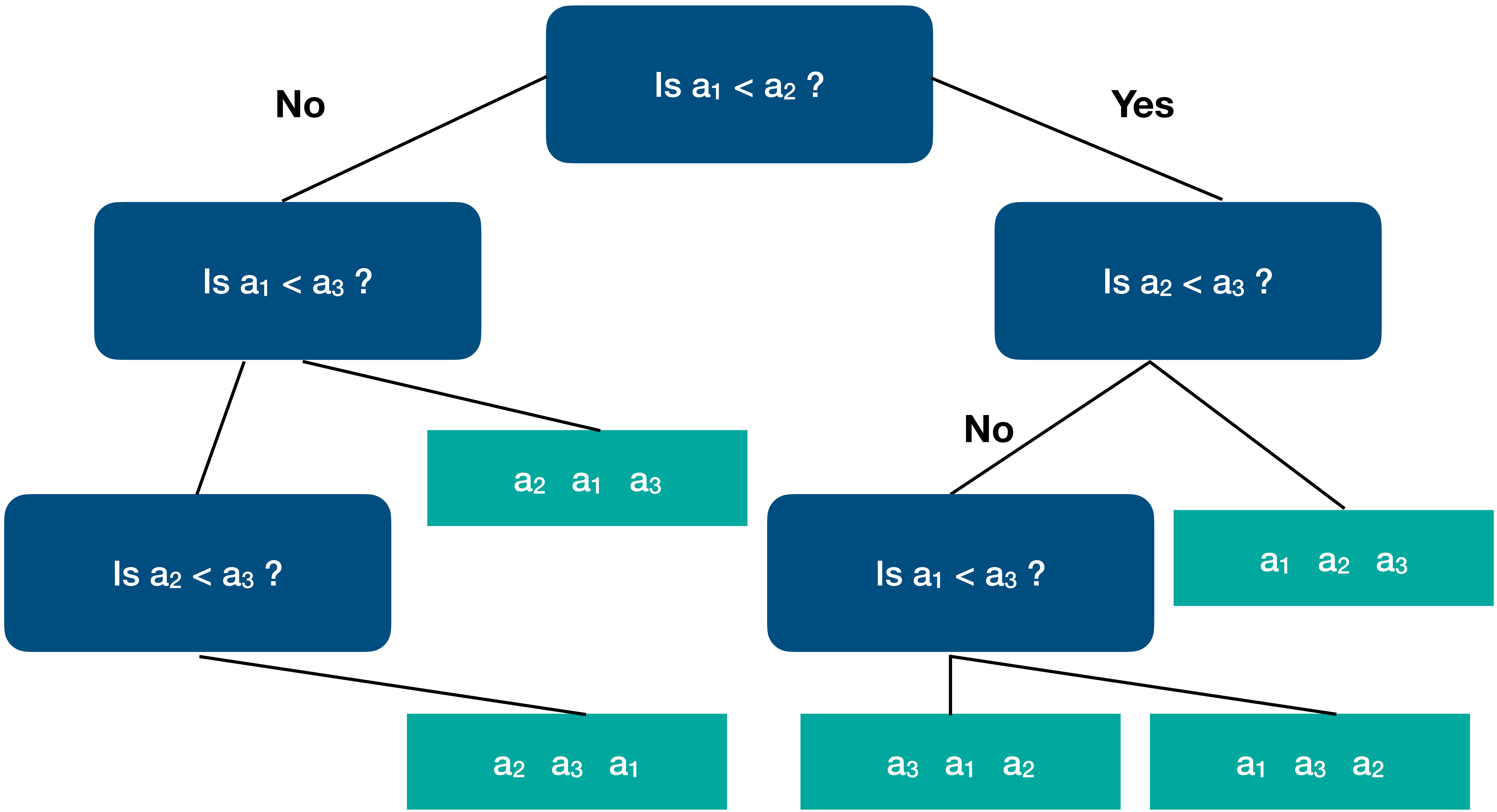

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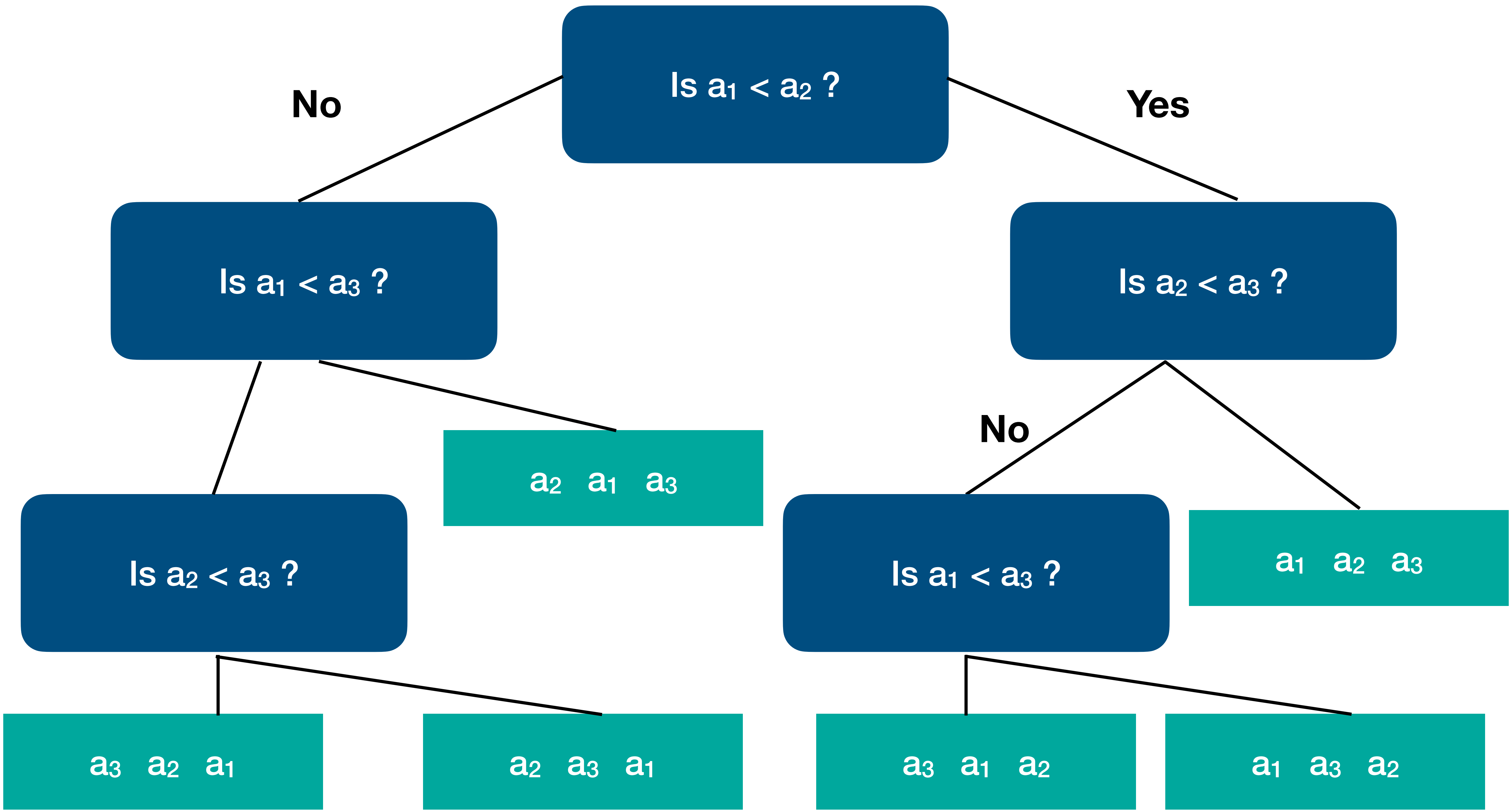

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Every possible permutation can appear as a leaf, since every possible permutation is a valid output.

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Fact: Every binary tree of depth d has at most 2^d leaves.

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Think about it at home! Try to prove it using induction.

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$$\begin{aligned}\log_2(n!) &= \log_2 (1 \cdot 2 \cdot, \ldots, \cdot n) \\ &= \log_2(1) + \log_2(2) + \ldots + \log_2(n) \\ &\geq \log_2(n/2) + \ldots + \log_2(n) \text{ *(half)*}\end{aligned}$$

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e.g., **CountingSort**

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Upper bound: We construct an algorithm that has performance $O(g(n))$ for criterion A.

Lower bound: We show that for any algorithm, the performance for criterion A is $\Omega(g(n))$.

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How do we find this function?

No easy answer!

We try to design algorithms which are as good as possible and when we feel that we can not improve more, we try to prove the *matching* lower bound.

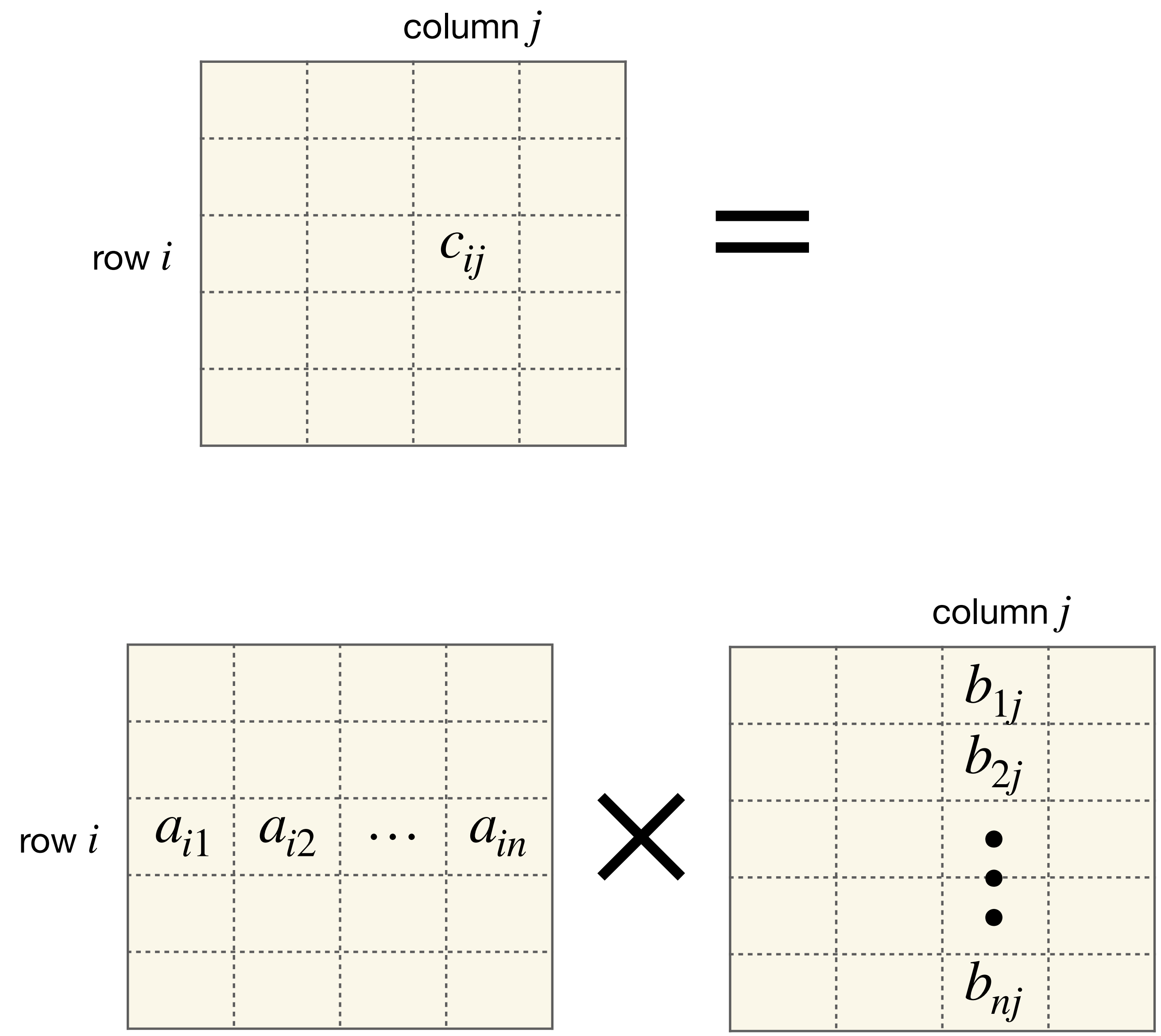
Matrix Multiplication

Matrix Multiplication

Assume that we have two square $(n \times n)$-matrices $A = (a_{ij})_{1 \le i,j \le n}$ and $B = (b_{ij})_{1 \le i,j \le n}$

The product of A and B is the $(n \times n)$-matrix $C = (c_{ij})_{1 \le i,j \le n}$ with entries

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$



A straightforward approach

Compute the sum of pairwise products $a_{ik} \cdot b_{kj}$ for each entry c_{ij} of C.

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for $i = 1$ **to** n **do**
    for $j = 1$ **to** n **do**
        $c_{ij} = 0$
        for $k = 1$ **to** n **do**
            $c_{ij} = c_{ij} + a_{ik} + b_{kj}$

return $C = (c_{ij})_{1 \leq i,j \leq n}$

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Running time: $\Theta(n^3)$

A naive D&C approach

Suppose we divide our matrices A and B as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

We can write C as:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ A_{21} \cdot B_{11} + A_{22} \cdot B_{21} & A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{pmatrix}$$

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We will assume from now on that $n = 2^k$ for some k.

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$$= \begin{pmatrix} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ A_{21} \cdot B_{11} + A_{22} \cdot B_{21} & A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{pmatrix}$$

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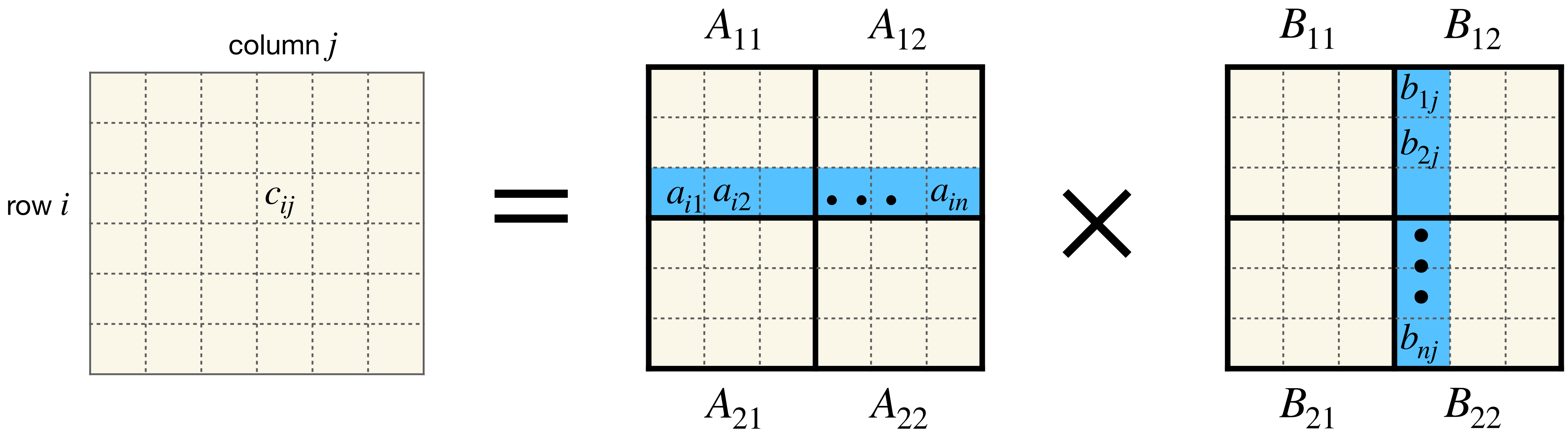

Suppose $i \leq n/2$ and $j > n/2$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n/2} a_{ik} b_{kj} + \sum_{n/2+1}^{n} a_{ik} b_{kj}$$

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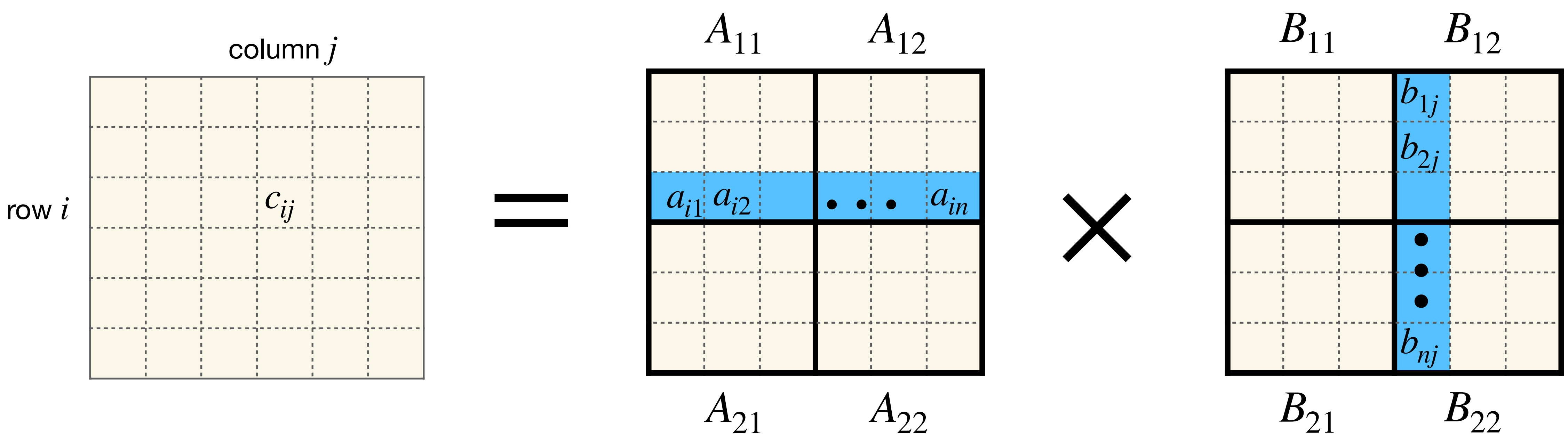

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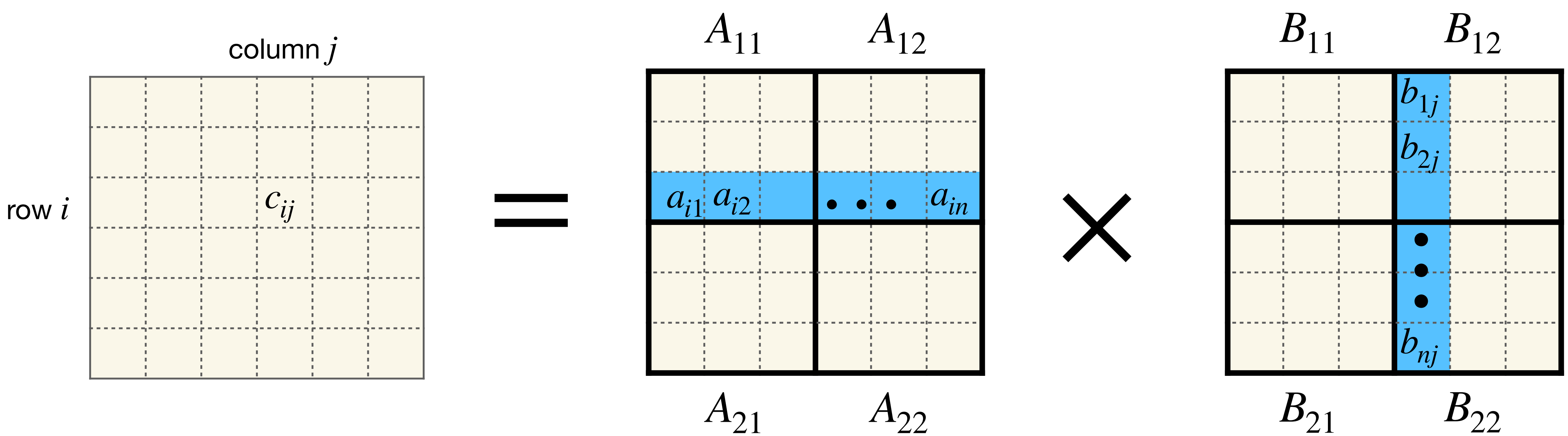

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The D&C algorithm

Matrix-Multiply-DC (A, B)

if $n = 1$, **do**

$c_{11} = a_{11} \cdot b_{11}$

return c_{11}

Partition $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ **and** $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$

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Running Time

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addition: $\Theta(n^2)$

Running Time

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Running Time

Recurrence relation: For some constant c,

$$T(n) = \begin{cases} 8T(n/2) + cn^2, \text{ when } n > 1. \\ c, \text{ when } n = 1. \end{cases}$$

The Master Theorem

The Master Theorem is a very general theorem for solving recurrence relations.

Suppose $T(n) \leq \alpha T\left(\lceil n/b \rceil\right) + O(n^d)$

for some constants $\alpha > 0,\ b > 1$ and $d \geq 0$.

$$\text{Then,}\quad T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b \alpha \\ O(n^d \log_b n), & \text{if } d = \log_b \alpha \\ O(n^{\log_b \alpha}), & \text{if } d < \log_b \alpha \end{cases}$$

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Strassen's remarkable algorithm to the rescue (next lecture)