Algorithms and Data Structures

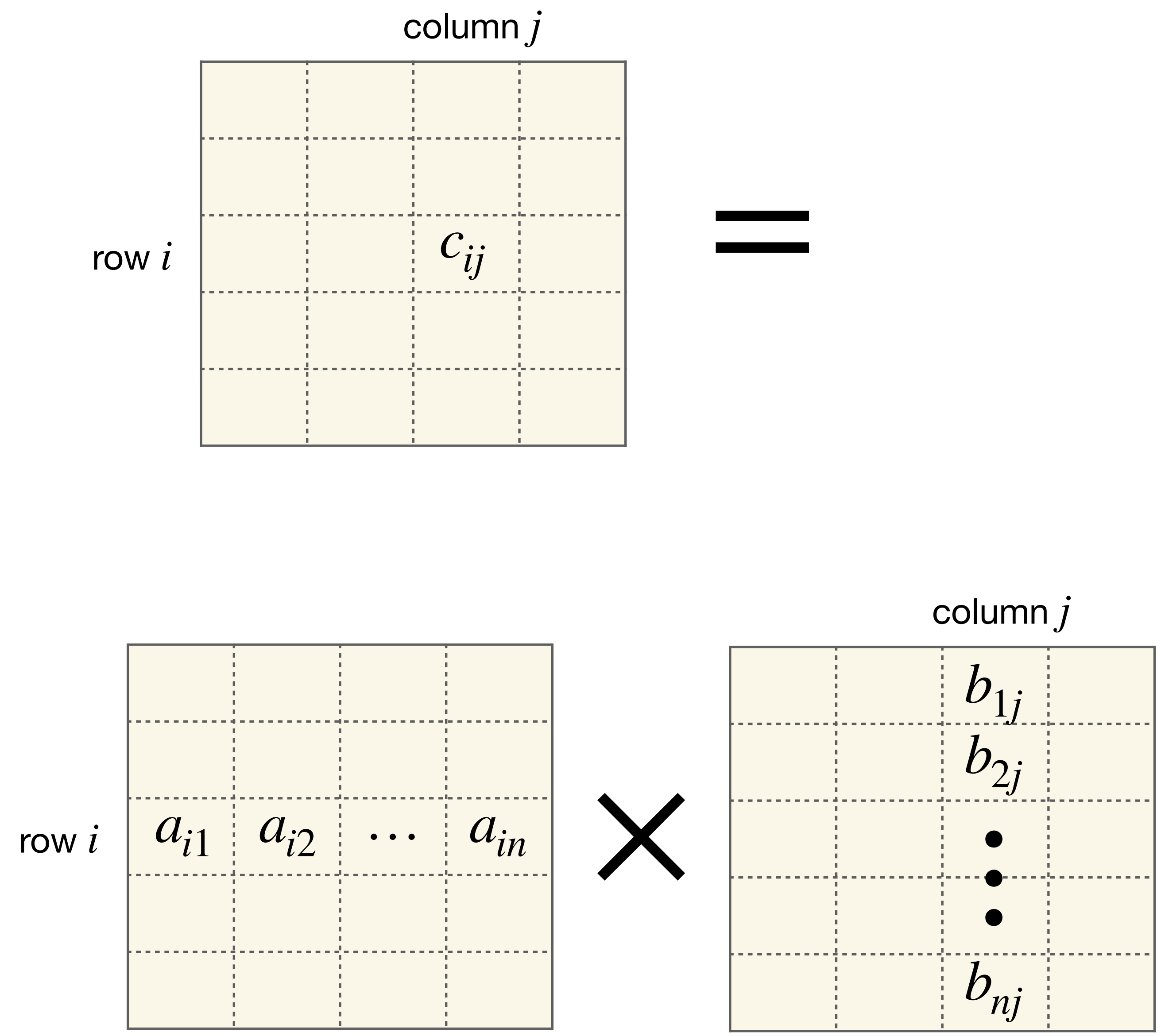
Upper and Lower Bounds for Sorting, Matrix Multiplication

Matrix Multiplication

Assume that we have two square $(n \times n)$-matrices $A = (a_{ij})_{1 \le i,j \le n}$ and $B = (b_{ij})_{1 \le i,j \le n}$

The product of A and B is the $(n \times n)$-matrix $C = (c_{ij})_{1 \le i,j \le n}$ with entries

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$



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Naive Divide & Conquer approach: $O(n^3)$

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Can we do better than that?

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For (possibly large matrices), multiplications are more expensive!

Divide and Conquer…

Suppose we divide our matrices A and B as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

We can write C as:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ A_{21} \cdot B_{11} + A_{22} \cdot B_{21} & A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{pmatrix}$$

…using fewer multiplications.

$$P_1 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$$

$$P_2 = (A_{21} + A_{22}) \cdot B_{11}$$

$$P_3 = A_{11} \cdot (B_{12} - B_{22})$$

$$P_4 = A_{22} \cdot (-B_{11} + B_{21})$$

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$$C_{11} = P_1 + P_4 - P_5 + P_7$$

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$$C_{21} = P_2 + P_4 \qquad C_{22} = P_1 + P_3 - P_2 + P_6$$

Let's calculate C_{11}

$P_1 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$

$P_2 = (A_{21} + A_{22}) \cdot B_{11}$

$P_3 = A_{11} \cdot (B_{12} - B_{22})$

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$$P_1 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22}$$

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$$P_1 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22}$$

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$$P_1 + P_4 = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{22} + A_{22} \cdot B_{21}$$

$$P_1 + P_4 - P_5 = A_{11} \cdot B_{11} + A_{22} \cdot B_{22} + A_{22} \cdot B_{21} - A_{12} \cdot B_{22}$$

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$P_1 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$
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Let's calculate C_{11}

$P_1 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$
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Try it at home: Check C_{12}, C_{21}, and C_{22}.

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How many multiplications do we need?

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We have 7 multiplications of $n/2 \times n/2$ matrices.

$$C_{11} = P_1 + P_4 - P_5 + P_7 \qquad C_{12} = P_3 + P_5$$
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Suppose $T(n) \leq \alpha T\left(\lceil n/b \rceil\right) + O(n^d)$ for some constants $\alpha > 0,\ b > 1$ and $d \geq 0$.

Then, $$T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b \alpha \\ O(n^d \log_b n), & \text{if } d = \log_b \alpha \\ O(n^{\log_b \alpha}), & \text{if } d < \log_b \alpha \end{cases}$$

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Timeline of matrix multiplication exponent

Year	Bound on omega	Authors
1969	2.8074	Strassen[1]
1978	2.796	Pan[10]
1979	2.780	Bini, Capovani [it], Romani[11]
1981	2.522	Schönhage[12]
1981	2.517	Romani[13]
1981	2.496	Coppersmith, Winograd[14]
1986	2.479	Strassen[15]
1990	2.3755	Coppersmith, Winograd[16]
2010	2.3737	Stothers[17]
2012	2.3729	Williams[18][19]
2014	2.3728639	Le Gall[20]
2020	2.3728596	Alman, Williams[21][22]
2022	2.371866	Duan, Wu, Zhou[23]
2024	2.371552	Williams, Xu, Xu, and Zhou[2]

Selection

The selection problem

Definition: The i^{th}-order statistic of a set of n (distinct) elements is the i^{th} smallest element.

i.e., the element which is larger than exactly $i - 1$ other elements.

The Selection Problem:

Selection(**A**[$1, \ldots, n$], i)

Input: A set of n (distinct) numbers (in an array **A**) and a number i, with $1 \leq i \leq n$.

Output: The i^{th}-order statistic of the set.

An easy solution

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Sort the numbers in $O(n \log n)$ time using **MergeSort**.

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Is sorting an overkill?

Divide and conquer

Split the input into smaller inputs.

Solve the problem for the smaller inputs recursively.

Combine the solutions into a solution for the original problem.

The Partition procedure

Procedure **Partition**(**A**[$i, \ldots, j$])

Choose a **pivot element** x of **A**

$k = i$

For $h = i$ to j do

If **A**[h] < x

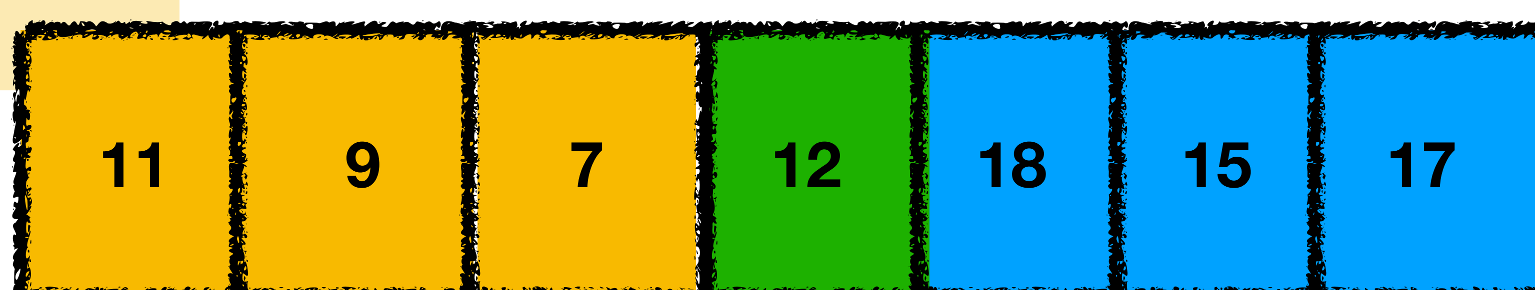
Swap **A**[k] with **A**[h]

$k = k + 1$

Swap **A**[k] with **A**[h]

Return k

Running time **O(n)**



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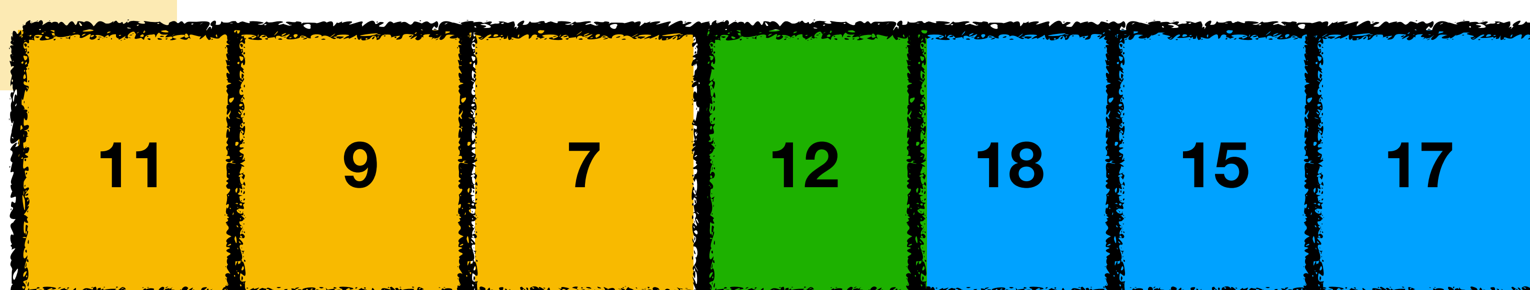
Running time **O(n)**



The Partition procedure

(with the pivot element as input)

Procedure **Partition**(**A**[$i, \ldots, j$], x)

~~Choose a **pivot element** x of **A**~~

$k = i$

For $h = i$ to j do

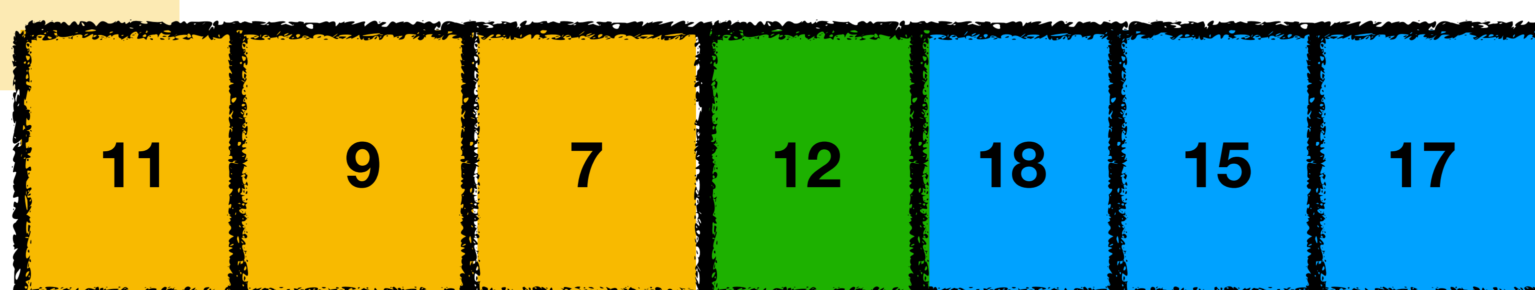
If **A**[h] < x

Swap **A**[k] with **A**[h]
$k = k + 1$

Swap **A**[k] with **A**[h]

Return k

Running time **O(n)**



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What is an algorithm for finding the median?

Selection(A[$1,\ldots,n$],$(n+1)/2$)

Let's try to do that…

Algorithm **Selection**(**A**$[1,\ldots,n]$, i)

x = **Selection**(**A**$[1,\ldots,n]$,$(n+1)/2$)

k = **Partition**(**A**$[1,\ldots,n]$, x)

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Algorithm Selection(**A**$[1,\ldots,n]$, i)

Do you see a problem?

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Before we conquer, we need to divide!

Are we stuck?

We need to partition the array into two using a good pivot element (*the median*).

Or otherwise the running time of the recursion will be bad!

But to find the median, we need an algorithm for selection!

Are we stuck?

We need to partition the array into two using a good pivot element (*something "close" to the median*).

Or otherwise the running time of the recursion will be bad!

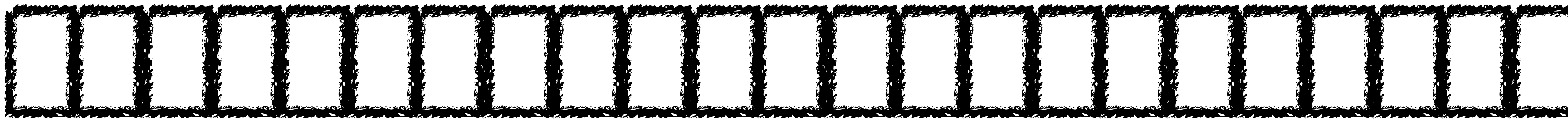
But to find the median, we need an algorithm for selection!

A good pivot element

Split the array **A** into sub-arrays with ***5*** *elements* each.

The last one might have fewer elements.

A good pivot element



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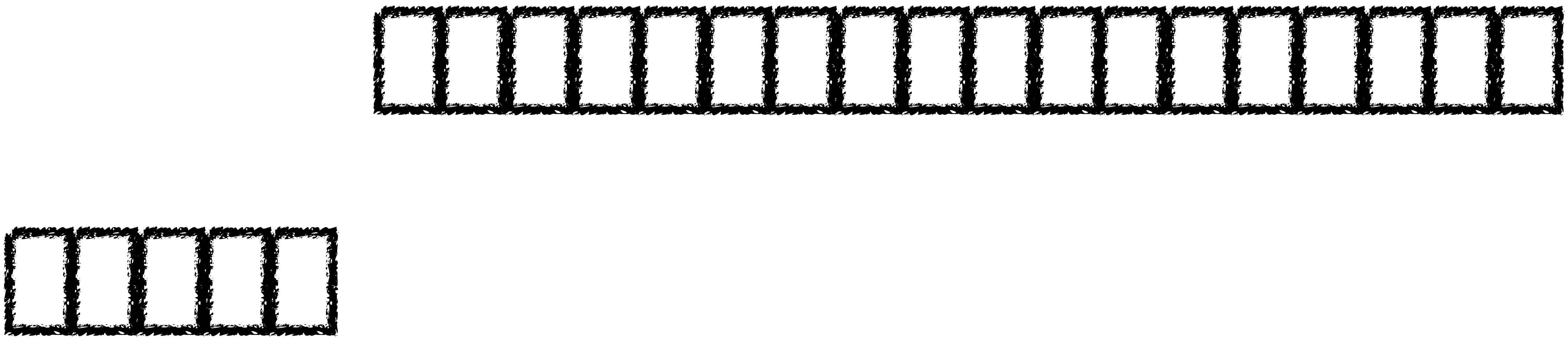

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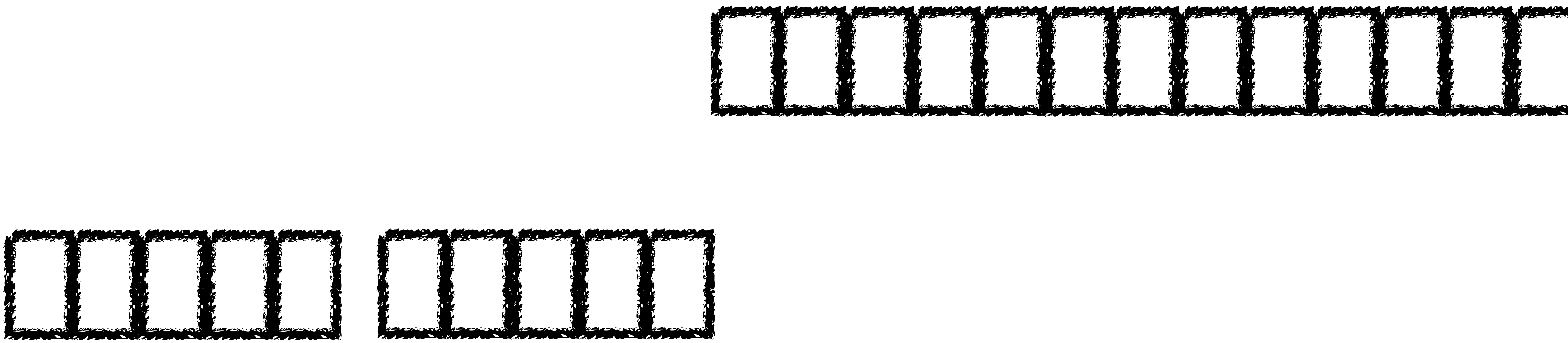

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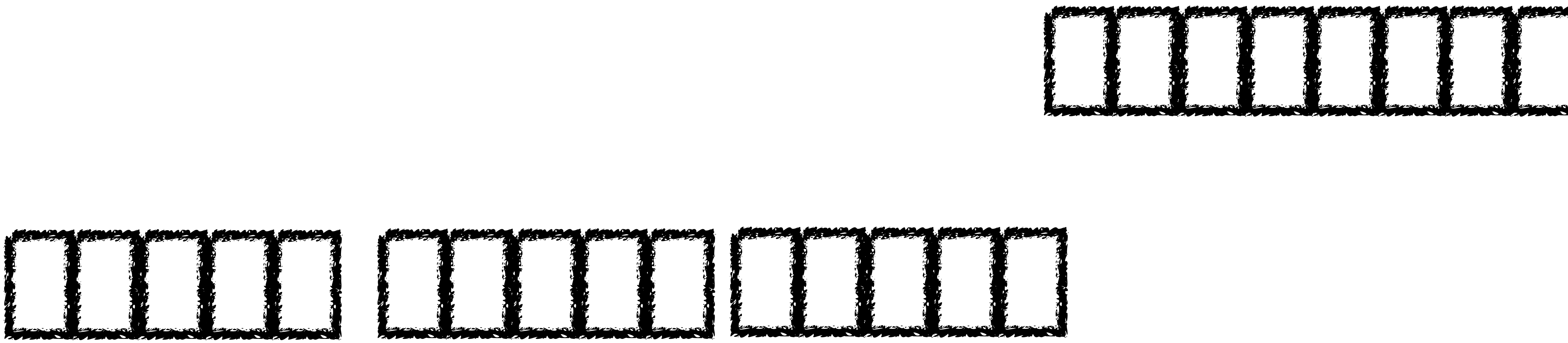

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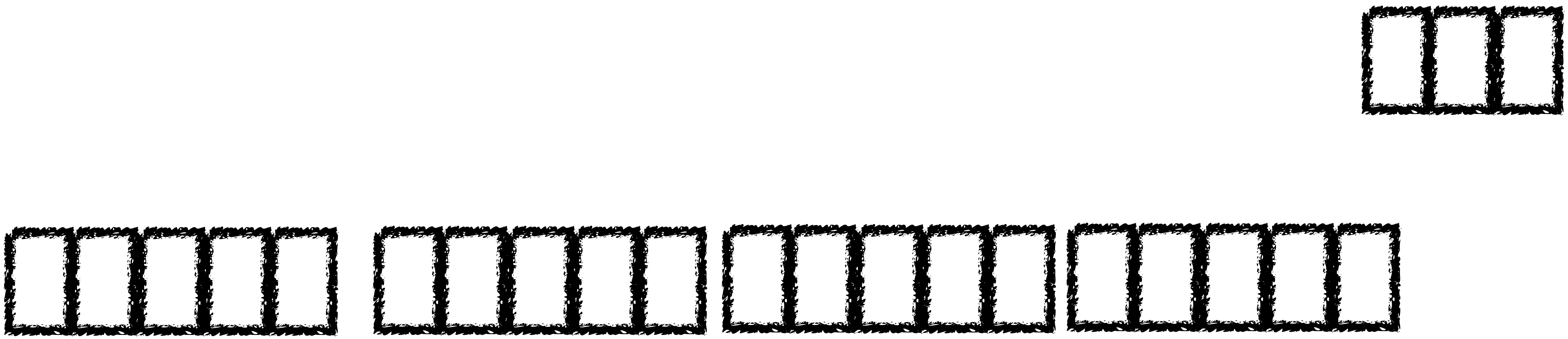

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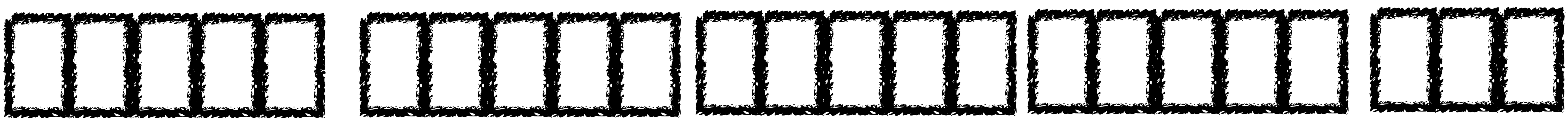

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Run **InsertionSort**

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Find the **median-of-medians**.

Median of medians

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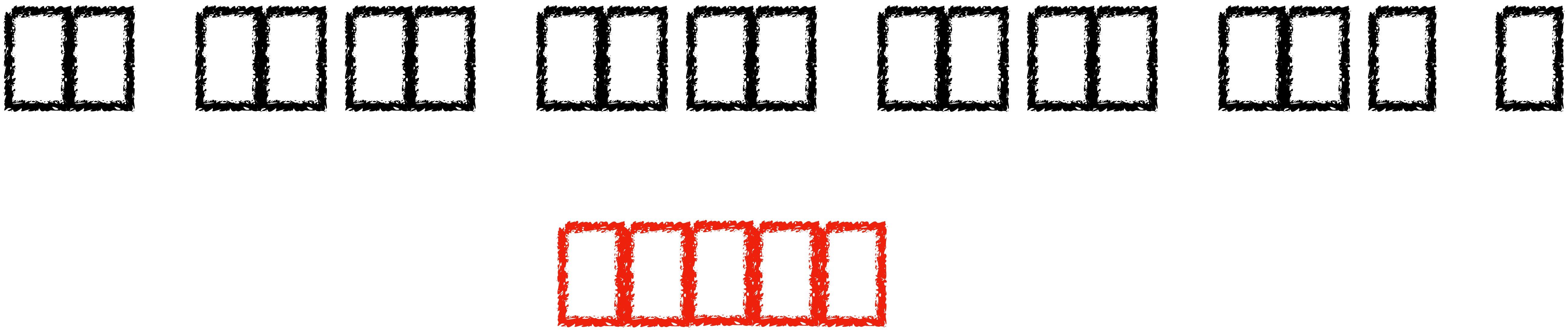

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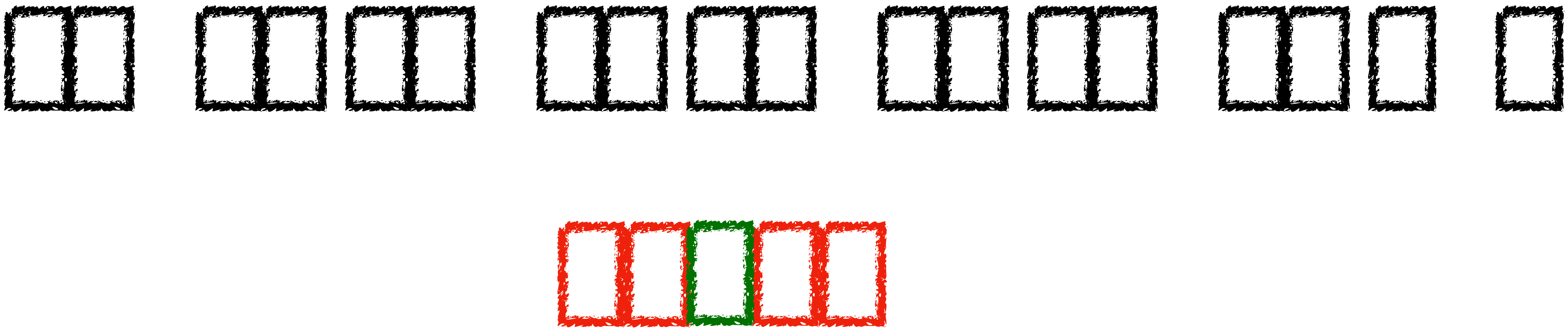

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Run **Selection**

This failed…

Algorithm **Selection**(**A**$[1,\ldots,n]$,i)

x = **Selection**(**A**$[1,\ldots,n]$,$(n+1)/2$)

k = **Partition**(**A**$[1,\ldots,n]$, x)

...but this won't.

Algorithm Selection(**A**[$1,\ldots,n$], i)

Split the array **A** into $n/5$ arrays of size 5
For each subarray **A$_i$**, find the *median.*
Let $m_1, m_2, \ldots, m_{n/5}$ be those medians

x = Selection(**A**[$m_1, \ldots, m_{n/5}$],$(n/5+1)/2$)

/*Find the median of medians */

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The Selection algorithm

Algorithm **Selection**(**A**[$1, \ldots, n$], i)

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$k - 1$ is the number of elements in the lower subarray.

If $i = k$, return x

If $i < k$, return **Selection**(**A**[$1, \ldots, k-1$], i)

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Zooming in

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We are looking for the i^{th}-order statistic.

If $i = k$, then x is the answer - it is larger than $k-1 = i-1$ elements.

If $i < k$, the answer cannot be in the second part, as then i would be larger than at least $k-1 = i-1$ elements.

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We are looking for the third smallest element ($i = 3$)

And in our case the pivot is in the fourth position, $k = 4$

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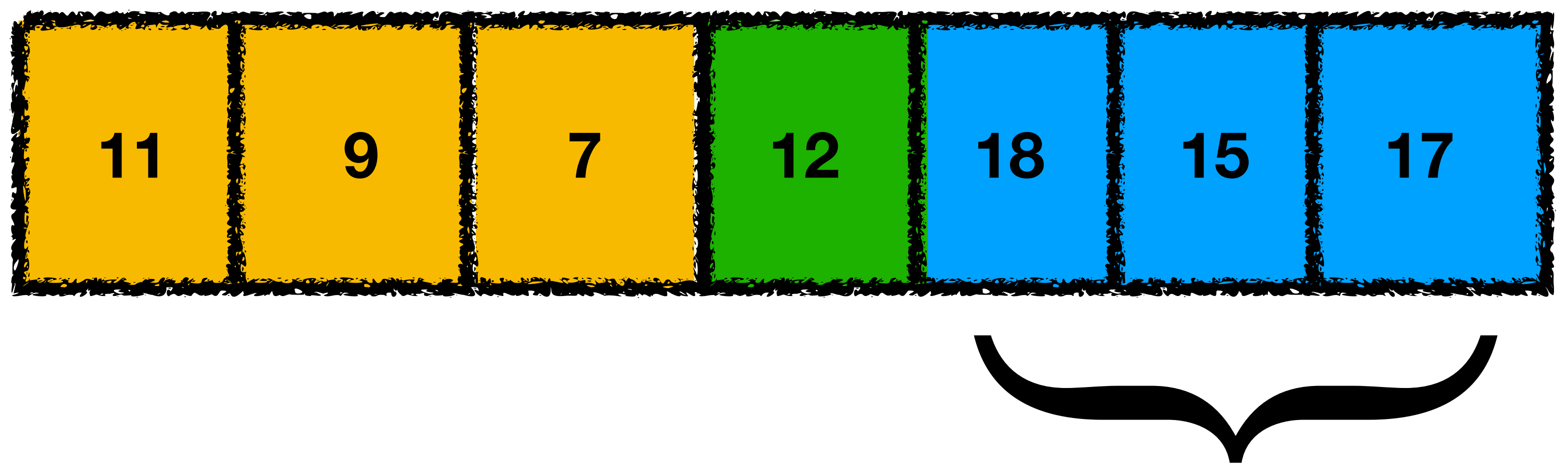

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For the same reason, if $i > k$, the answer cannot be in the first part.

Running Time

Algorithm **Selection**(**A**$[1,\ldots,n]$, i)

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$T(|S_{\max}|)$ If $i < k$, return **Selection**(**A**[$1,\ldots,k-1$], i)

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Running time

Running time

$$T(n) \leq T(n/5) + T(|S_{\max}|) + bn$$

Running time

$T(n) \leq T(n/5) + T(|S_{\max}|) + bn$

Before we proceed, we have to bound $|S_{\max}|$.

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x is a median of medians.

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At least (…) subarrays have "baby medians" ≥ x.

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Median of medians

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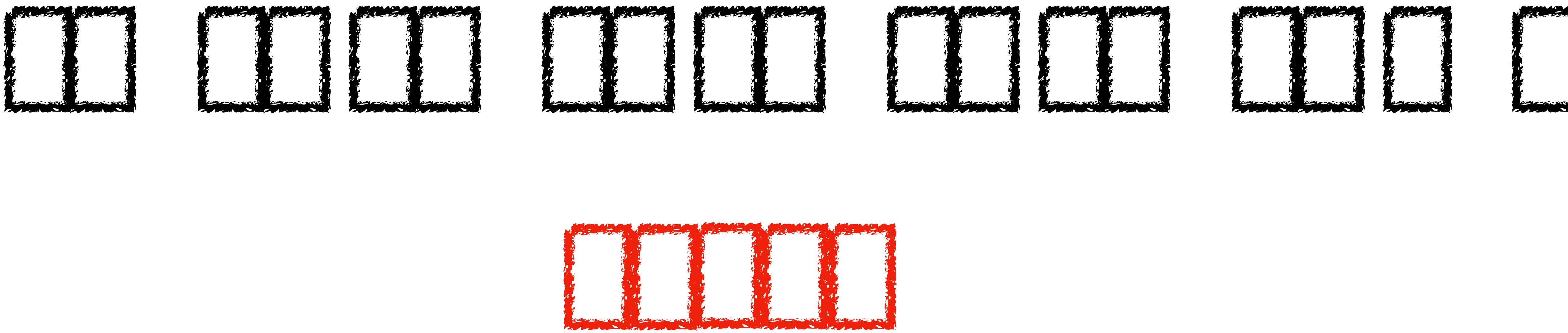

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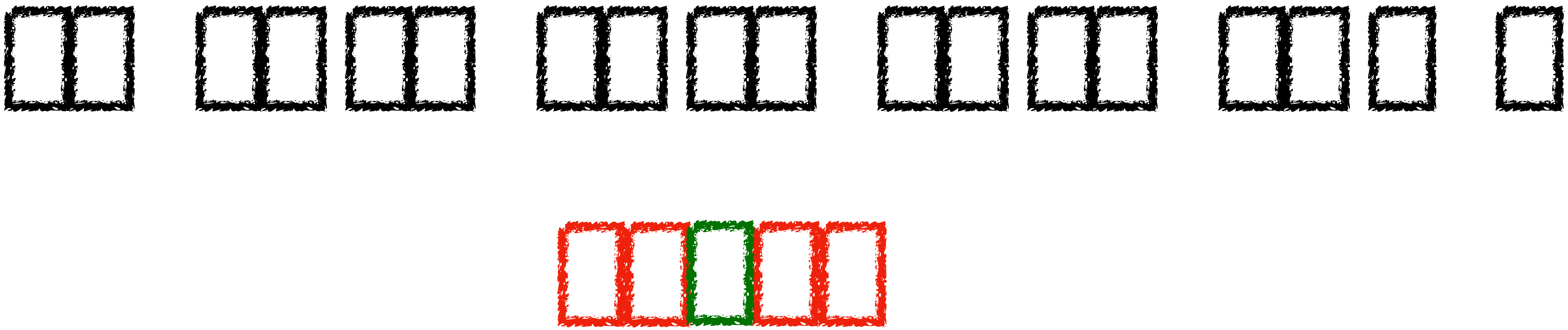

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Each one of these groups has at least (…) elements > x.

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Each one of these groups has at least 3 elements > x.

Median of medians

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x is a median of medians.

At least *half of the* subarrays have "baby medians" $\geq x$.

Each one of these groups has at least 3 elements $> x$.

Because $x \leq$ their "baby median".

Except possibly

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At least *half of the* subarrays have "baby medians" $\geq x$.

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Bounding the size of the subarrays

x is a median of medians.

At least *half of the* subarrays have "baby medians" $\geq x$.

Each one of these groups has at least 3 elements $> x$.

Because $x \leq$ their "baby median".

Except possibly the group containing x and the group that has fewer than 5 elements.

Median of medians

Bounding the size of the subarrays

What is the total number of elements larger than x?

$$3\left(\left\lceil \frac{1}{2} \cdot \left\lceil \frac{n}{5} \right\rceil \right\rceil - 2\right) \geq \frac{3n}{10} - 6$$

elements > x in each of those groups

groups with “baby medians” > x

groups

groups who could be exceptions

This means that the size of the lower subarray is at most

$$7n/10 + 6$$

Bounding the size of the subarrays

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Back to the recurrence:

T(n) ≤ T($n/5$) + T ($|S_{\max}|$) + bn = T($n/5$) + T ($7n/10 + 6$) + bn

Solving the recurrence

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Let's guess that $T(n) \le cn$, for some constant c.

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We get that

$$
\begin{aligned}
T(n) &\leq c(n/5) + c(7n/10 + 6) + bn \\
&= 9cn/10 + 6c + bn \\
&= cn + (-cn/10 + 6c + bn)
\end{aligned}
$$

Solving the recurrence

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$$T(n) \leq c(n/5) + c(7n/10 + 6) + bn$$
$$= 9cn/10 + 6c + bn$$
$$= cn + (-cn/10 + 6c + bn)$$

This is at most cn whenever $-cn/10 + 6c + bn \leq 0$, or equivalently, when $c \geq 10bn/(n - 60)$.

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This is at most cn whenever $-cn/10 + 6c + bn \le 0$, or equivalently, when $c \ge 10bn/(n-60)$.

If $n \ge 120$, then $n/(n-60) \le 2$ and then, it suffices to have $c \ge 20b$.

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Let $a = \max\{ T(n) / n , n \leq 120\}$ and let $c = \max\{a, 20b\}$.

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We will prove the statement by induction.

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We will prove the statement by induction.

Base case: For every $n \leq 120$, $T(n) \leq \max\{ T(n) / n , n \leq 120\}\, n$
$= an \leq \max\{a, 20b\}n = cn$

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We want to show that there is some constant c > 0, such that $T(n) \leq cn$ for all n > 0.

Let a = max{ $T(n) / n$, $n \leq 120$} and let c = max{a, 20b}.

We will prove the statement by induction.

Base case: For every $n \leq 120$, $T(n) \leq \max\{ T(n) / n , n \leq 120\}\, n$ $= an \leq \max\{a, 20b\}n = cn$

Inductive Step: Suppose that it holds for all n up to $k = 120$. Then for $n = k + 1$, we have $T(n) \leq cn + (-cn/10 + 6c + bn)$

This follows from the previous slide and the fact that $n > 120$ and $c \geq 20b$.