Algorithms and Data Structures

Max Flow in Polynomial Time: The Edmonds-Karp Algorithm

Ford-Fulkerson analysis

Feasibility

Does the algorithm produce a flow if it terminates?

Termination

Does the algorithm always terminate?

Running Time

What is the running time of the algorithm?

Optimality / Correctness

Does the algorithm produce a maximum flow?

Ford-Fulkerson analysis

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Running Time

The running time of *FF* is $O(m\mathrm{F})$, where F is the value of the maximum flow.

Since F $\leq c$, this is in fact $O(mc)$.

Is this an efficient algorithm?

The running time is *pseudopolynomial*, as it runs in time polynomial in n and the unary representation of the total capacity c.

It is fairly efficient, if in the numbers involved in the input are reasonably small.

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Return (f)

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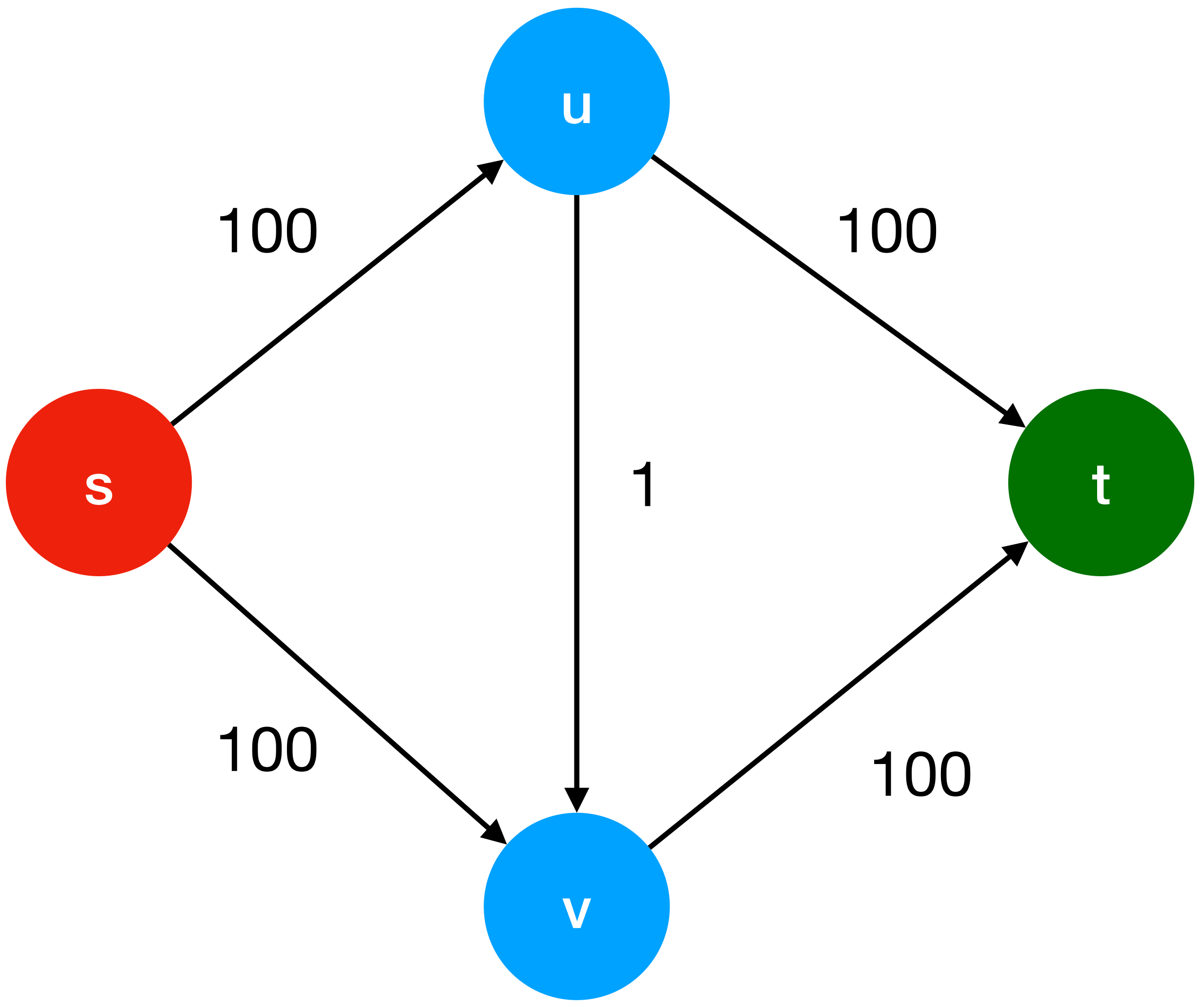

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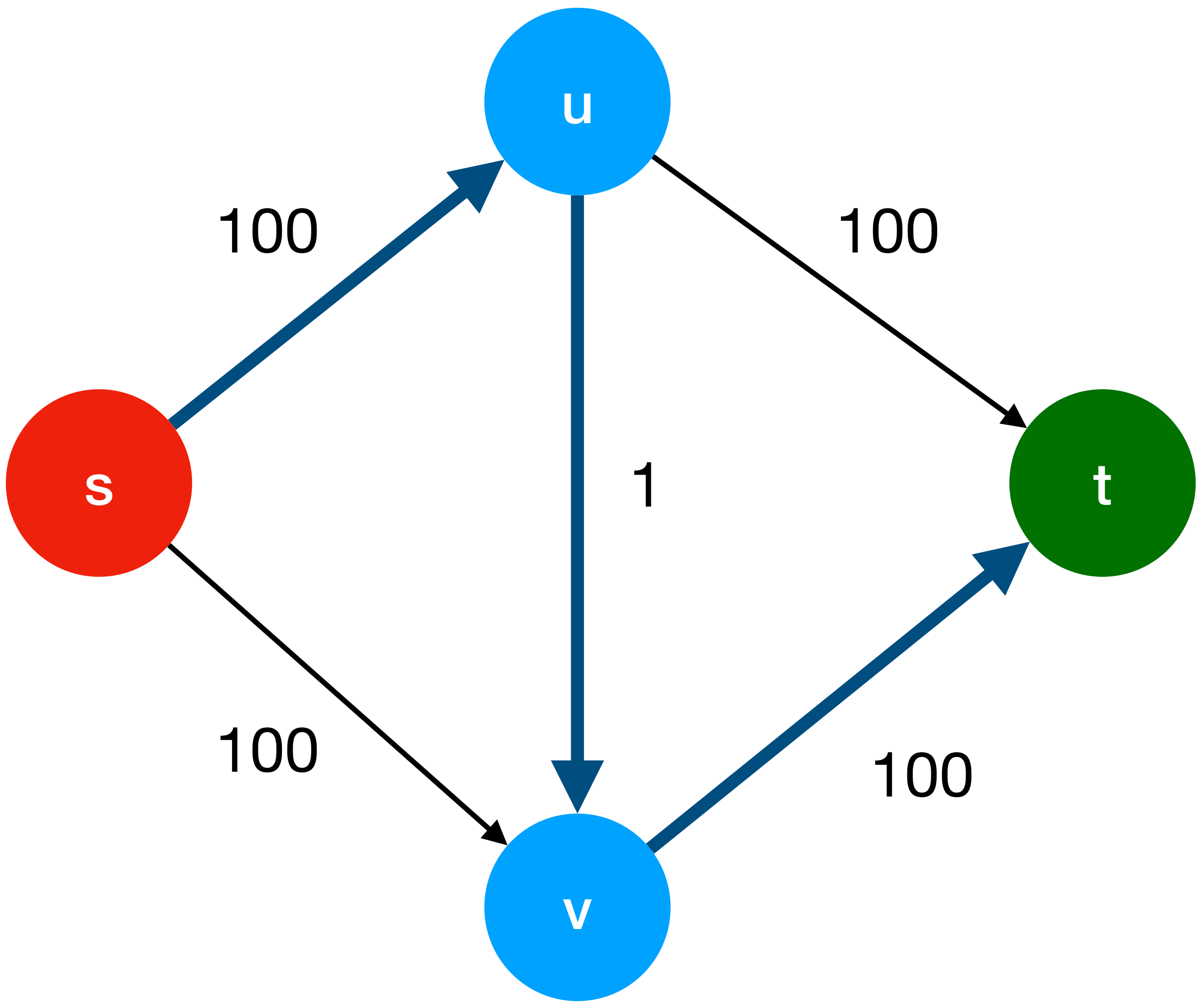

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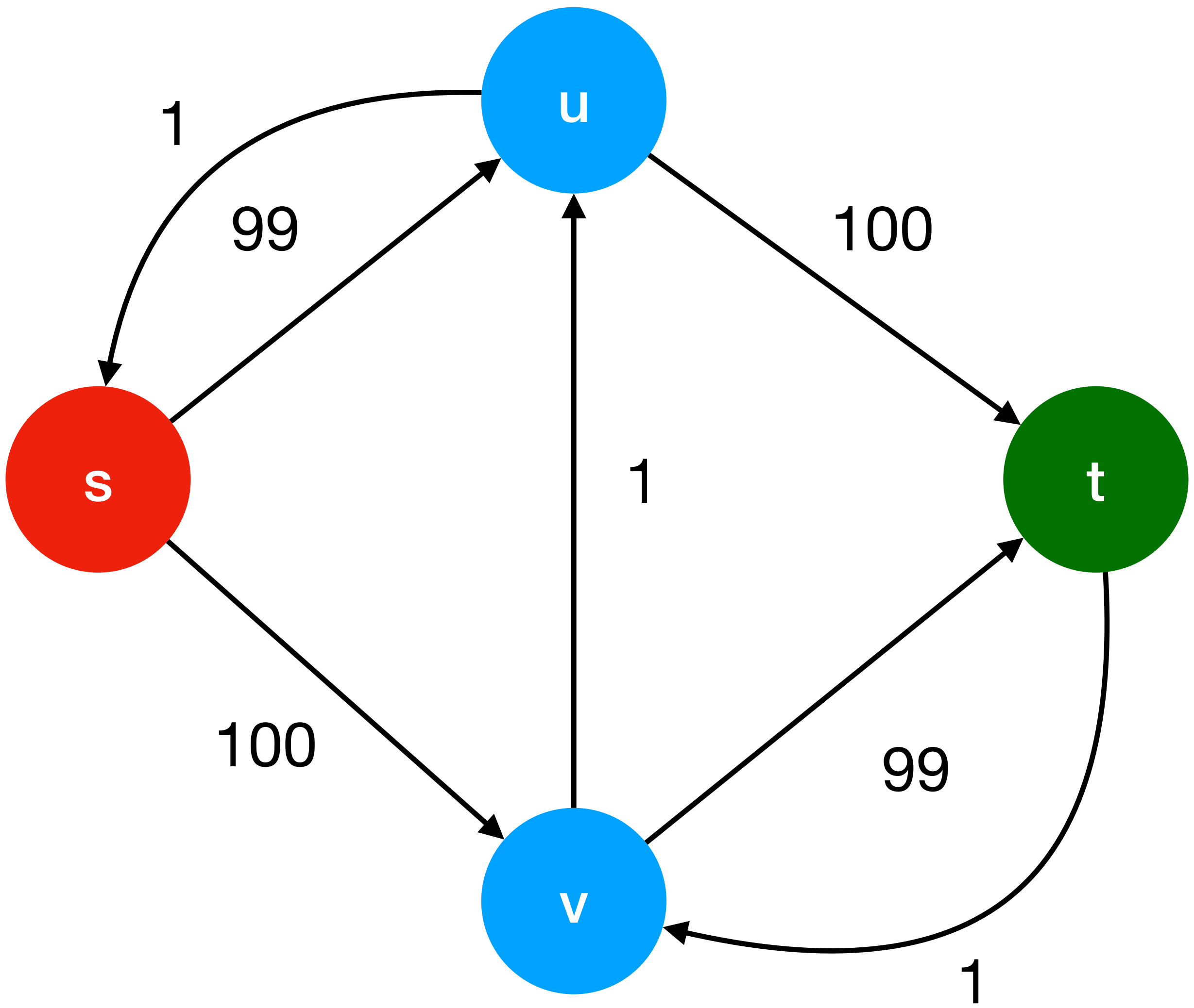

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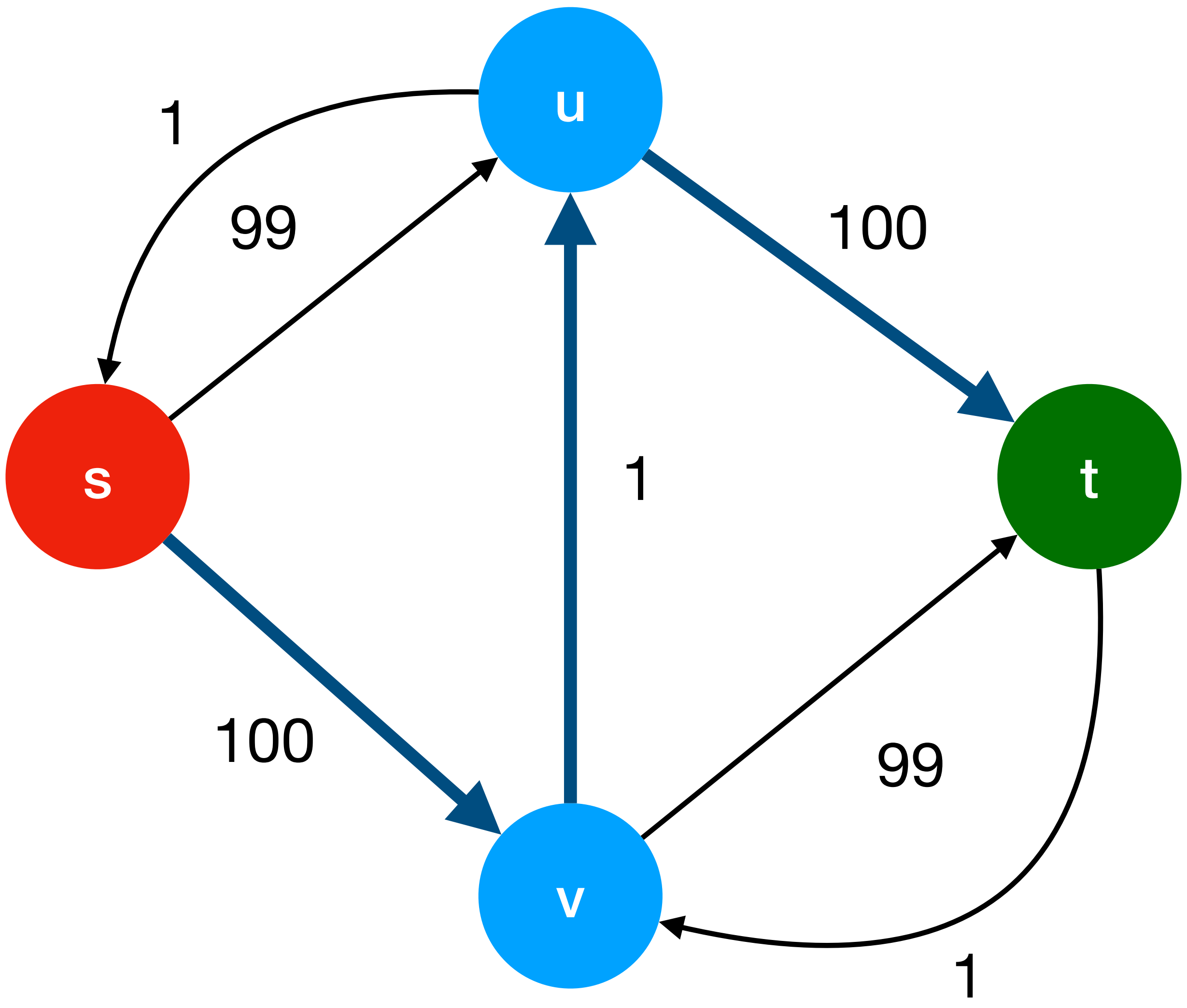

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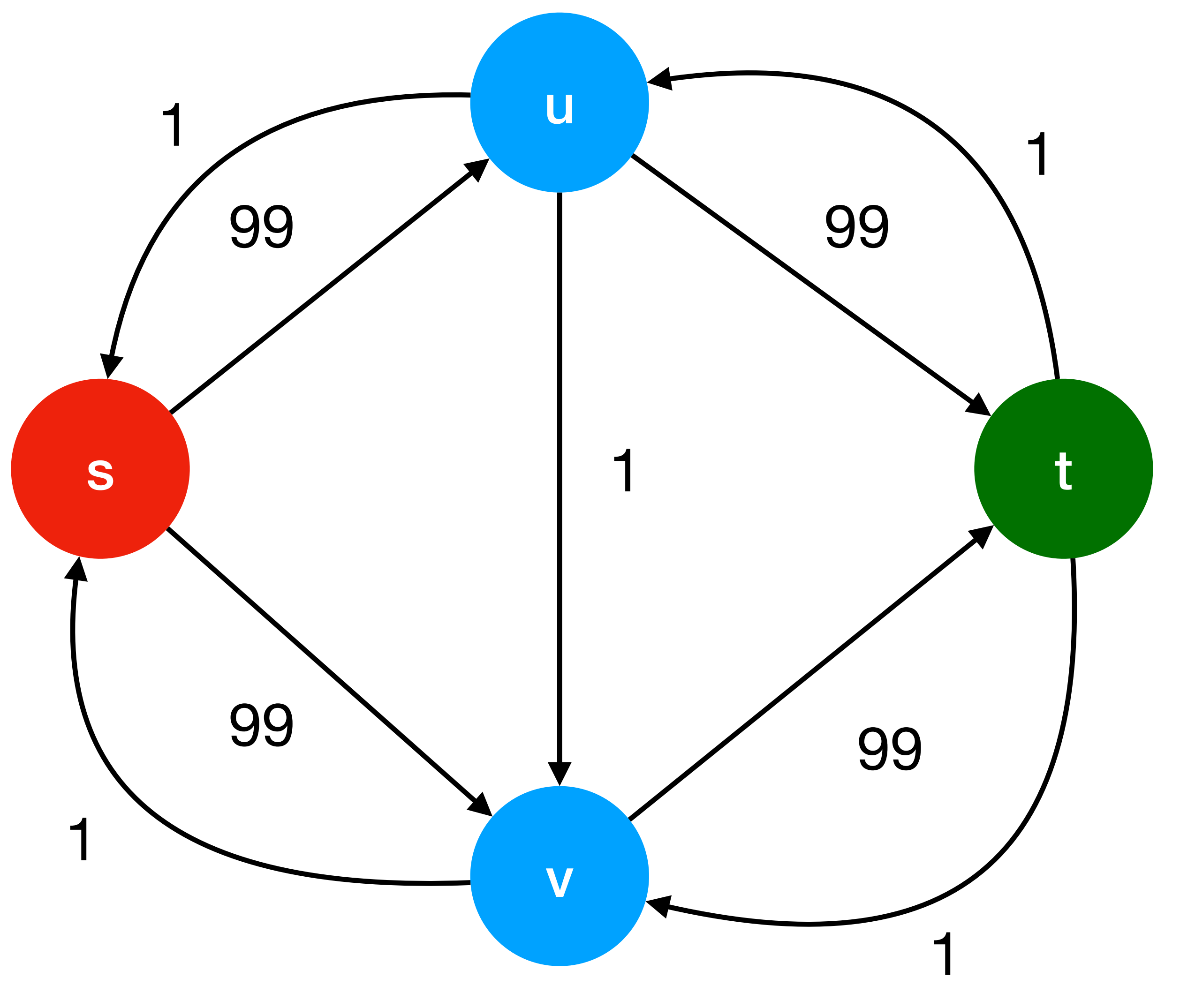

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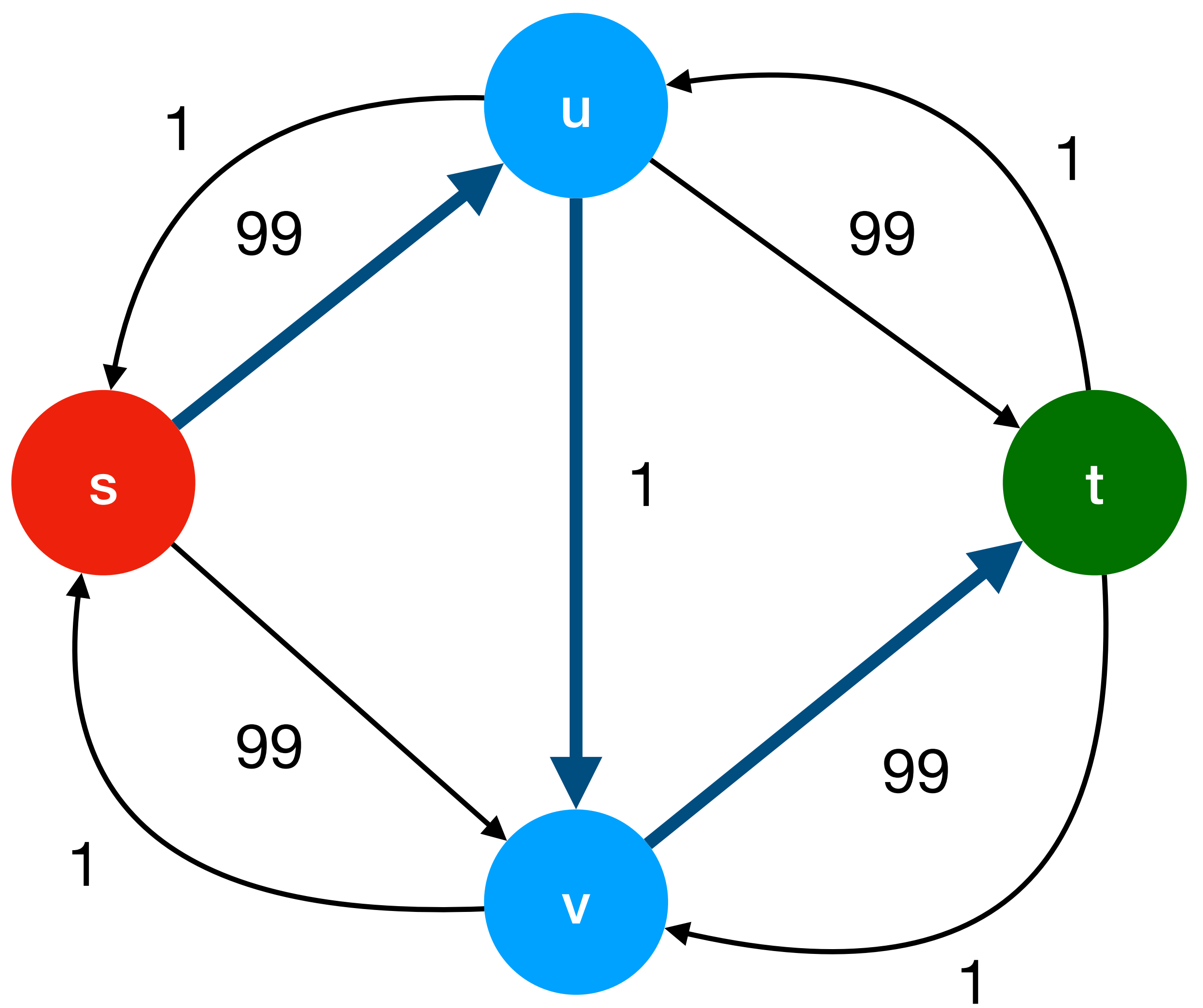

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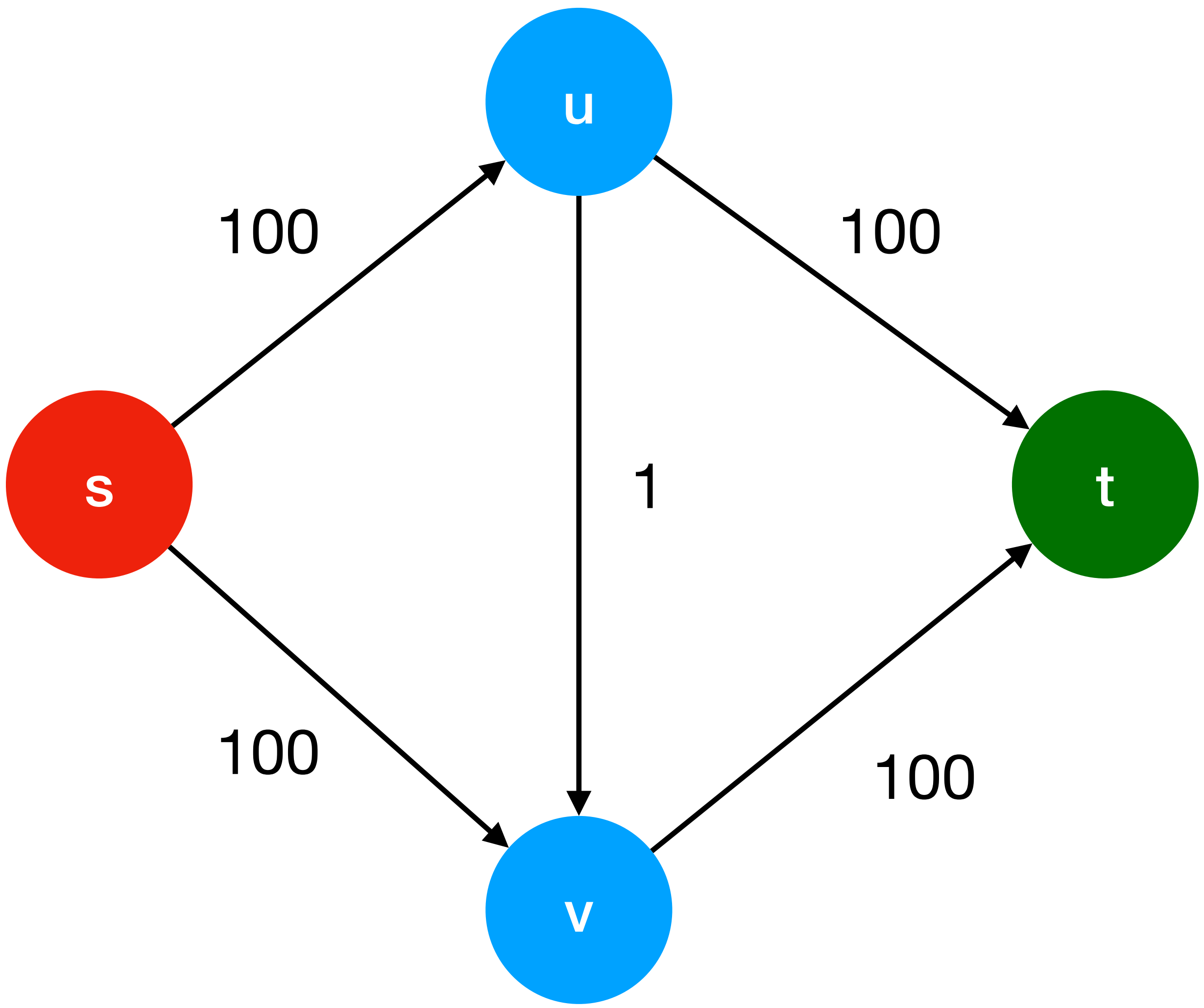

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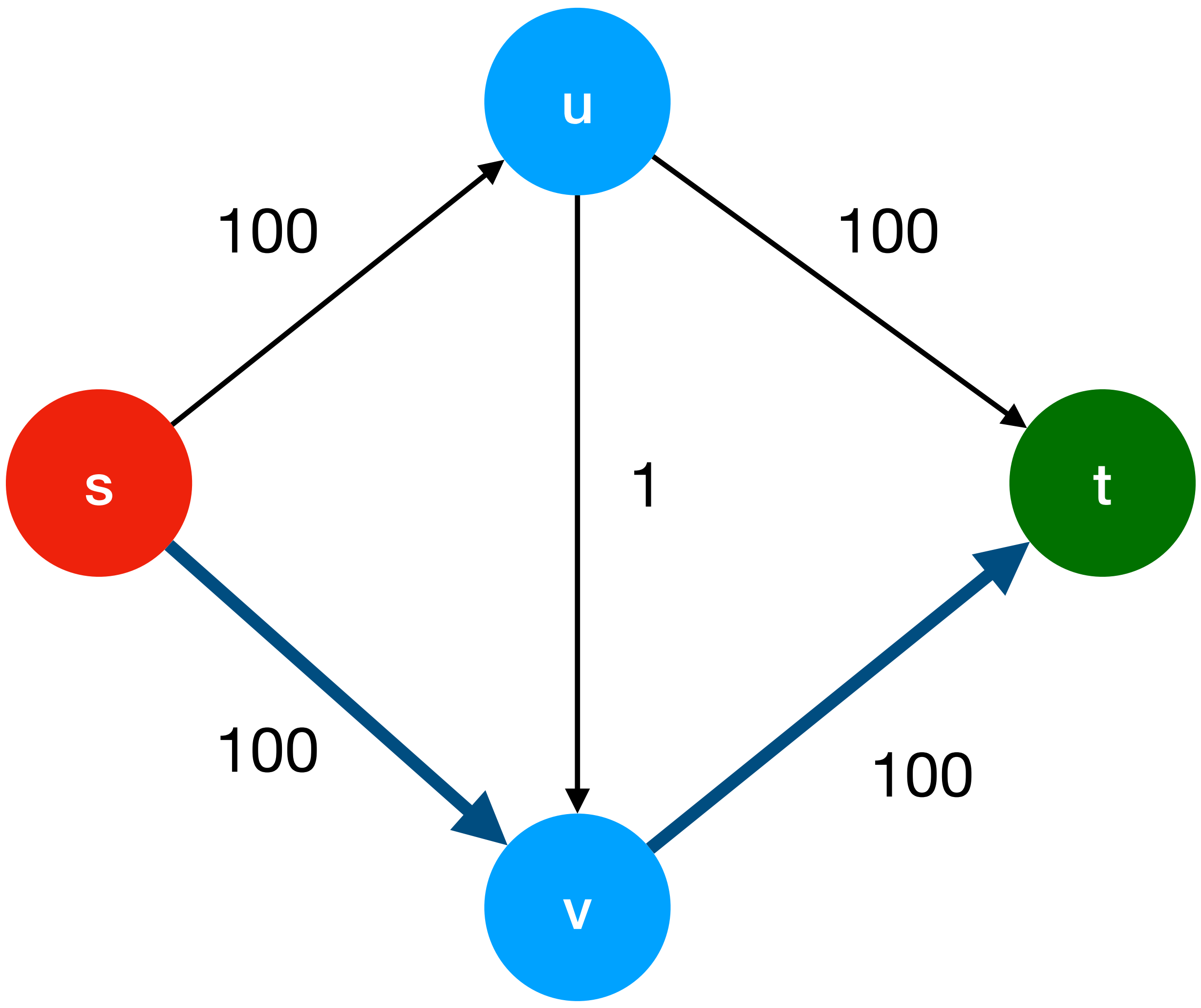

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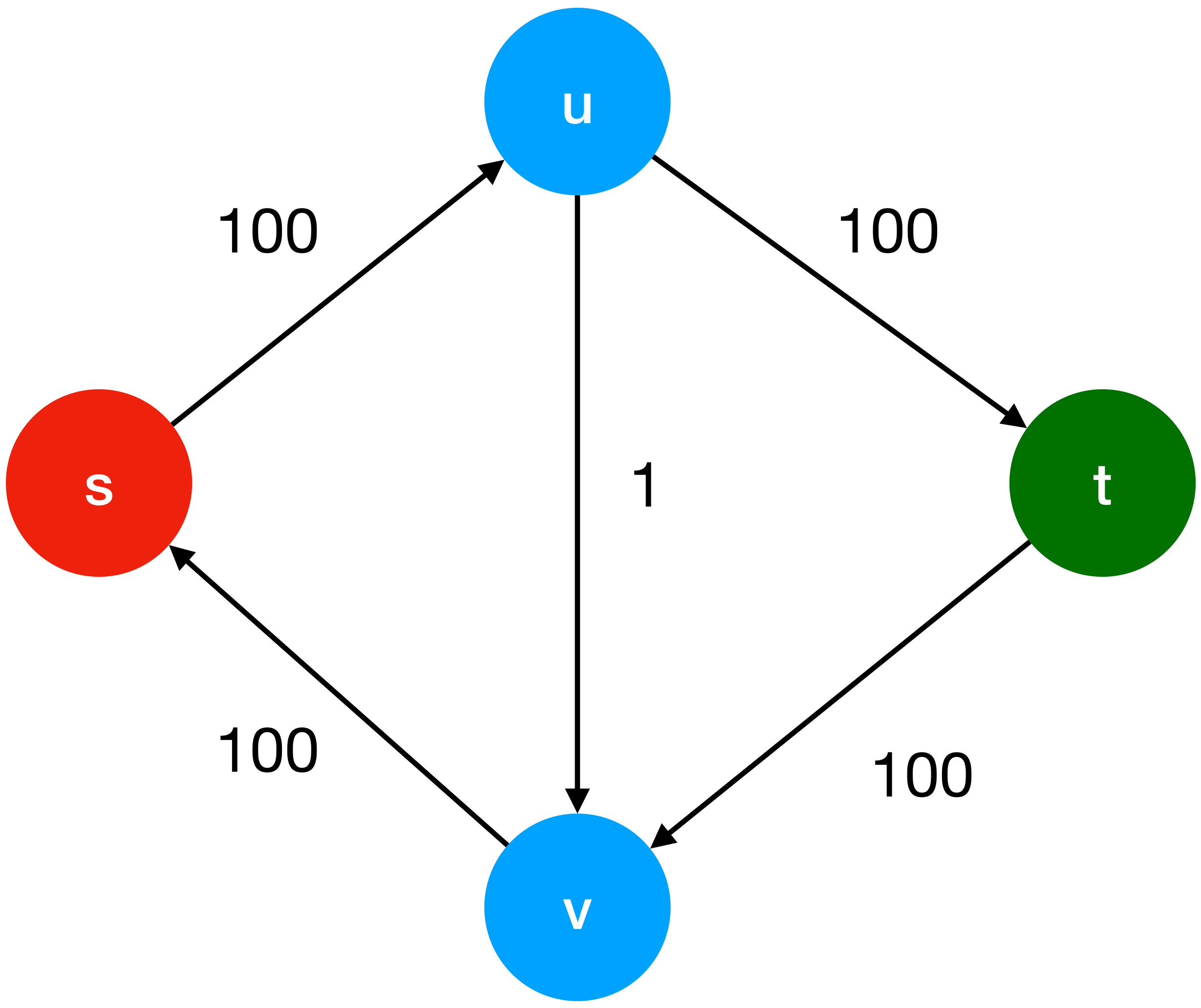

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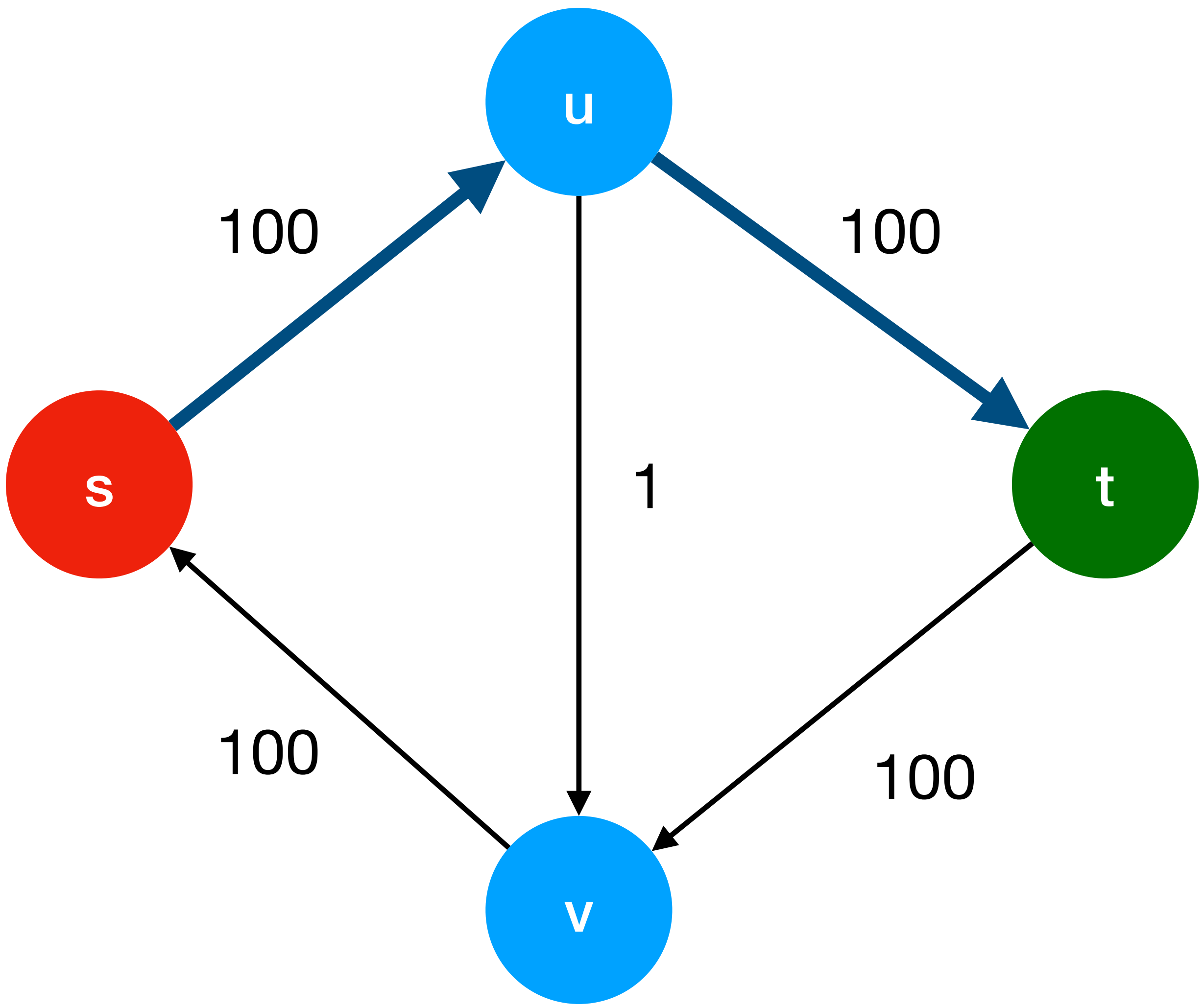

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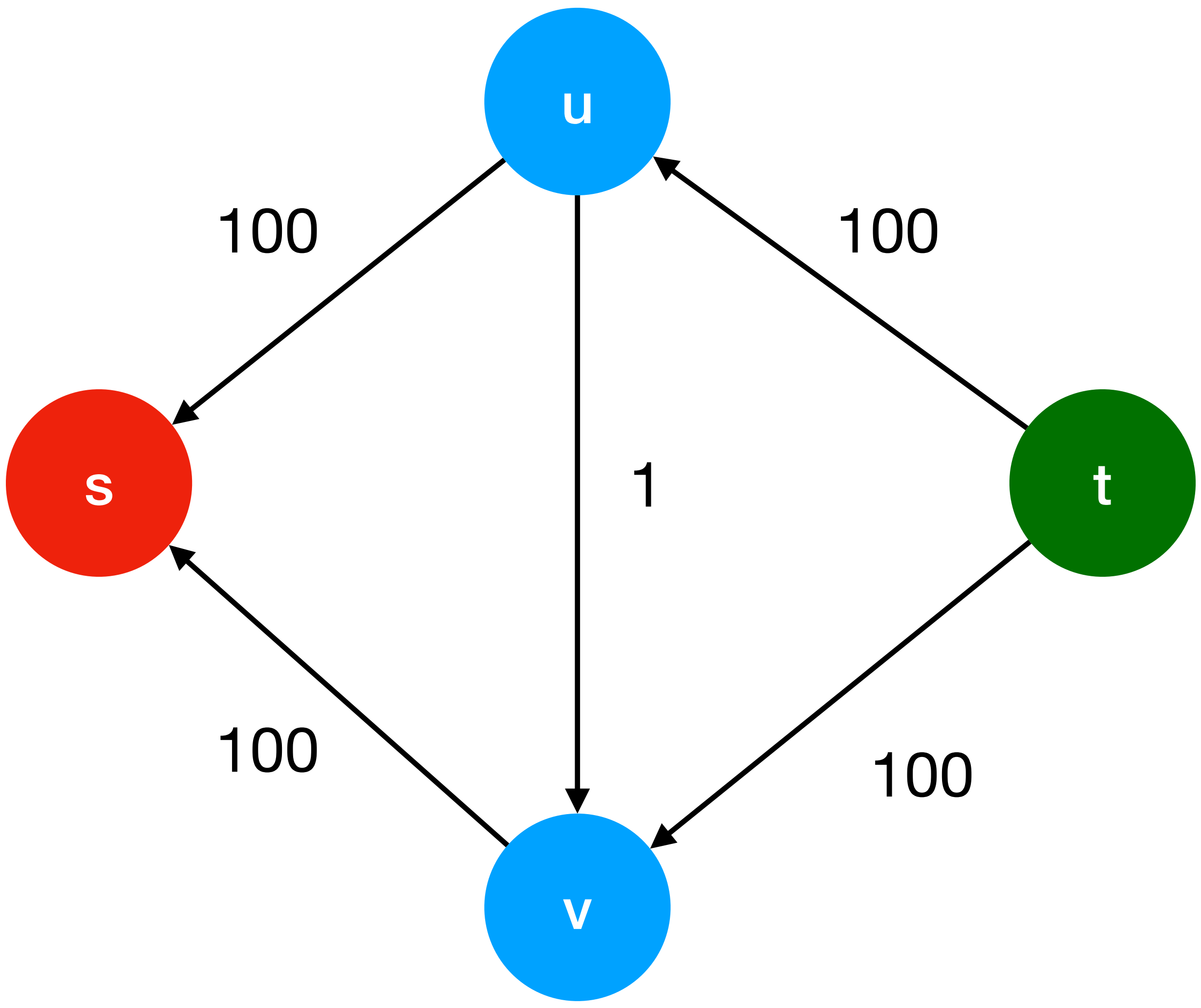

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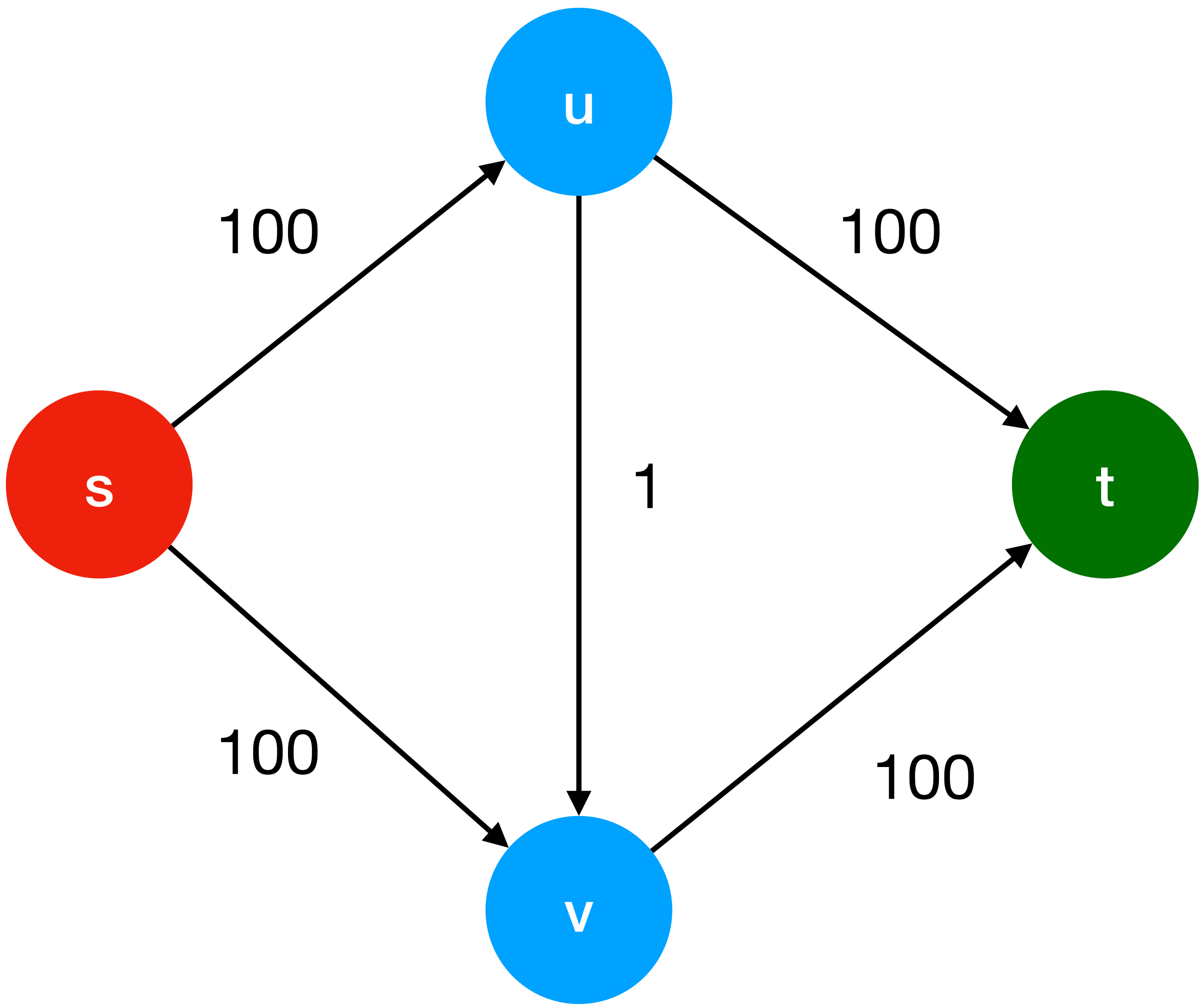

Max-Flow in polynomial time

We made the algorithm must faster by simply selecting the shortest path with available capacity.

Can we always hope to do that?

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Useful Lemma

Lemma: Suppose that we run the Edmonds-Karp algorithm on a flow network $G = (V, E)$. For every node $u \in V \setminus \{s, t\}$, the length of the shortest path $d_f(s, u)$ from s to u in the residual graph G_f given by f increases with each flow augmentation.

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This is a contradiction, hence $(u, v) \notin E_f$

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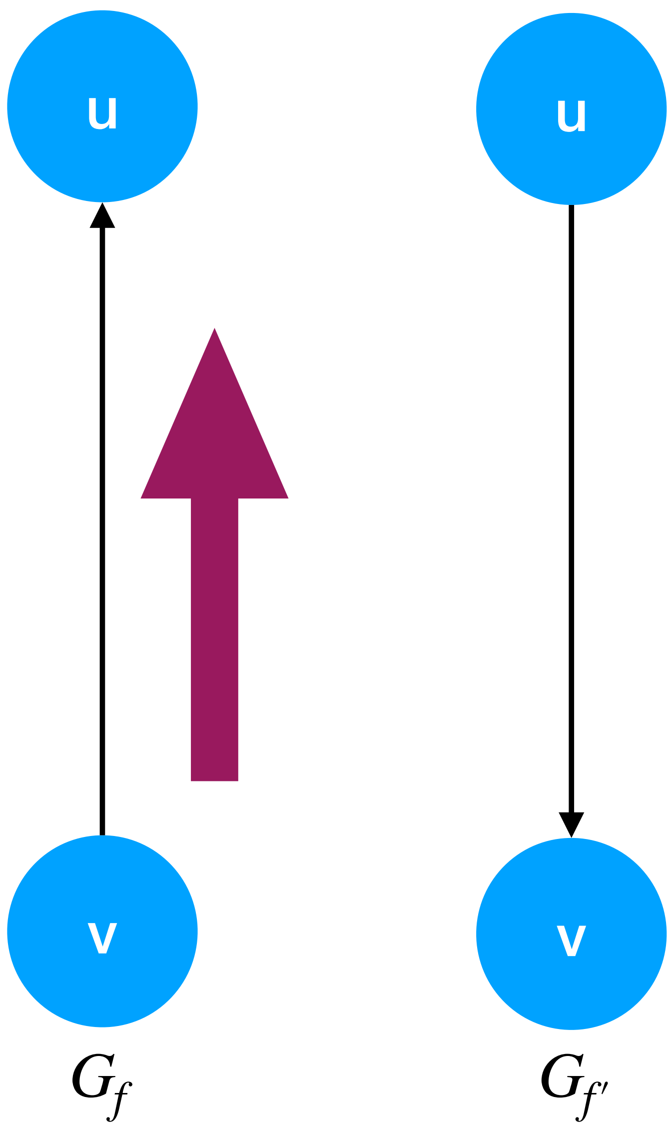

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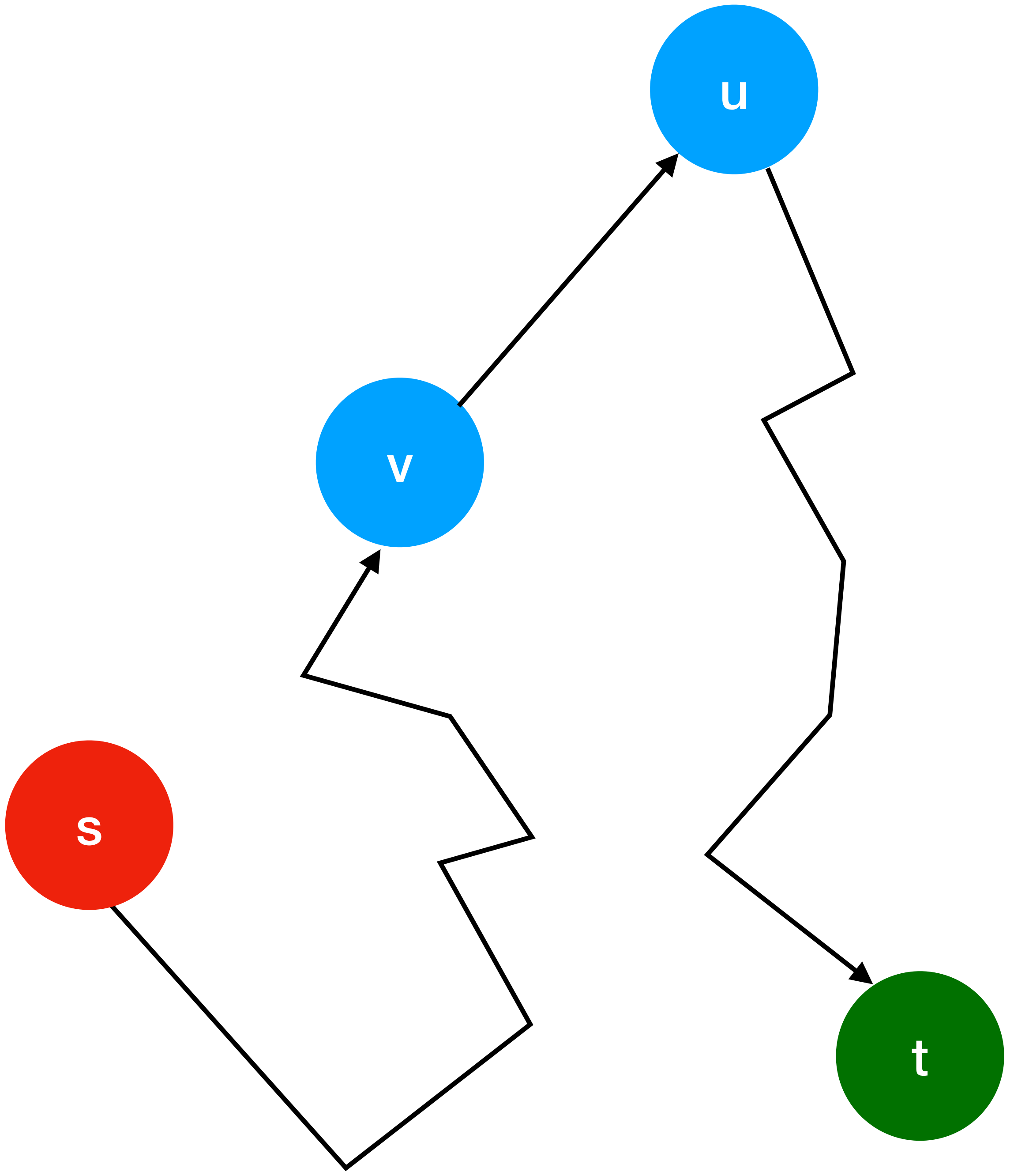

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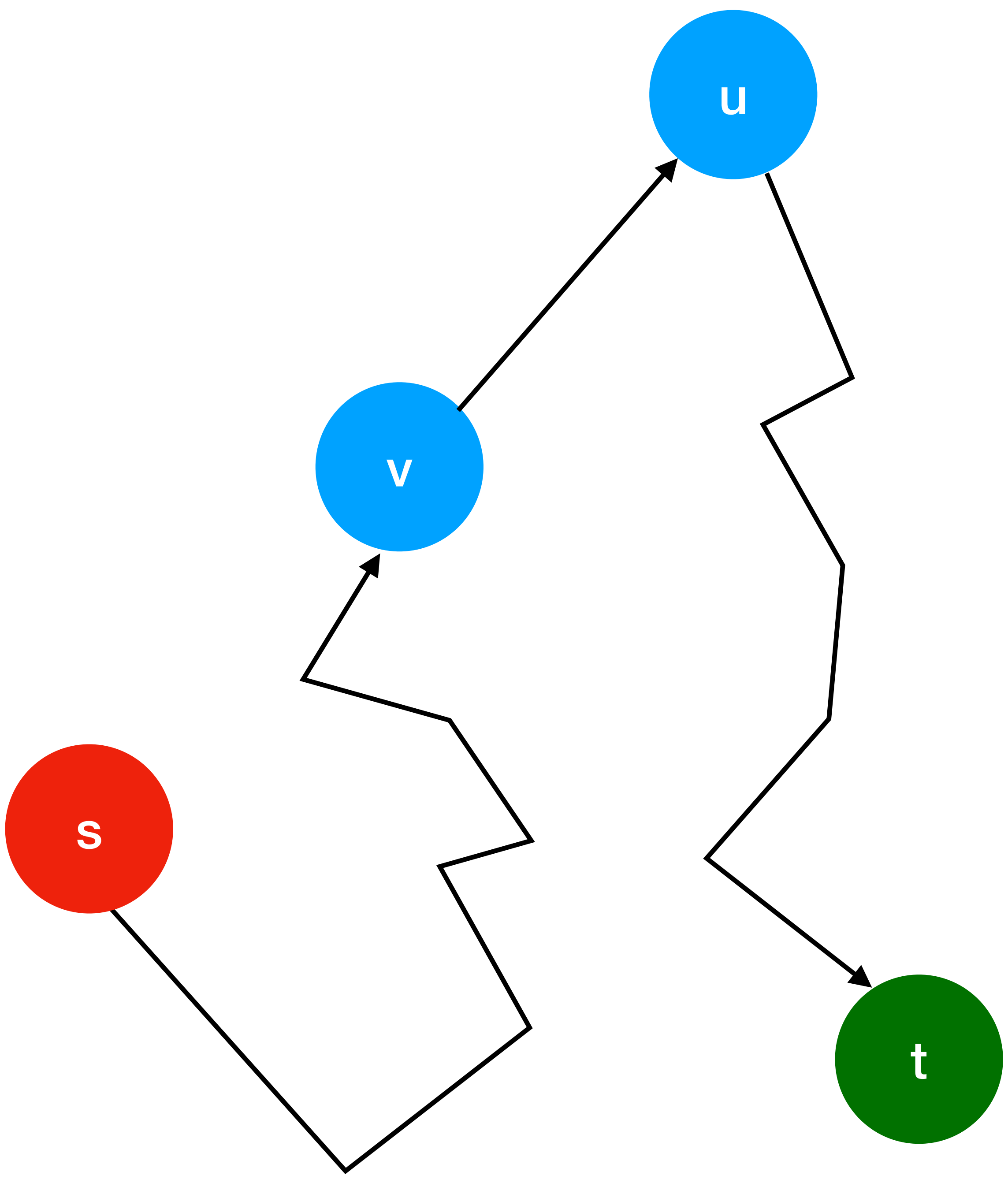

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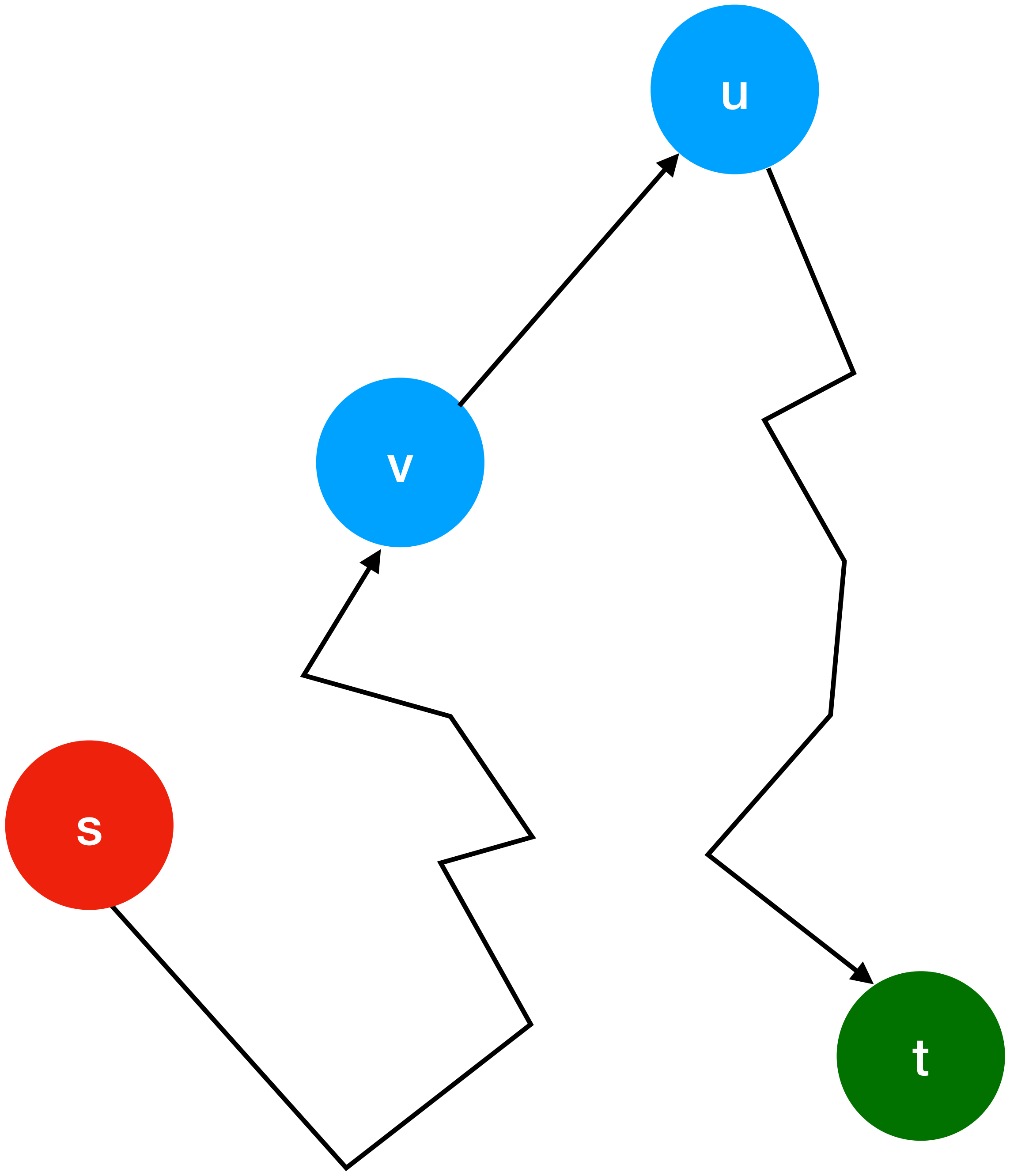

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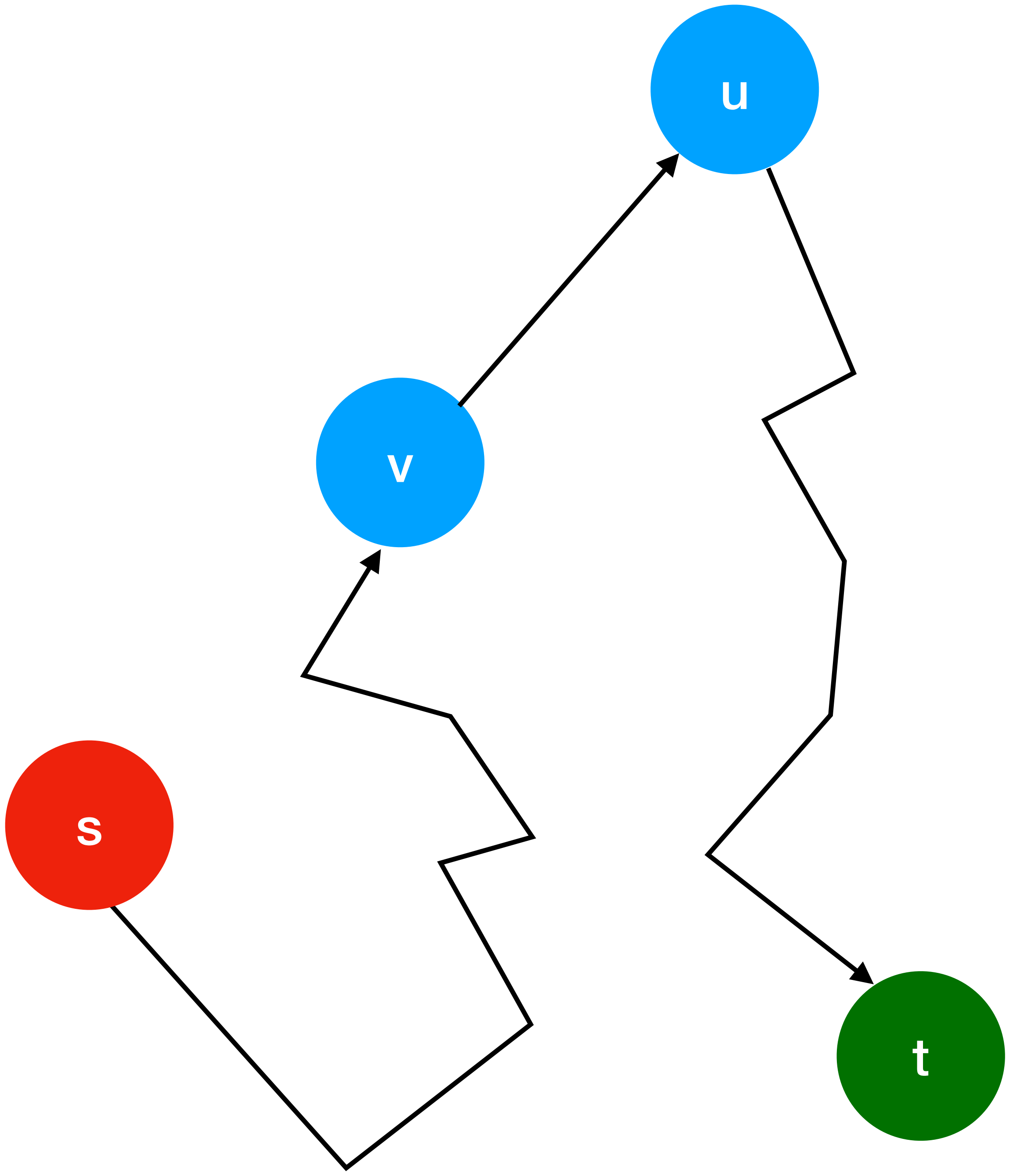

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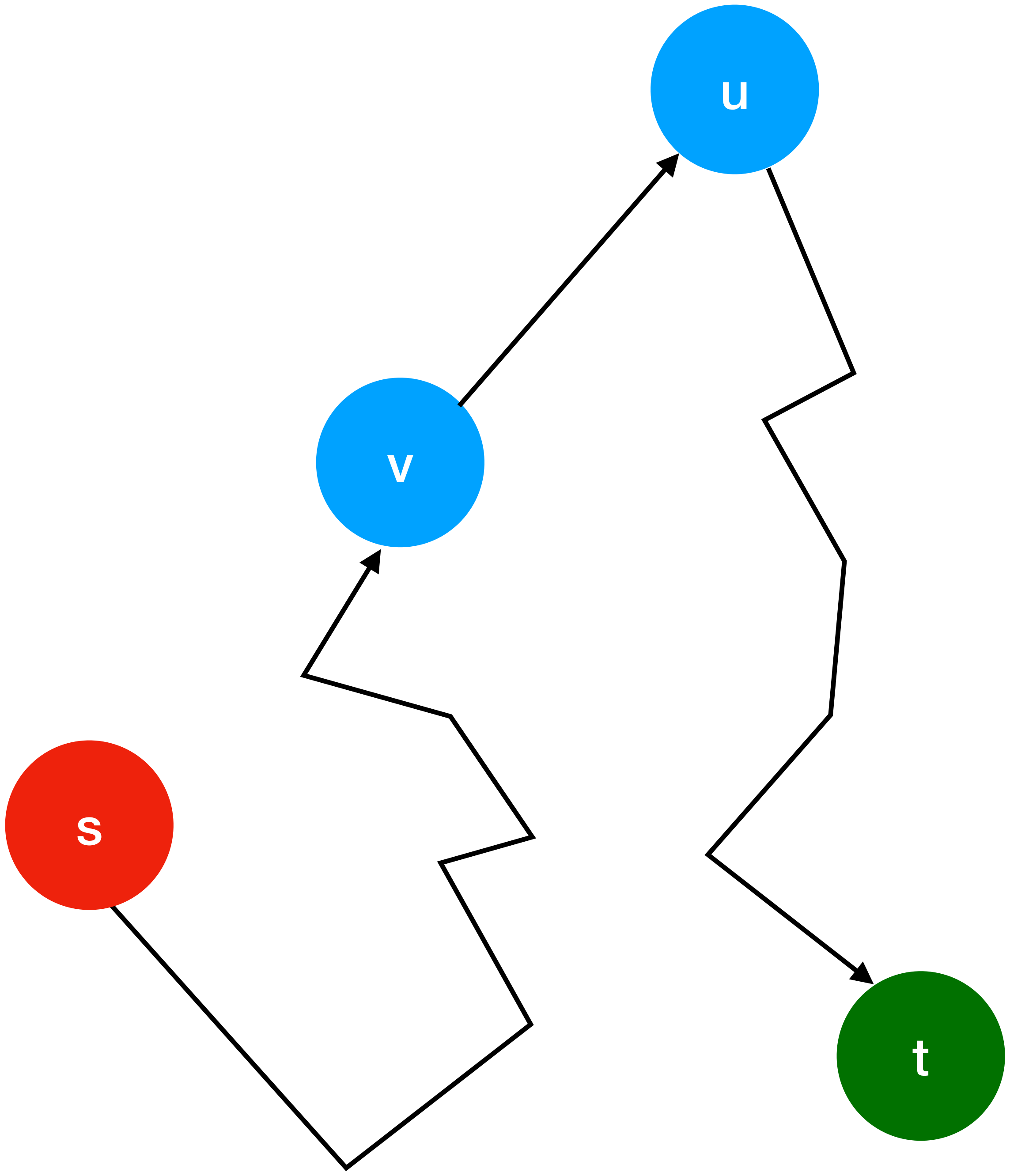

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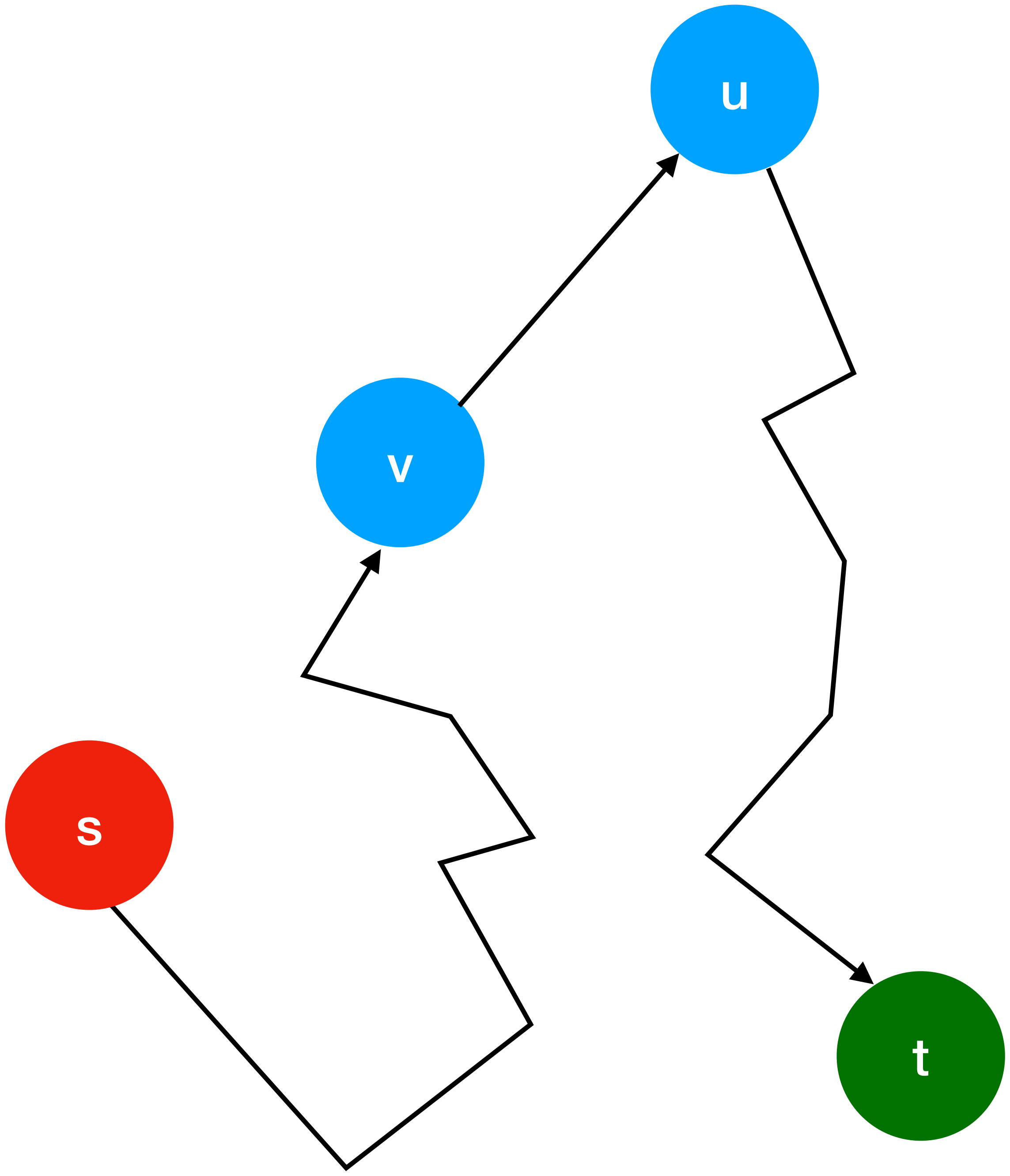

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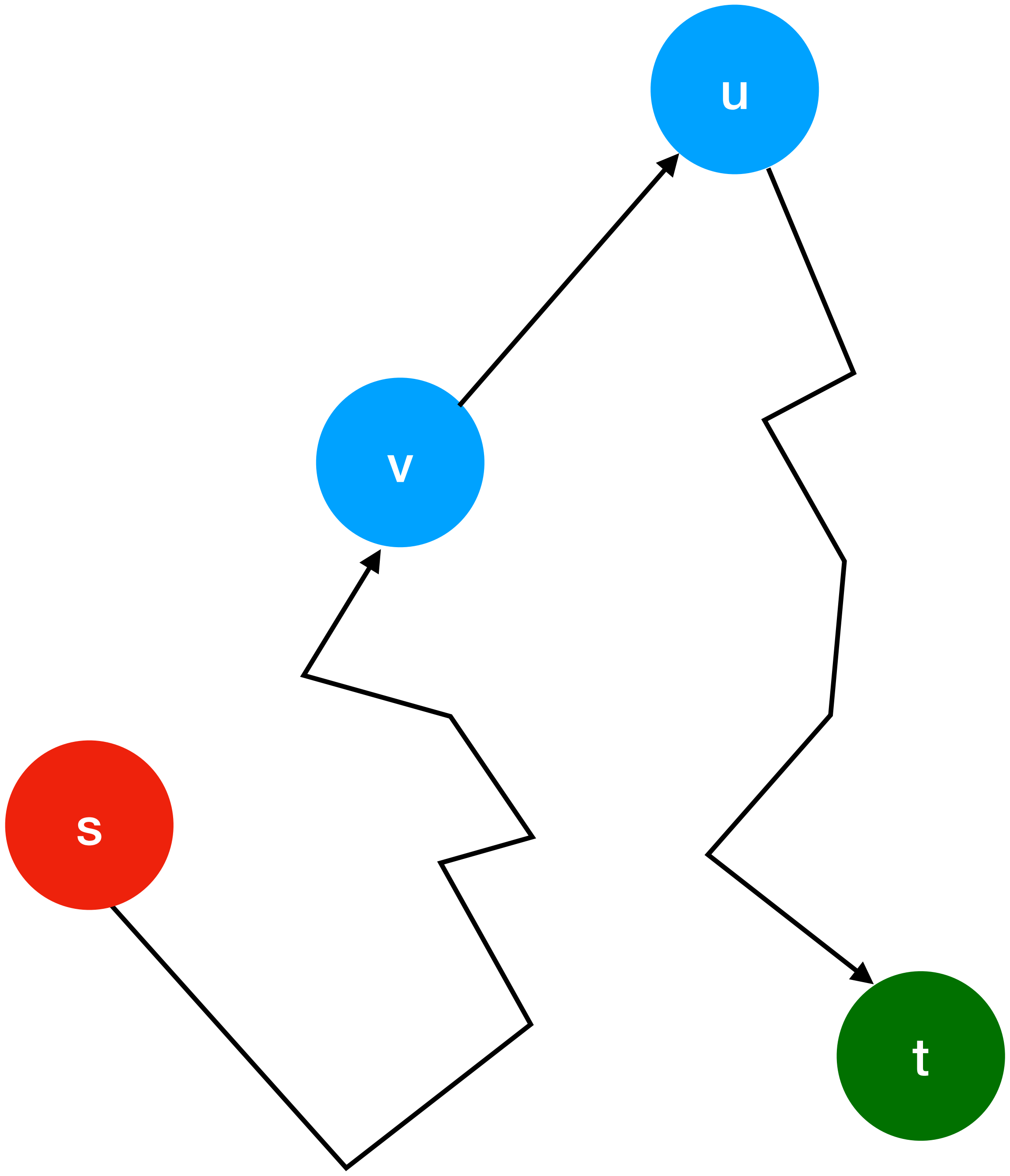

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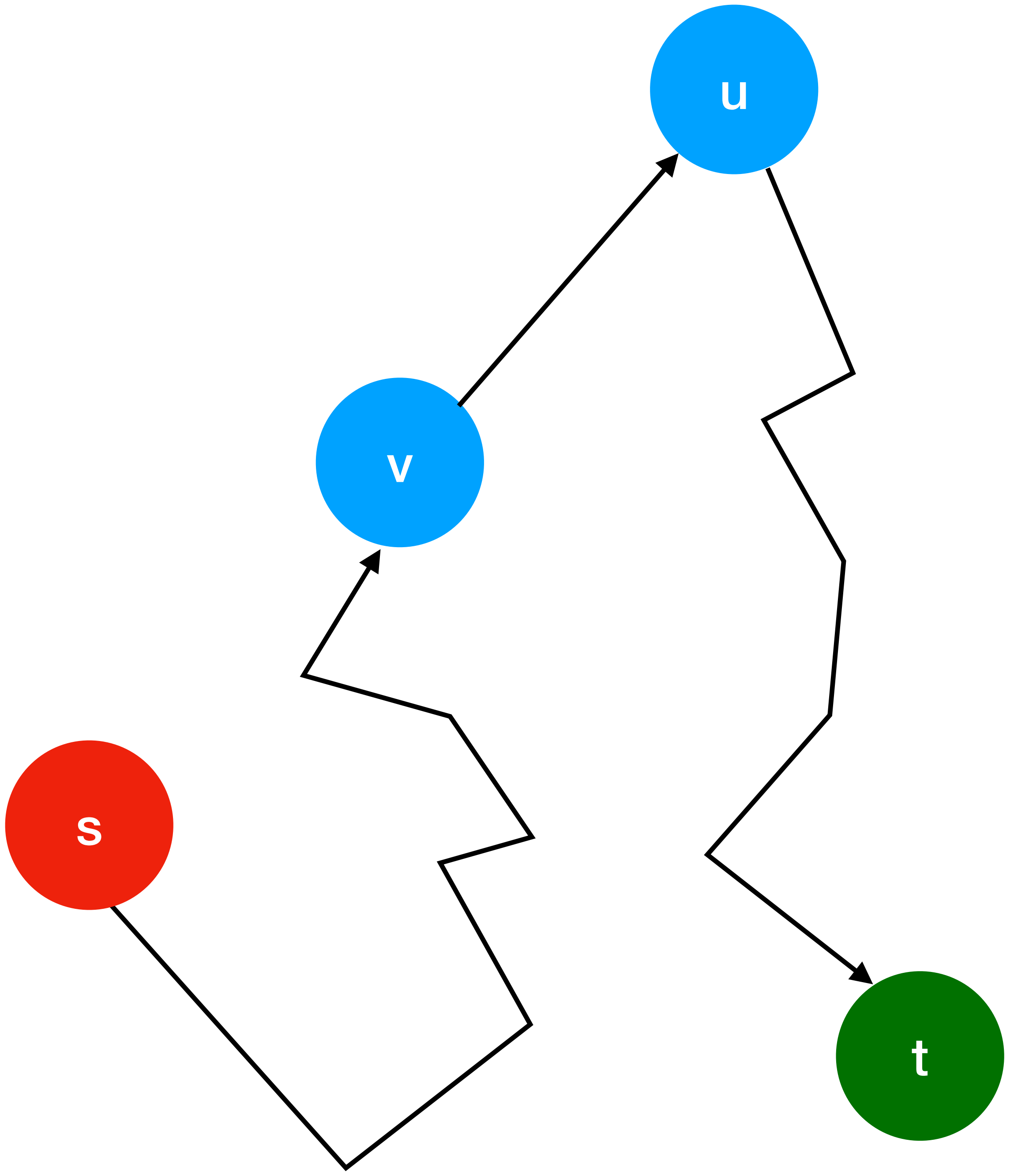

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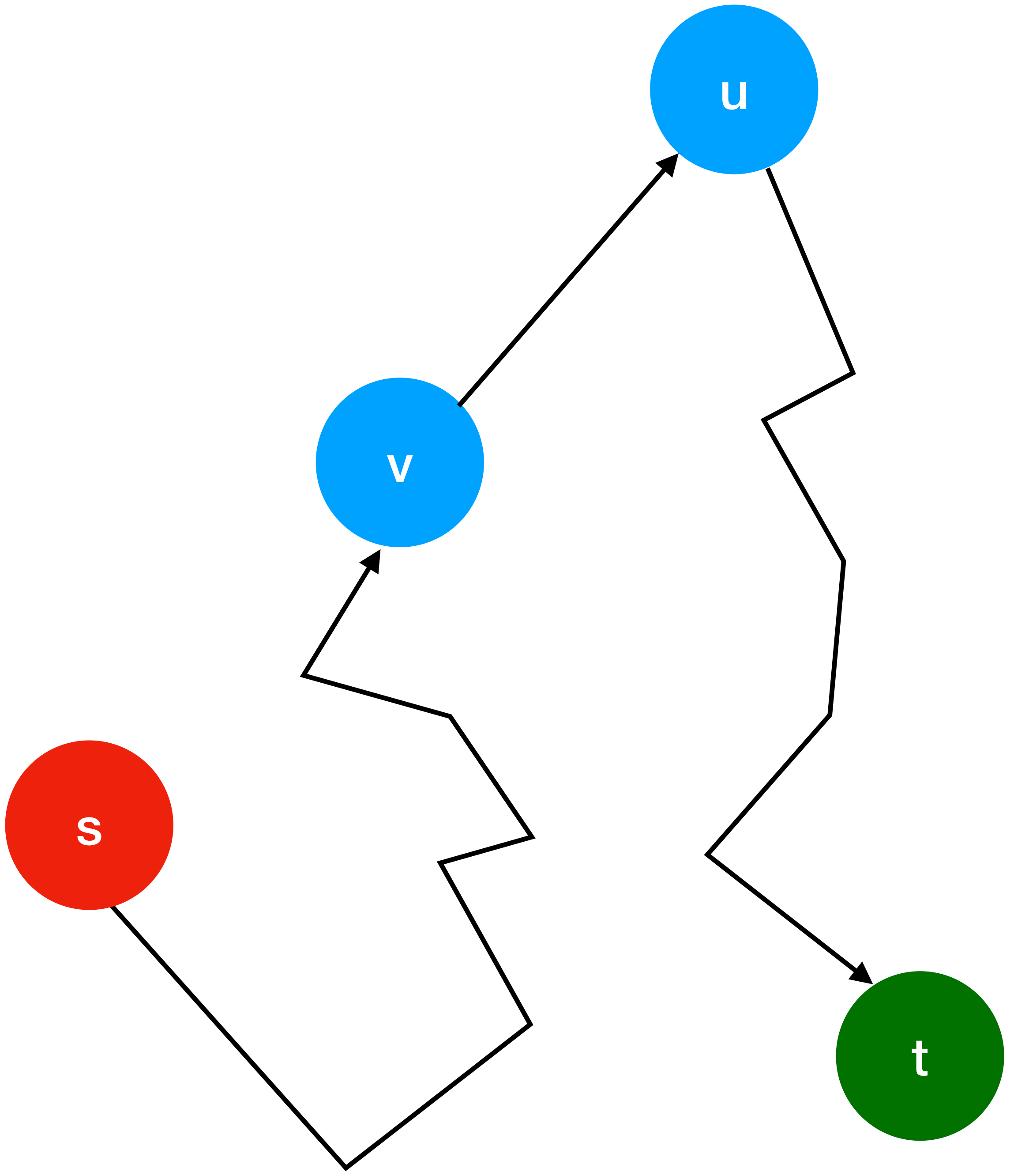

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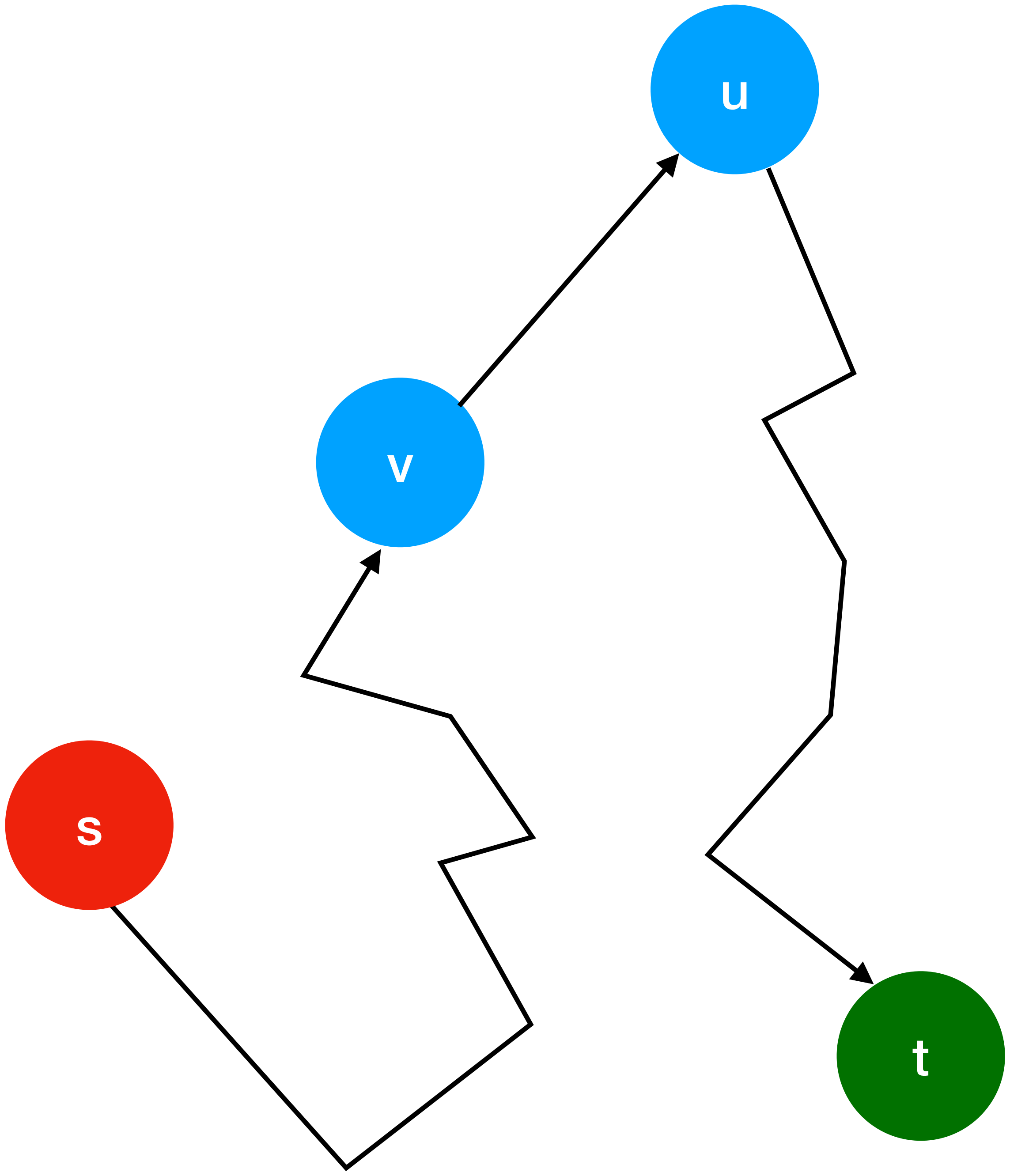

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Lemma: Suppose that we run the Edmonds-Karp algorithm on a flow network $G = (V, E)$. For every node $u \in V \backslash \{s, t\}$, the length of the shortest path $d_f(s, u)$ from s to u in the residual graph G_f given by f increases with each flow augmentation.

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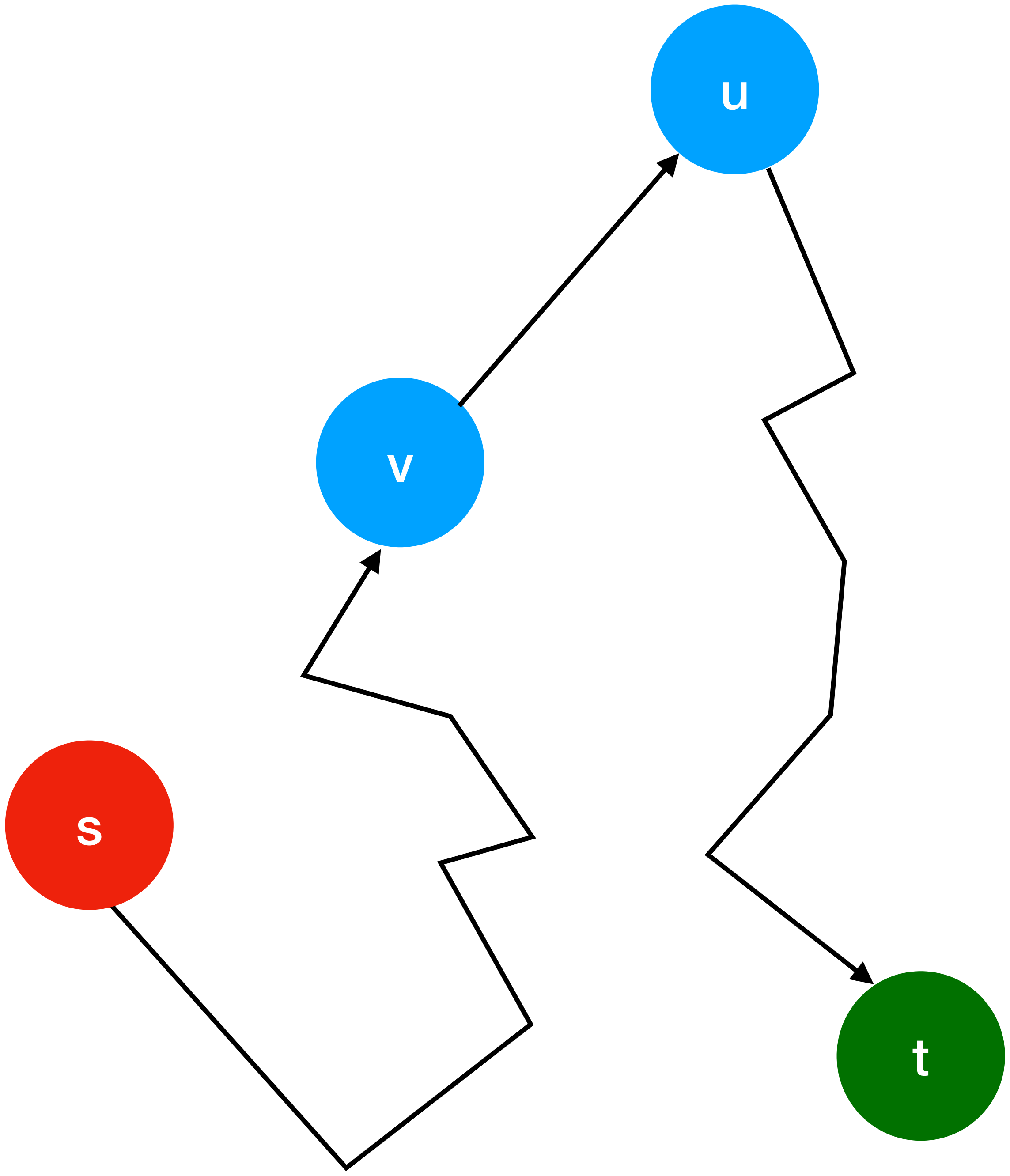
Consider the sub-path (s, u) of the (s, t) path above. This is also a shortest (s, u) path.

We have

$$\begin{aligned} d_f(s, v) = {} & d_f(s, u) - 1 && \text{obvious} \\ & \leq d_{f'}(s, u) - 1 && \text{why?} \\ & = d_{f'}(s, v) - 2 && \text{why?} \end{aligned}$$



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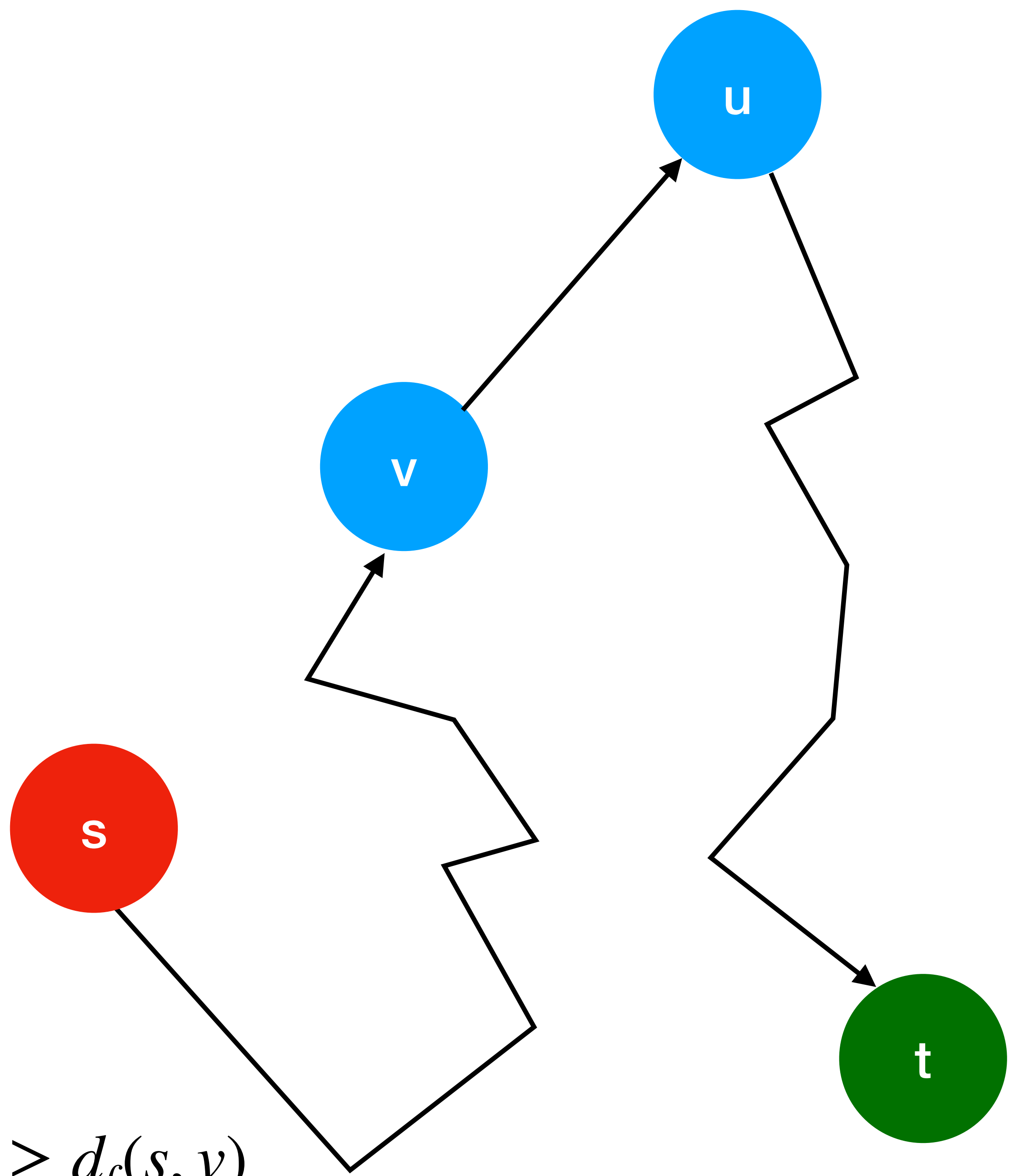

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How many times can an edge e become critical during the execution of the algorithm?

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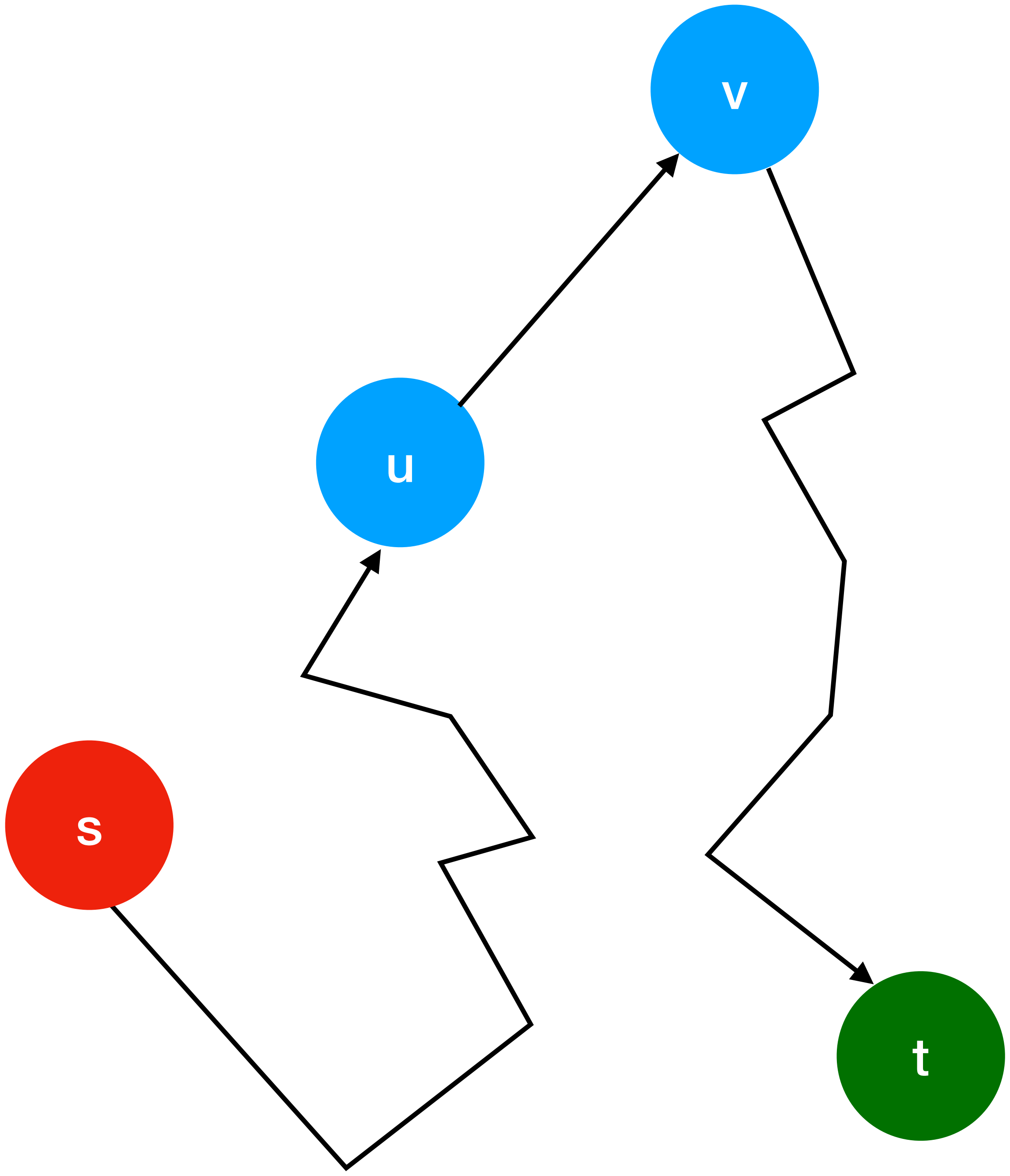
Let (u, v) be an edge. Consider the first time it becomes critical, at which point the residual graph is G_f.



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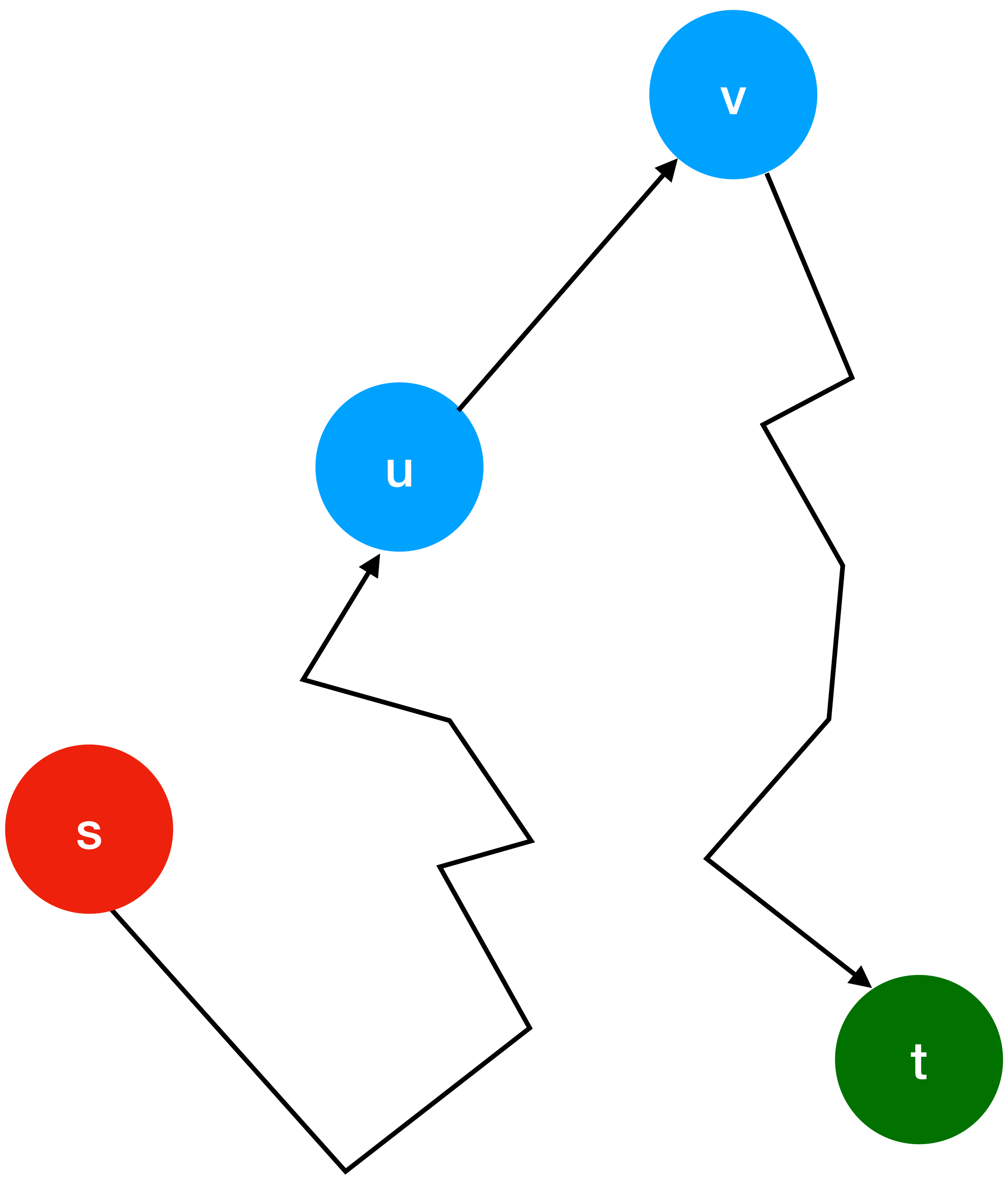

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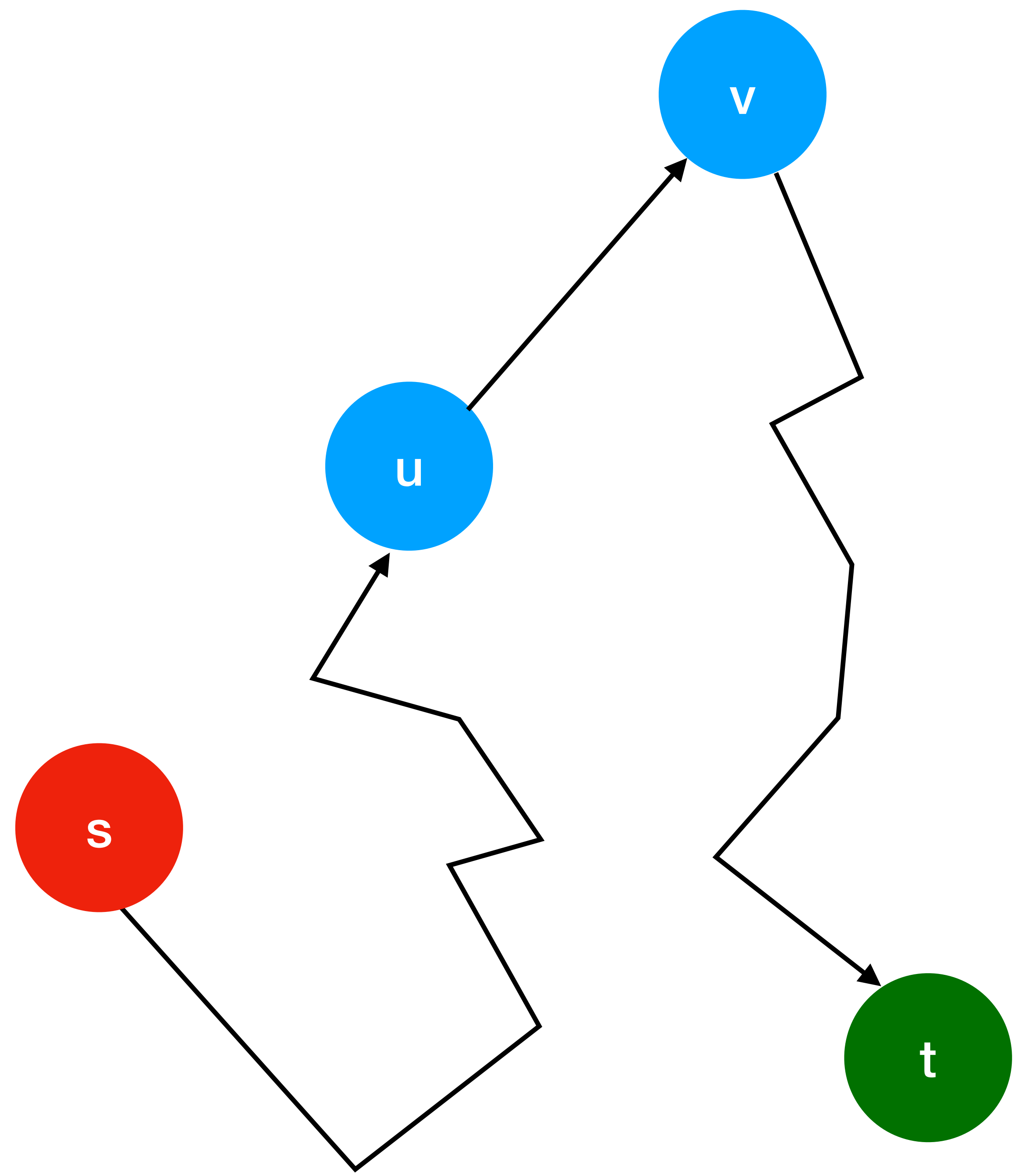

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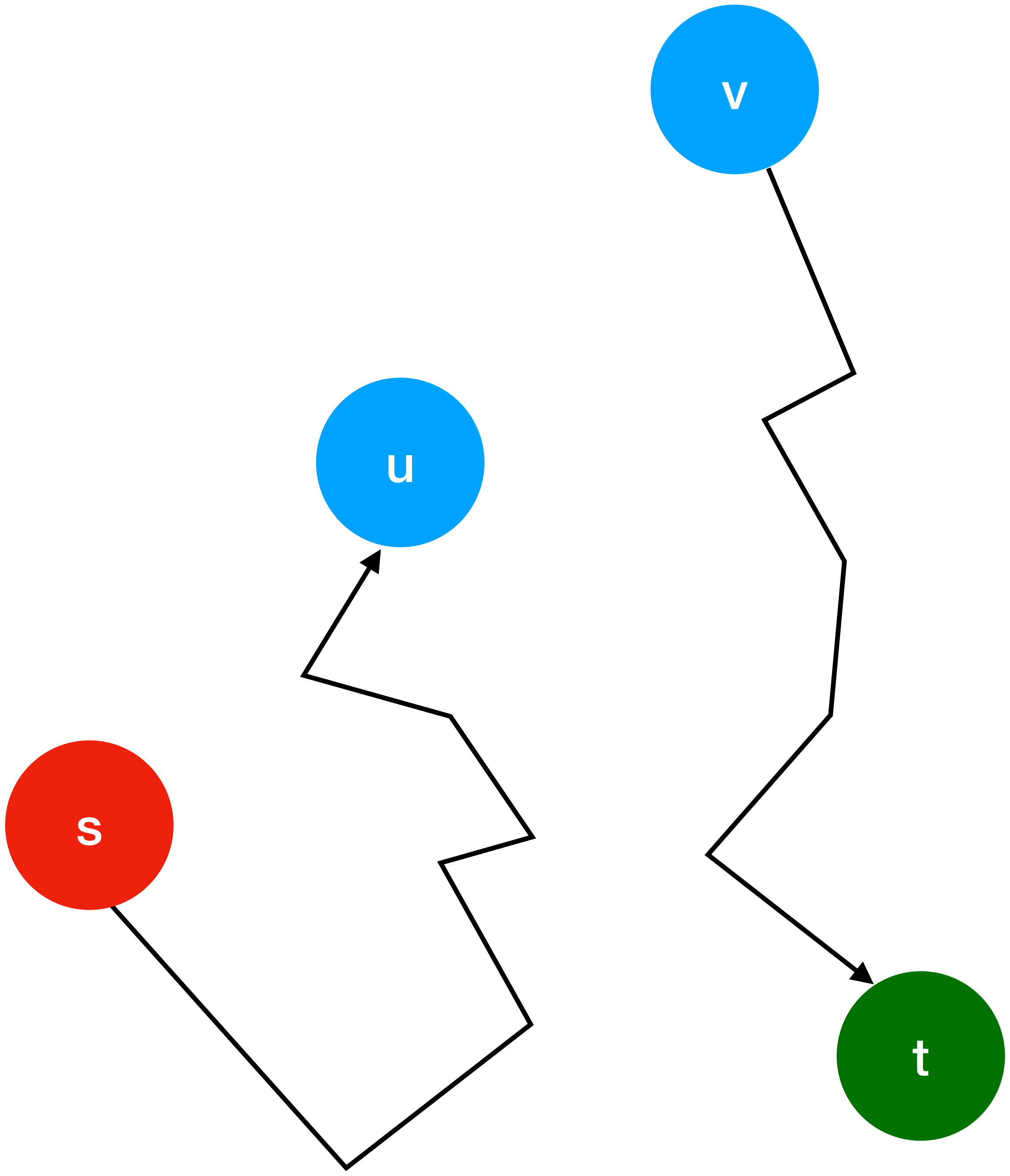

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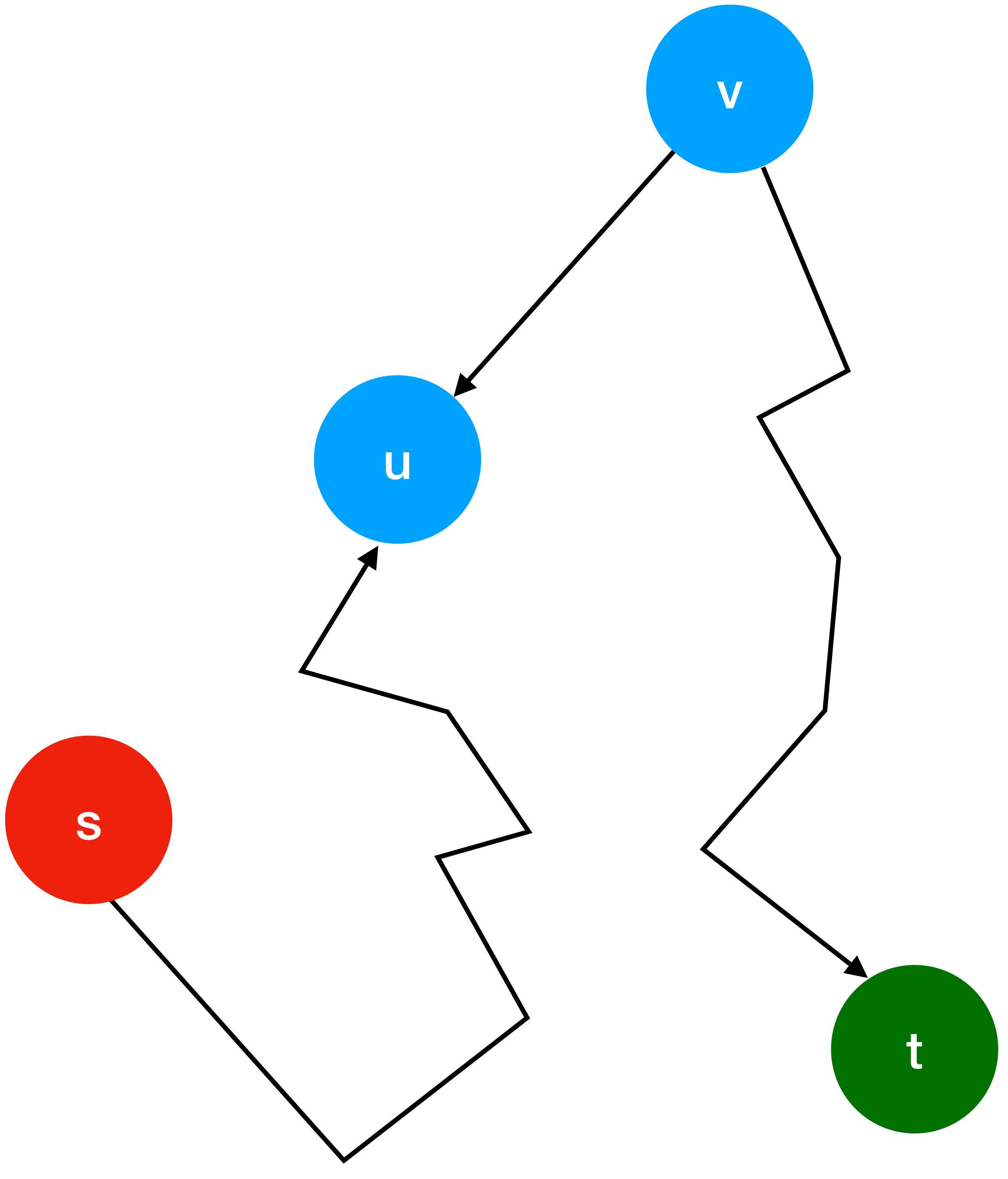

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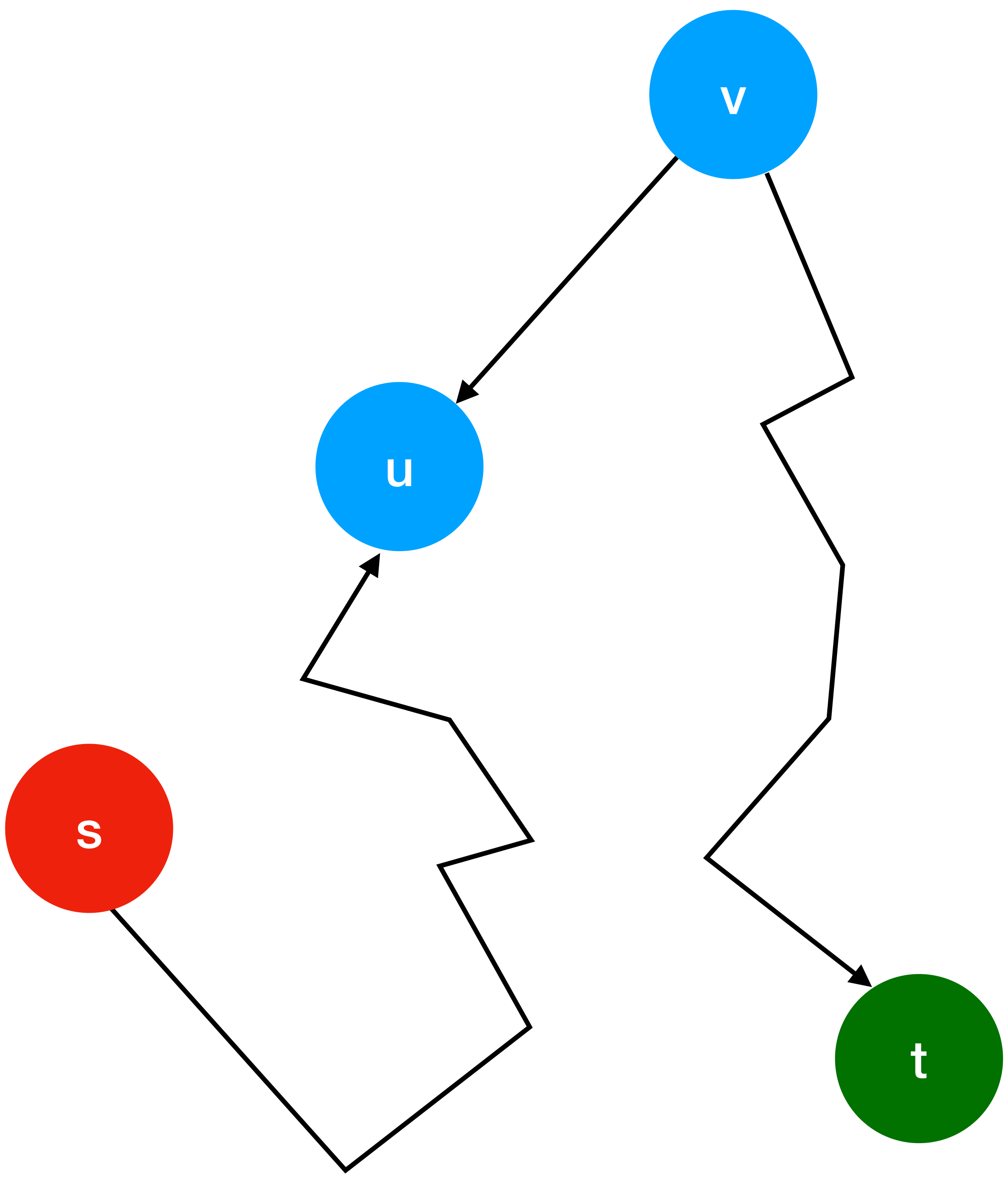

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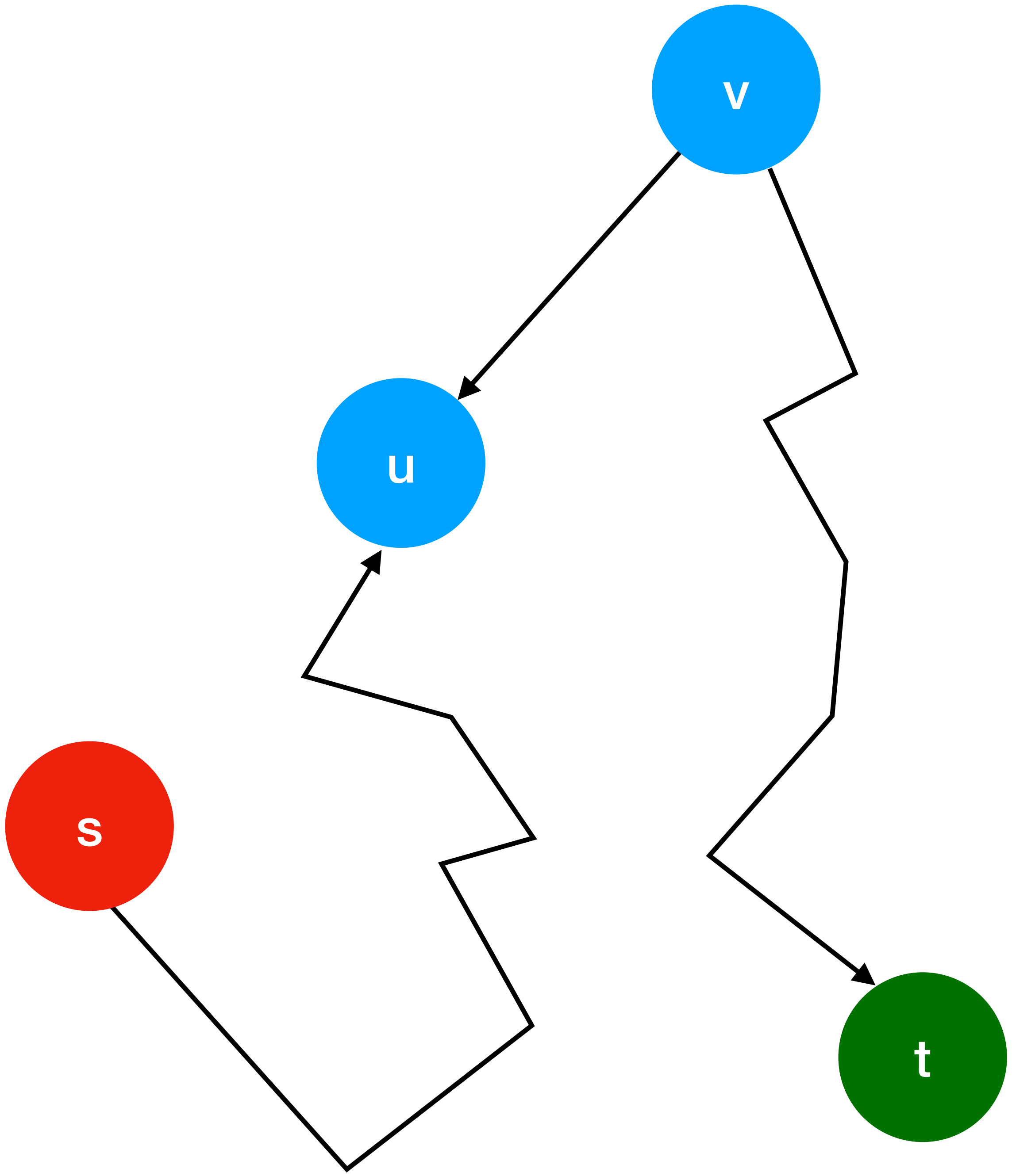

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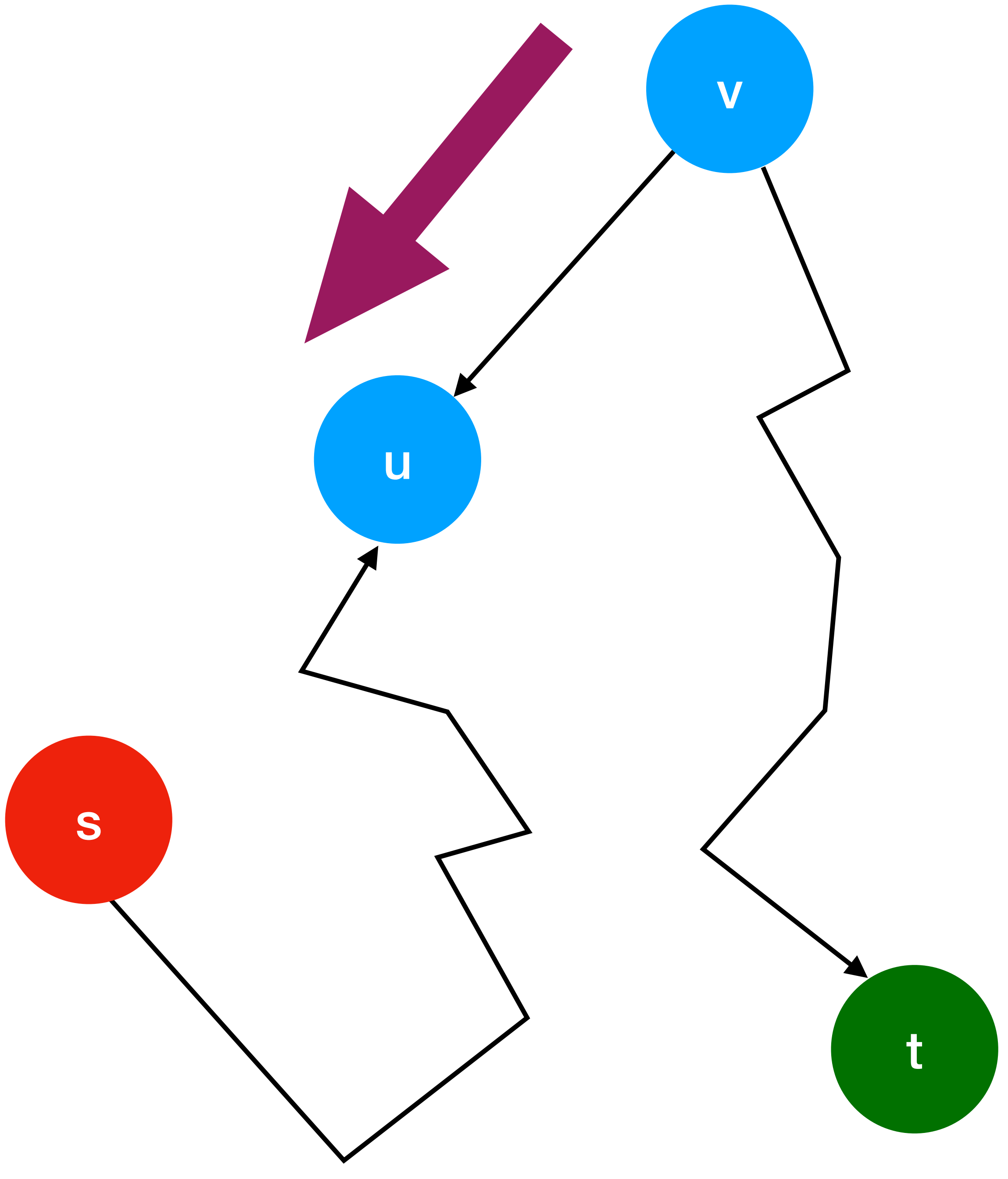

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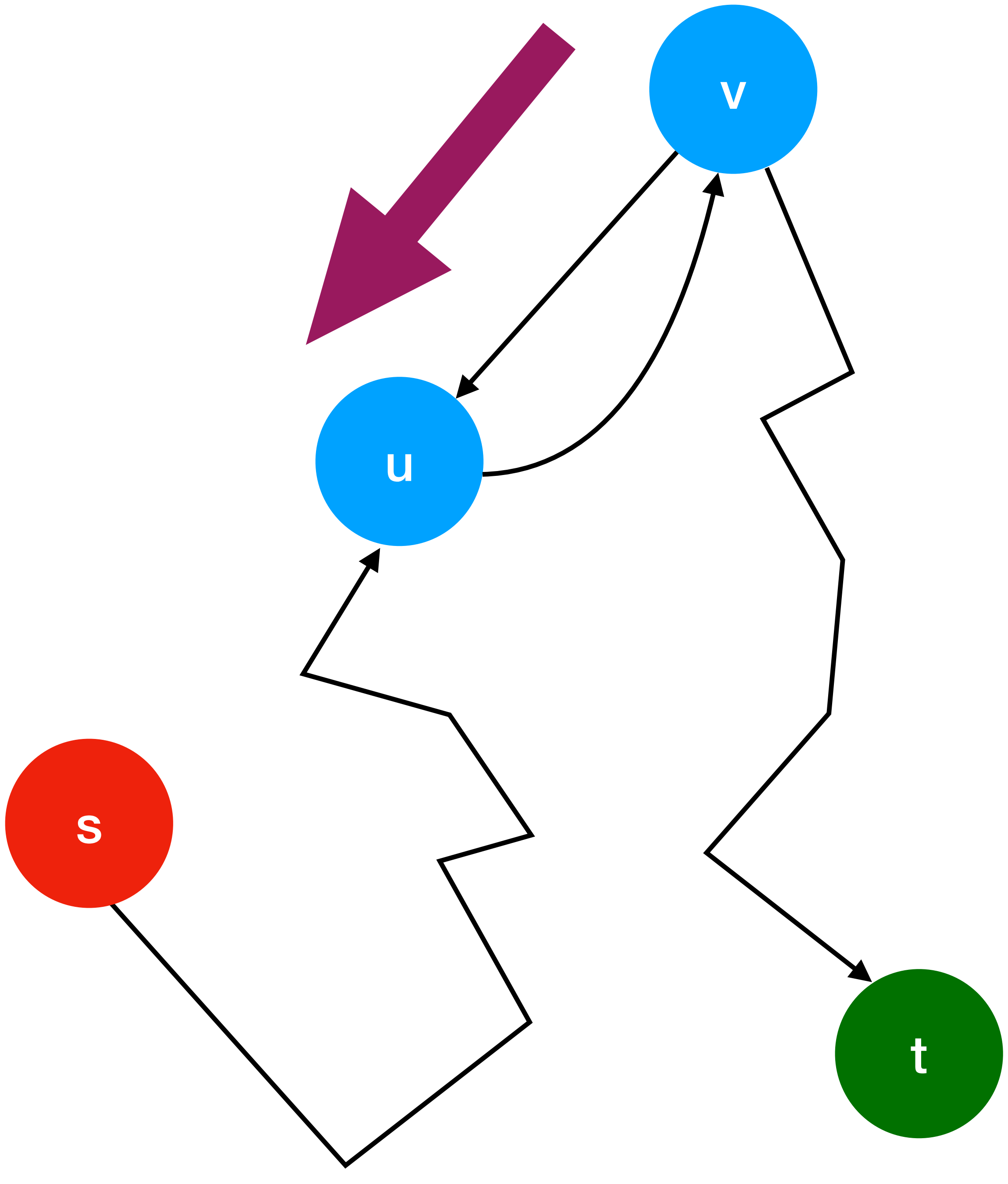

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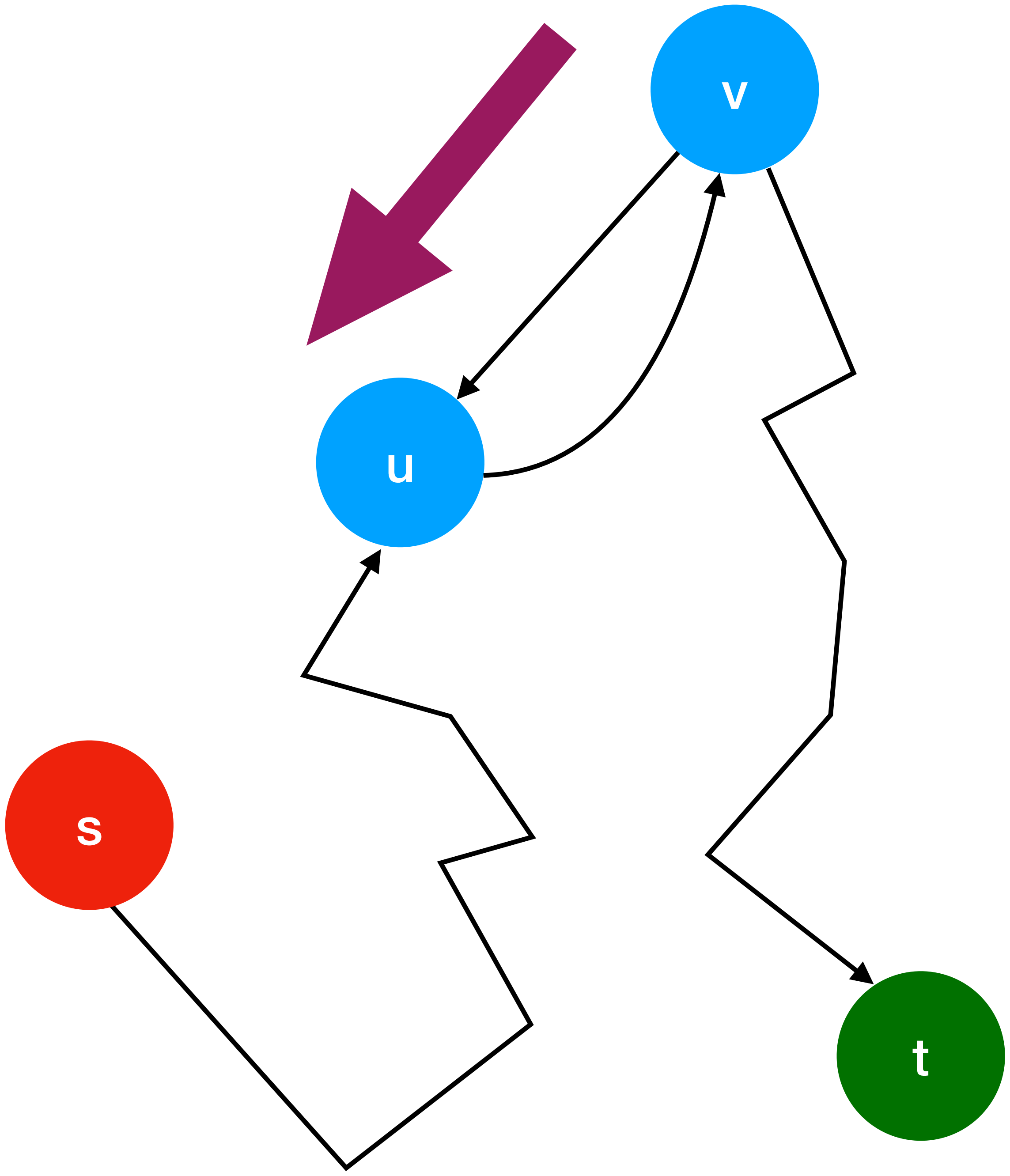

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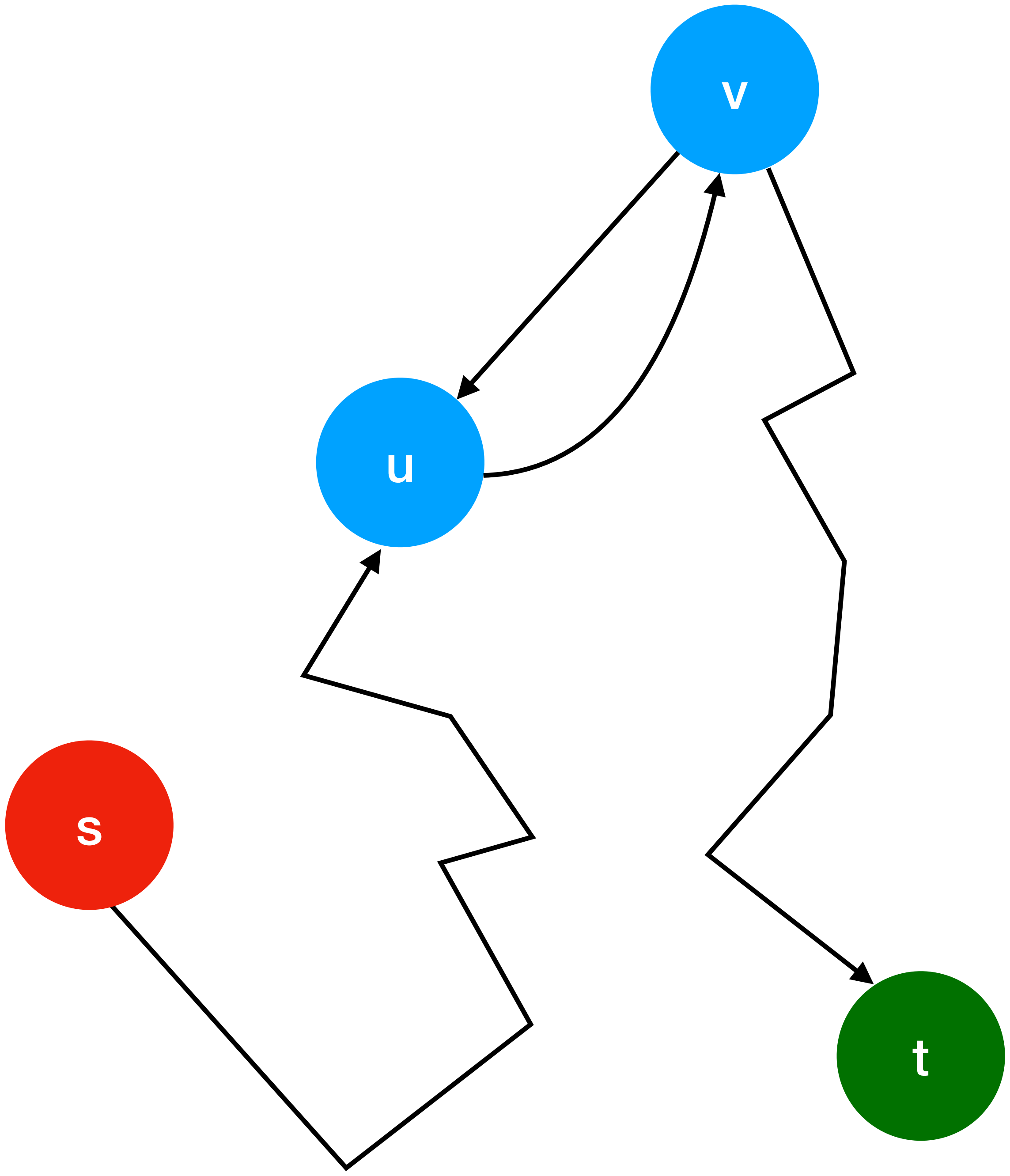

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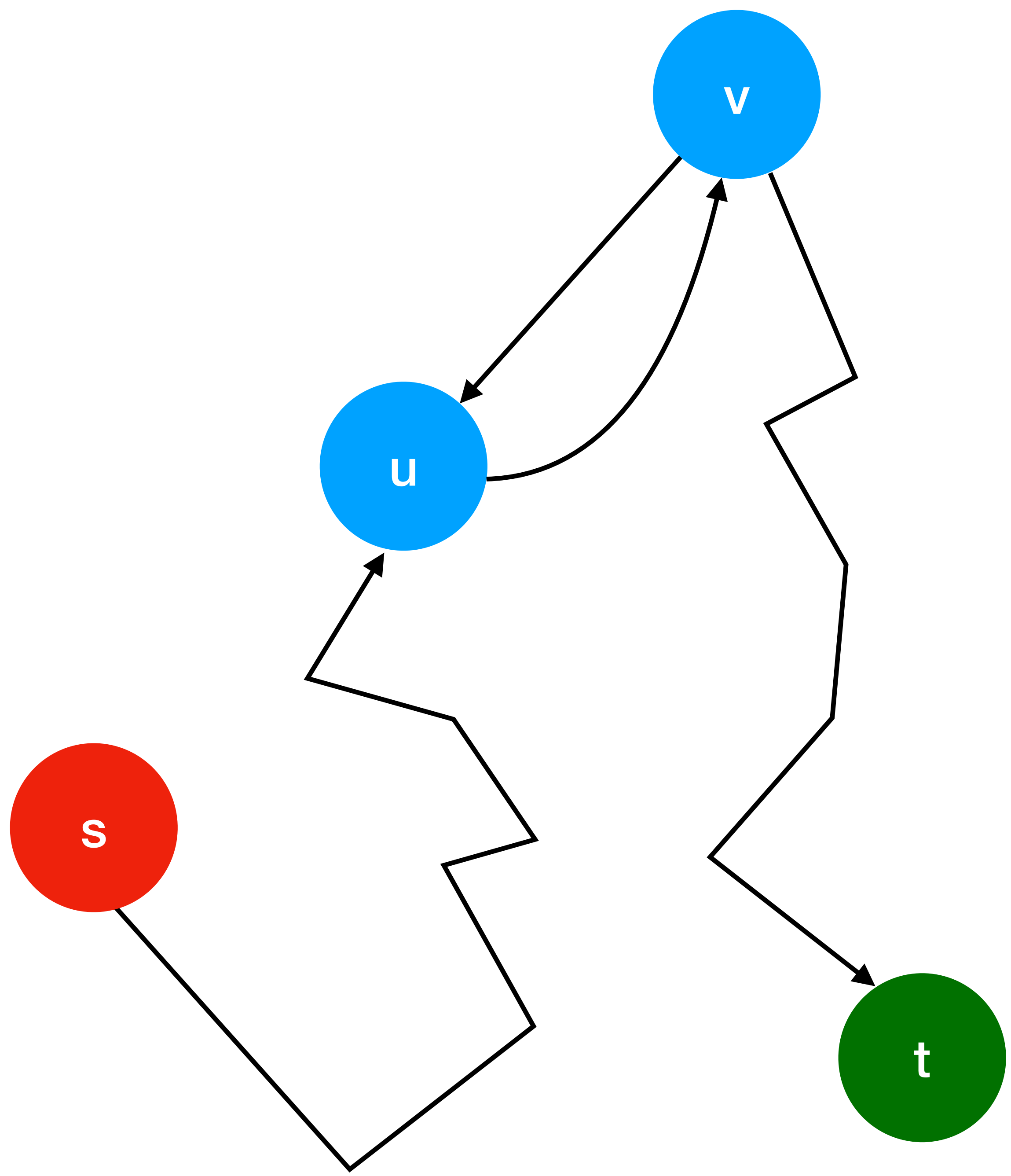

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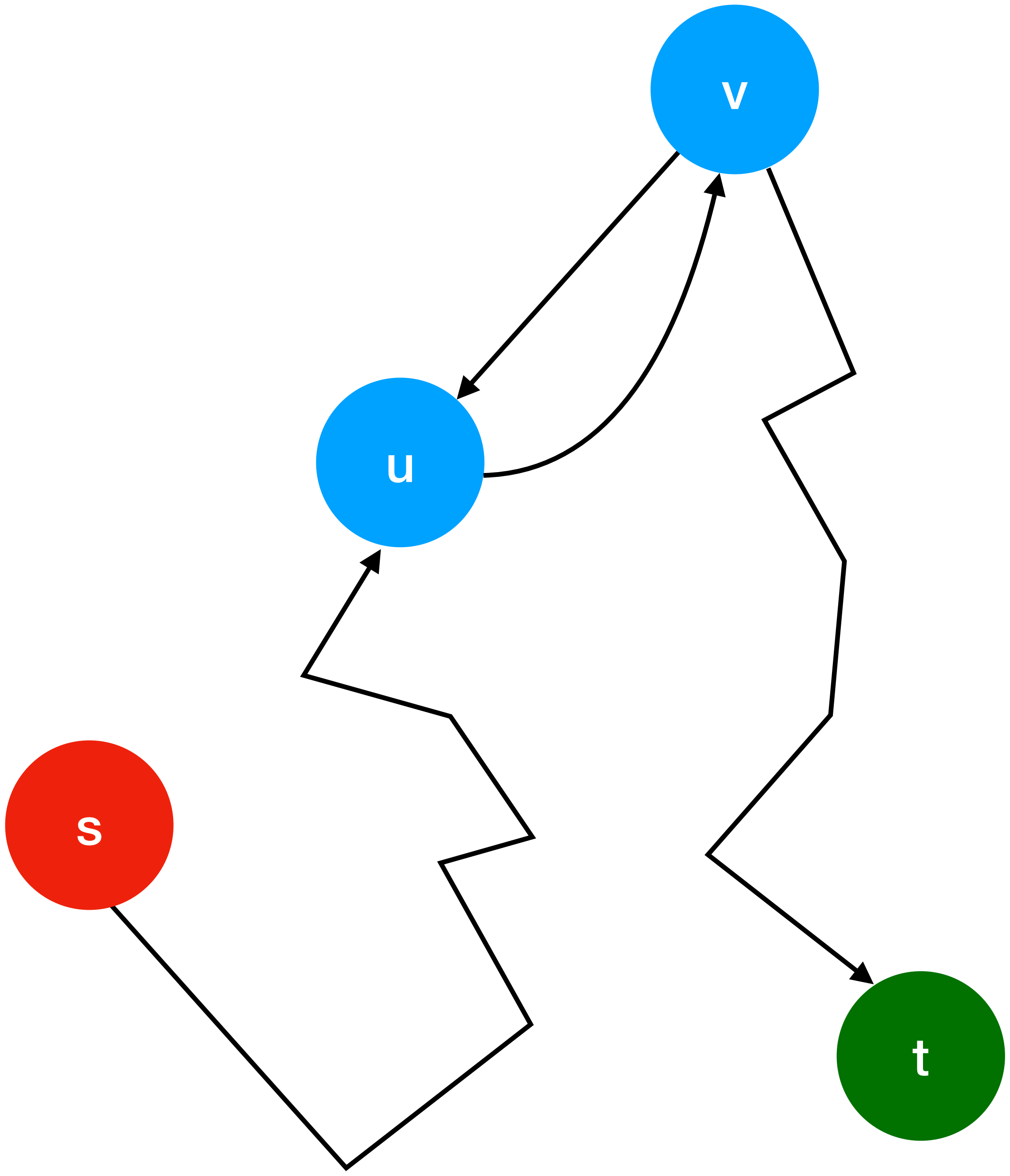

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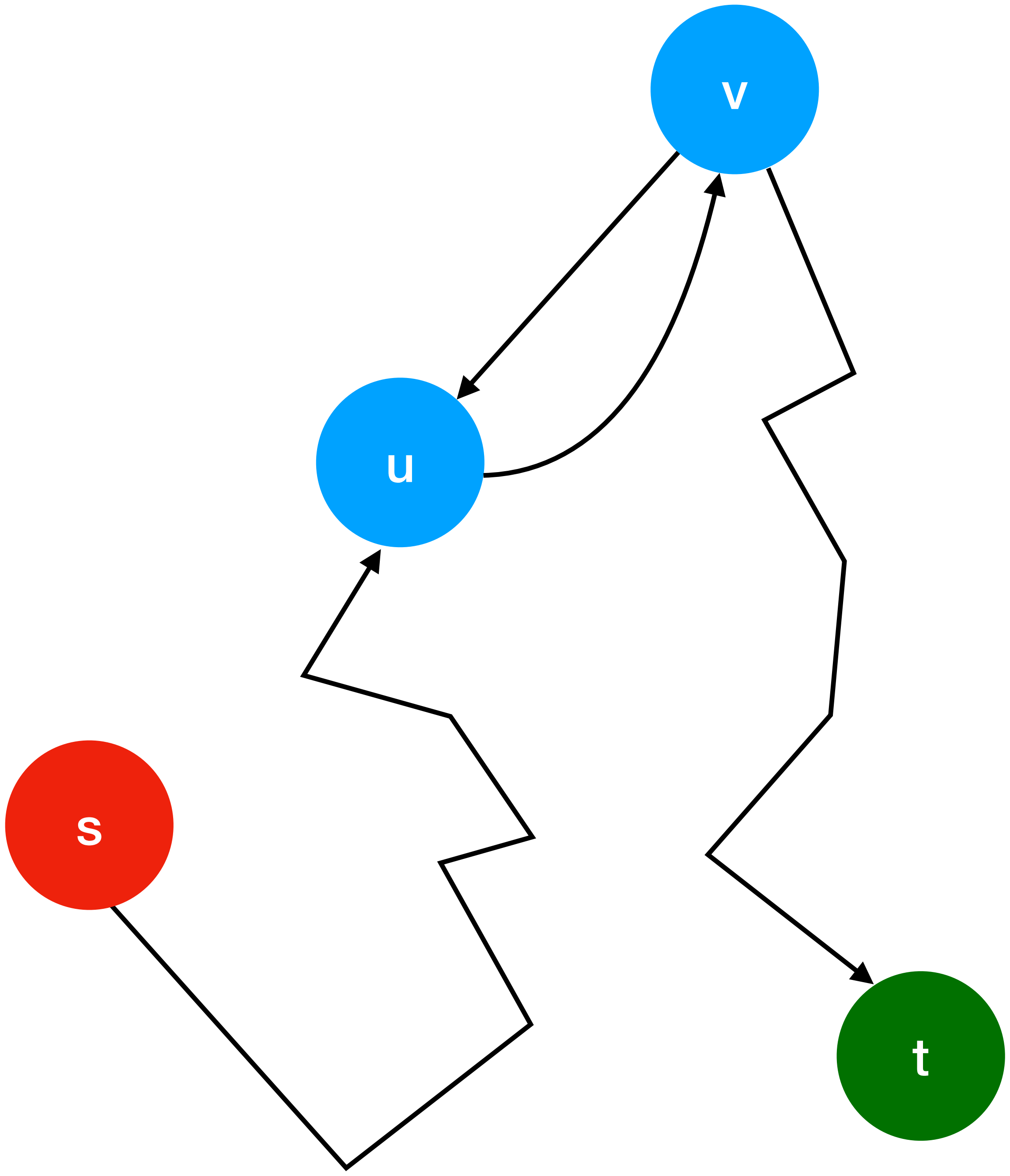

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Edmonds-Karp Running Time

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Edmonds-Karp Running Time

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Total running time?

$O(nm^2)$.