Algorithms and Data Structures

Minimum cuts and the correctness of Ford-Fulkerson

Ford-Fulkerson analysis

Feasibility ✓

Does the algorithm produce a flow if it terminates?

Termination ✓

Does the algorithm always terminate?

Running Time ✓

What is the running time of the algorithm?

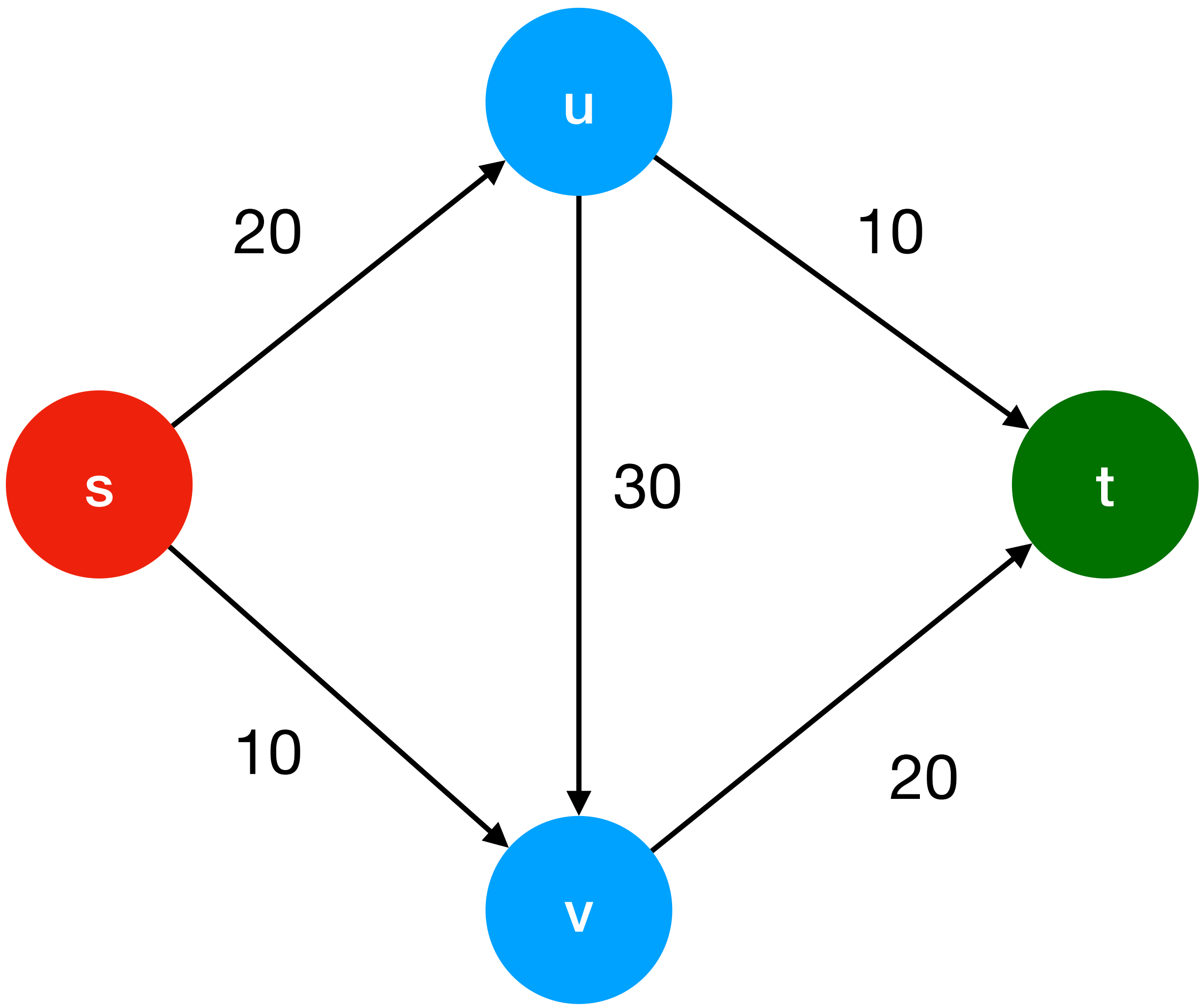
Optimality / Correctness ?

Does the algorithm produce a maximum flow?

Minimum Cut

- A *cut* C is a partition of the nodes of G into two sets S and T, such that s is in S and t is in T.
- The capacity c(S,T) of a cut C is the sum of capacities of all edges "out of S".
  - these are edges (u, v) where u is in S and v is in T.

Example



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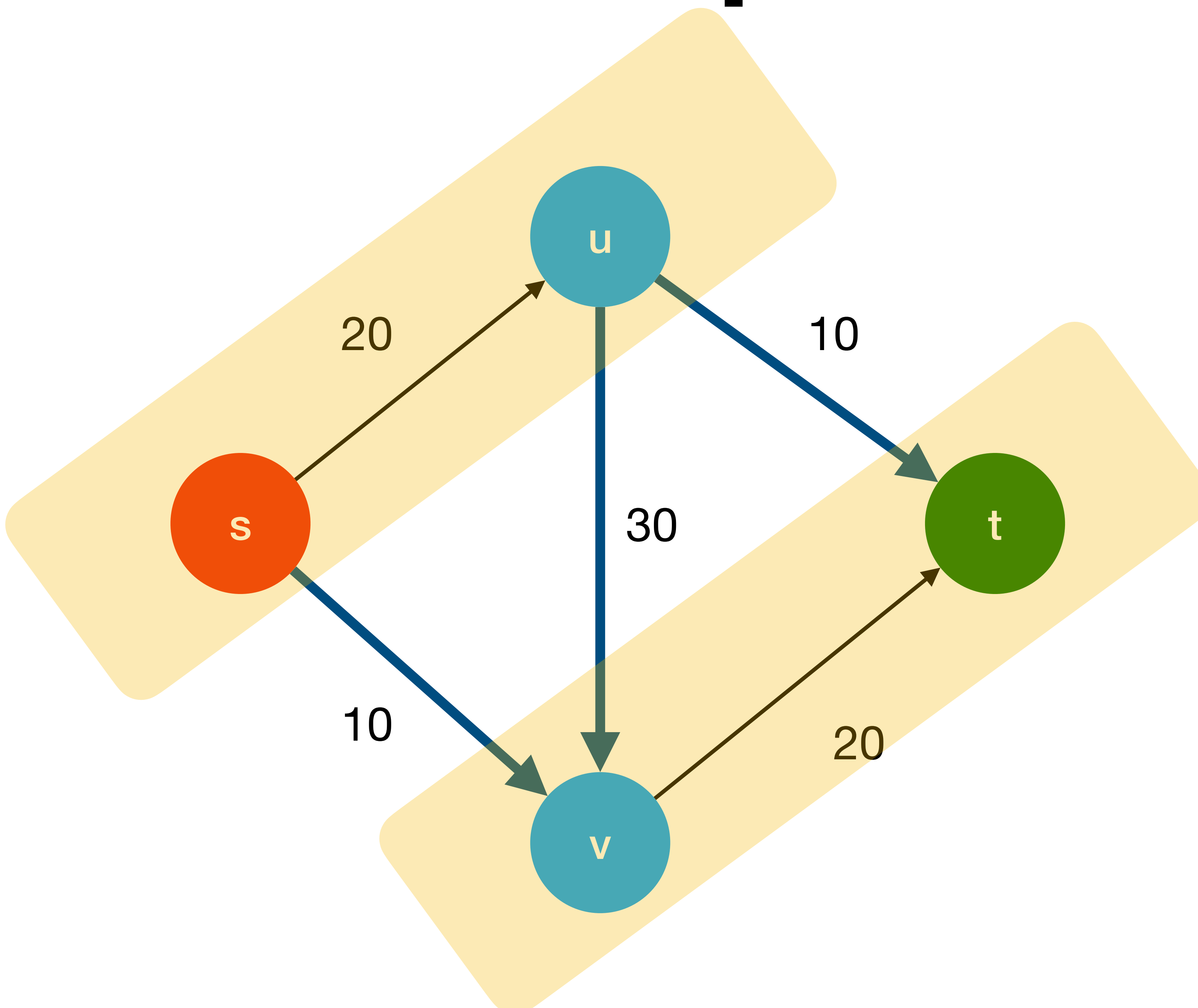

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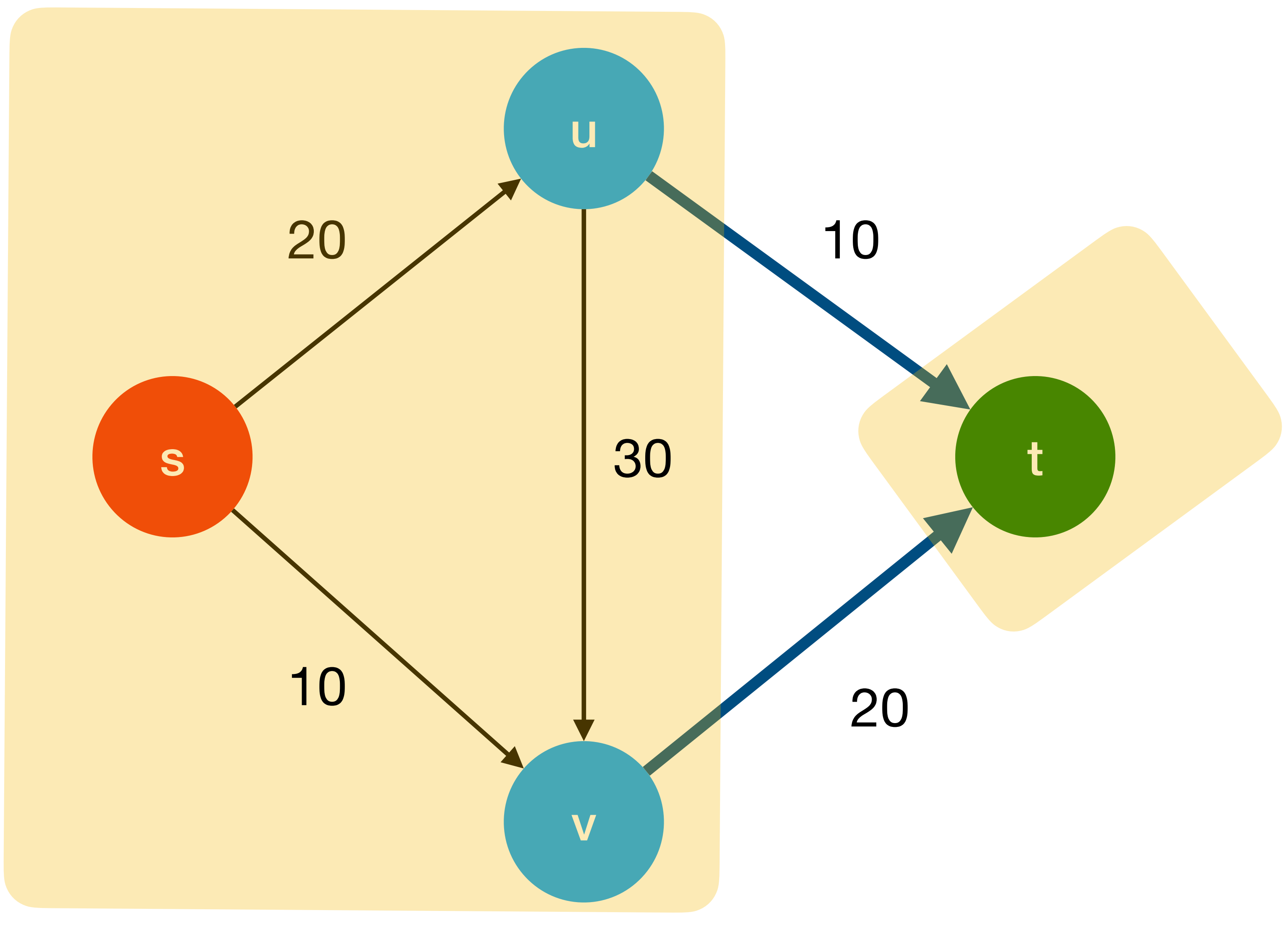
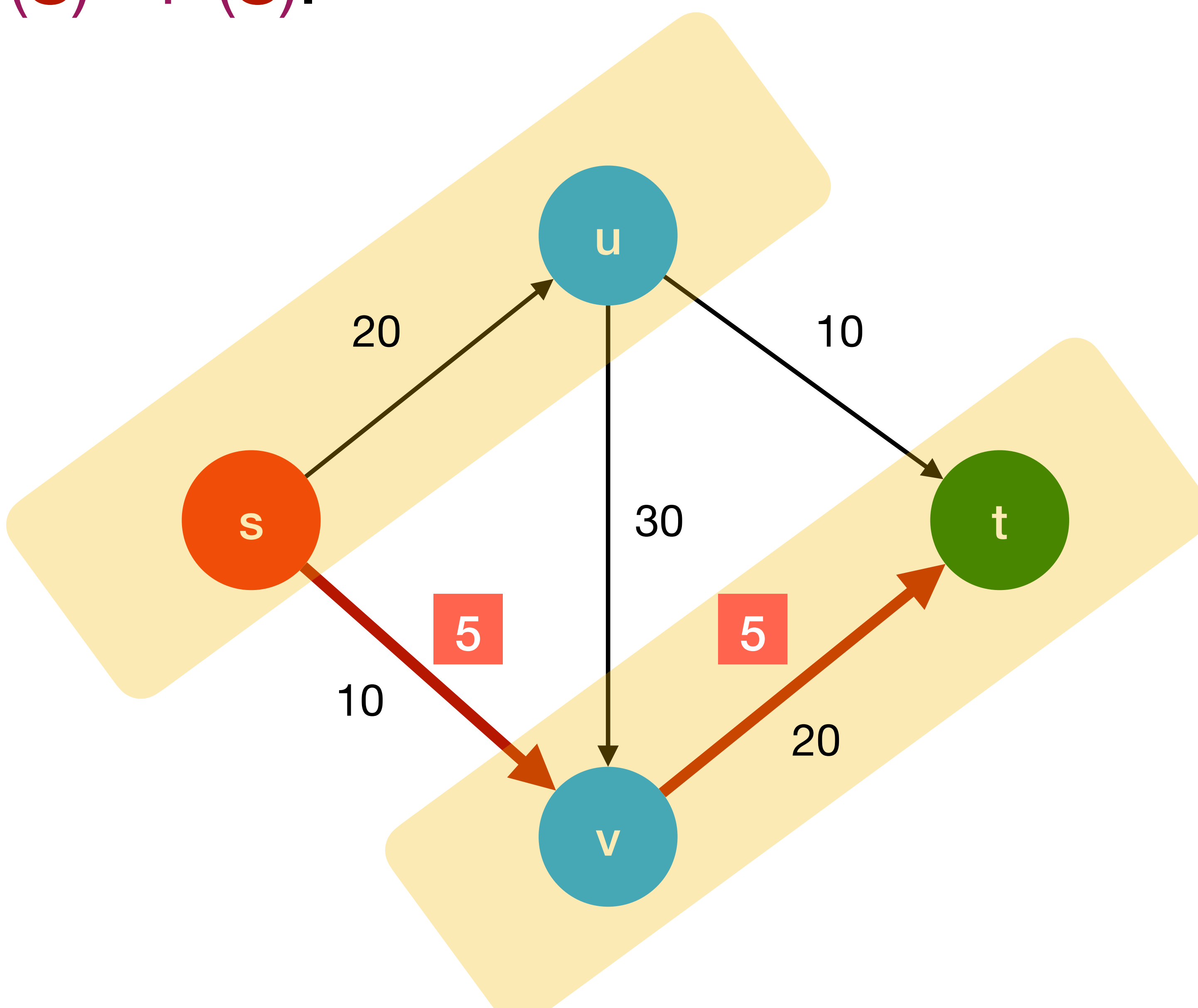


The Max-Flow Min-Cut Theorem

Theorem: In every flow network, the value of the maximum flow is *equal* to the capacity of the minimum cut.

A series of facts

Fact 1: Let f by any (s-t) flow and (S, T) be any (s-t) cut. Then $v(f) = f^{out}(S) - f^{in}(S)$.



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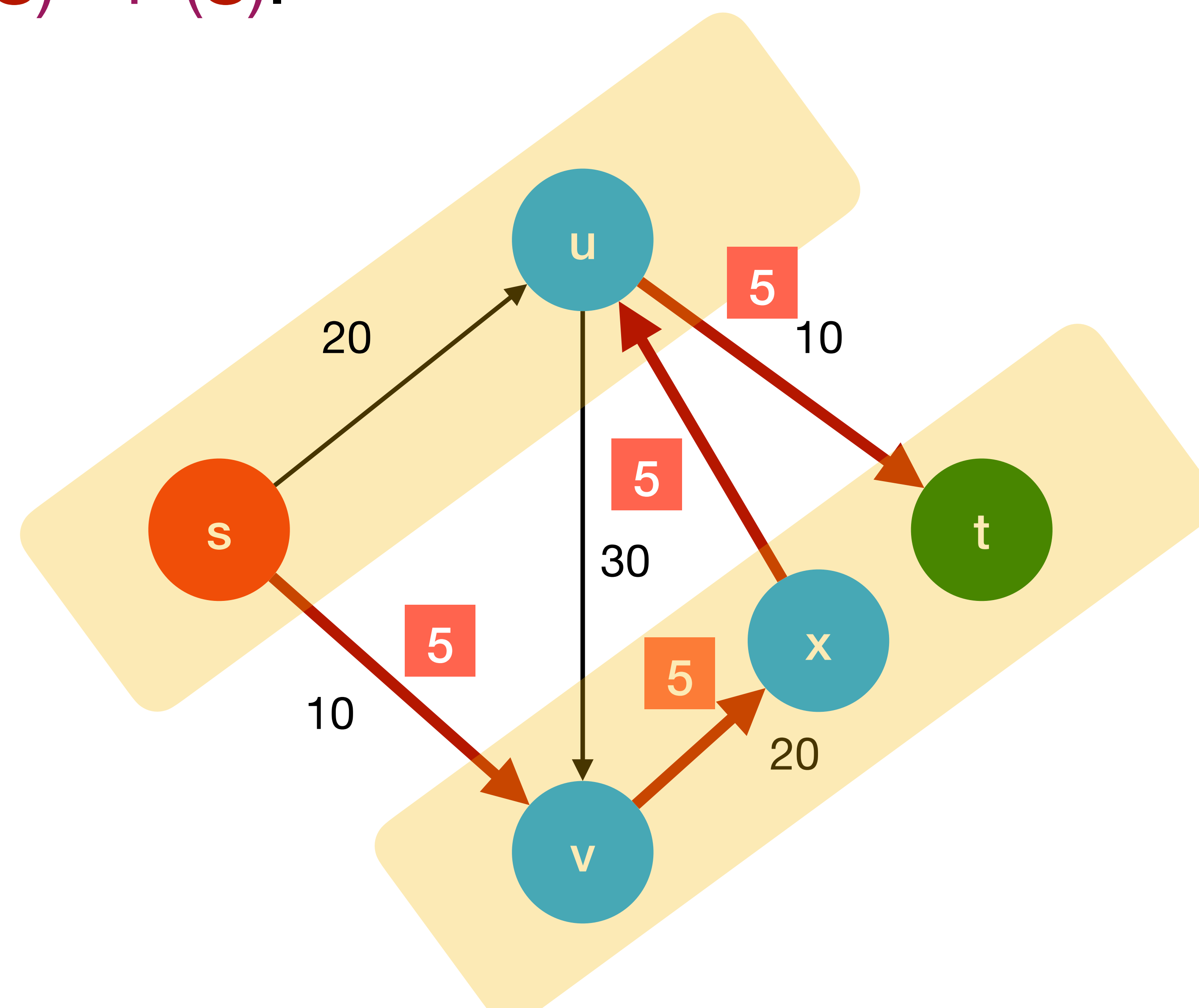

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Therefore we have:

$$v(f) = \sum_{v \in S} \left(f^{\mathbf{out}}(v) - f^{\mathbf{in}}(v) \right)$$

Fact 1 - Rewriting the sums

Fact 1: Let f by any (s-t) flow and (S, T) be any (s-t) cut.
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Otherwise the edge does not appear in the sum.

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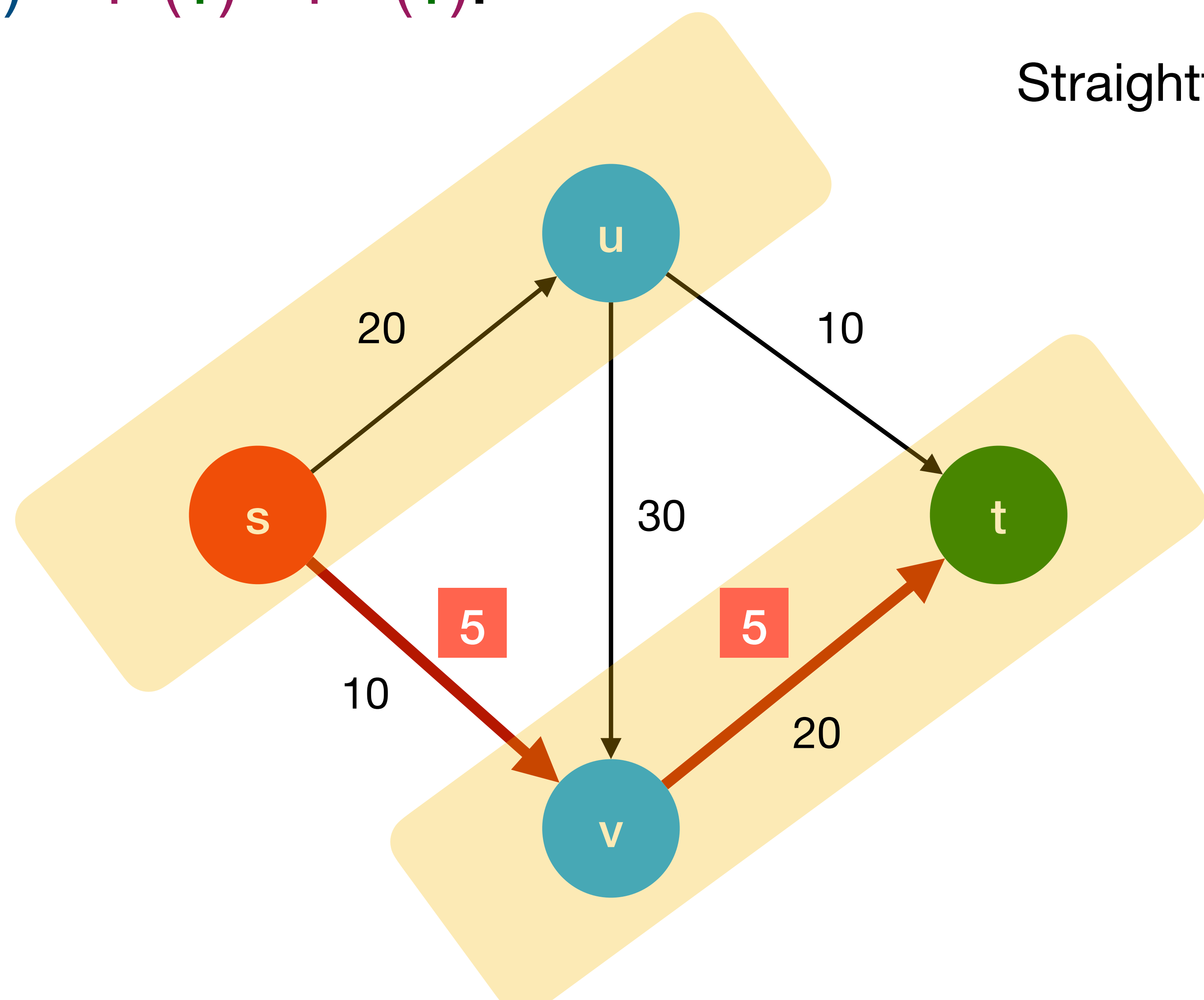
We can write

$$v(f) = \sum_{v \in S} \left(f^{\textbf{out}}(v) - f^{\textbf{in}}(v) \right) = \sum_{e \textbf{ out of } S} f(e) - \sum_{e \textbf{ into } S} f(e) = f^{\textbf{out}}(S) - f^{\textbf{in}}(S)$$

A series of facts

Fact 2: Let f by any (s-t) flow and (S, T) be any (s-t) cut. Then $v(f) = f^{in}(T) - f^{out}(T)$.

Straightforward by Fact 1.



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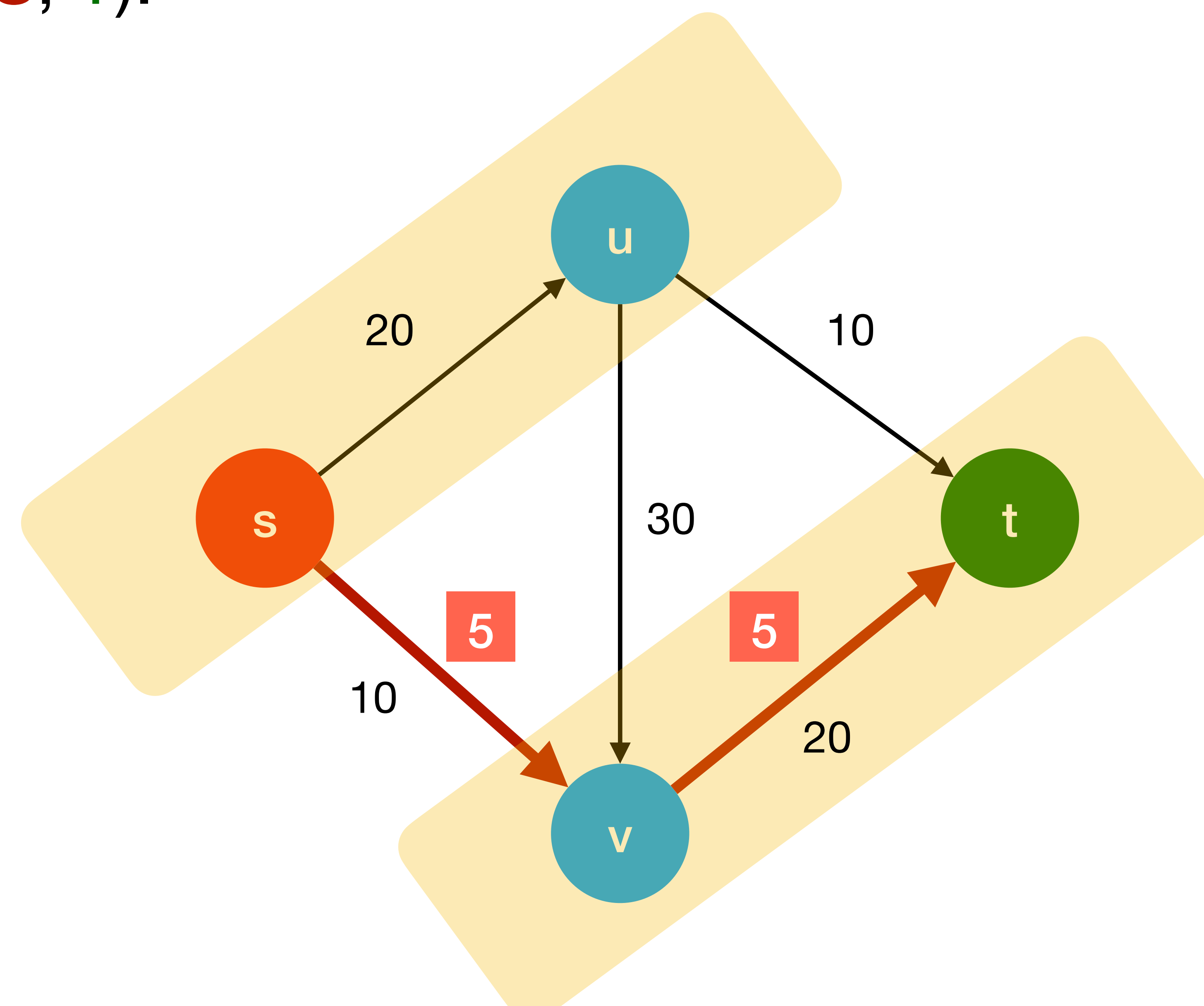
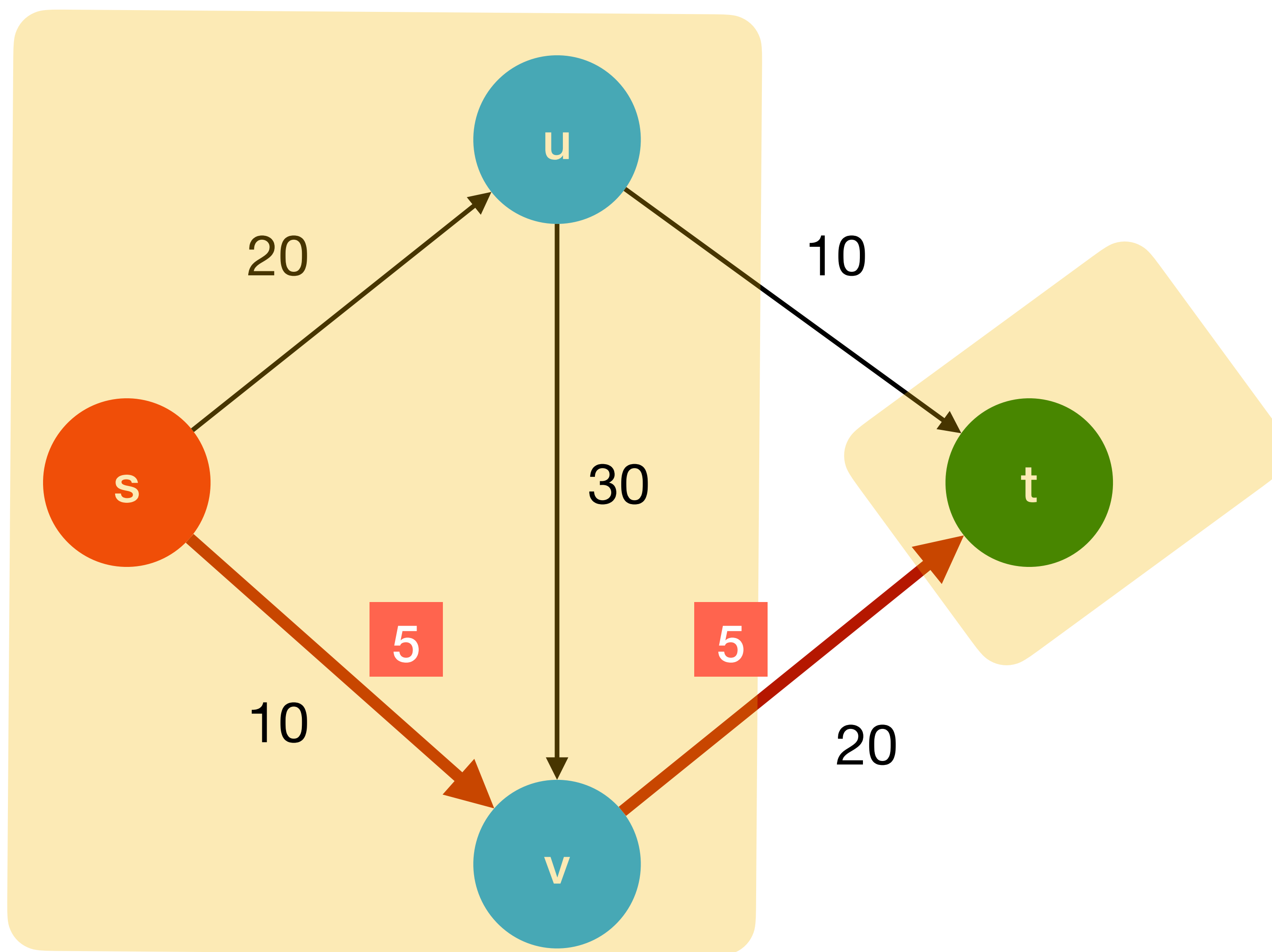
Fact 3: Let f by any (s-t) flow and (S, T) be any (s-t) cut. Then $v(f) \leq c(S, T)$.



Another (s-t) cut



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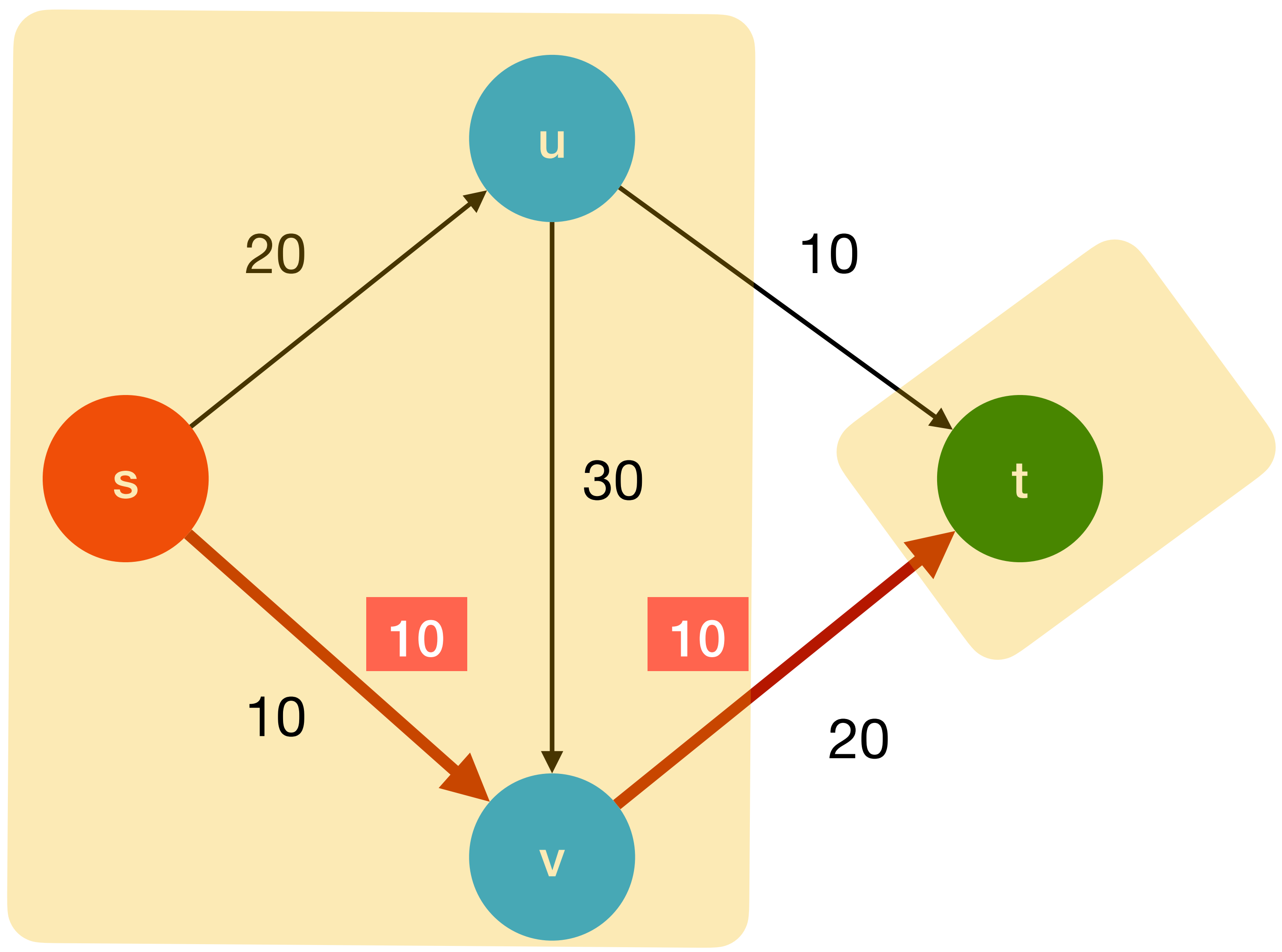

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Comparing facts

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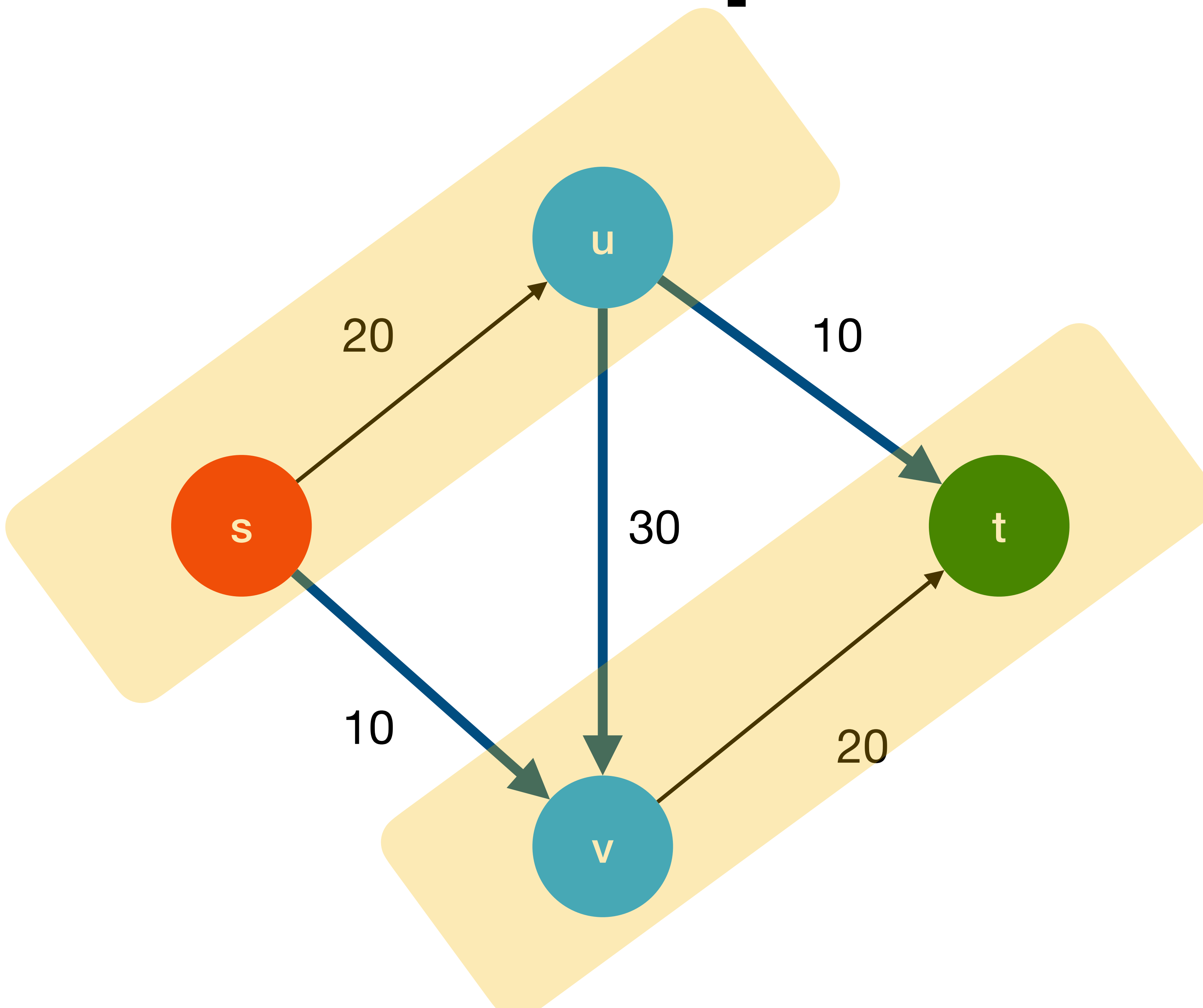
Example



What can we safely say about the maximum flow?

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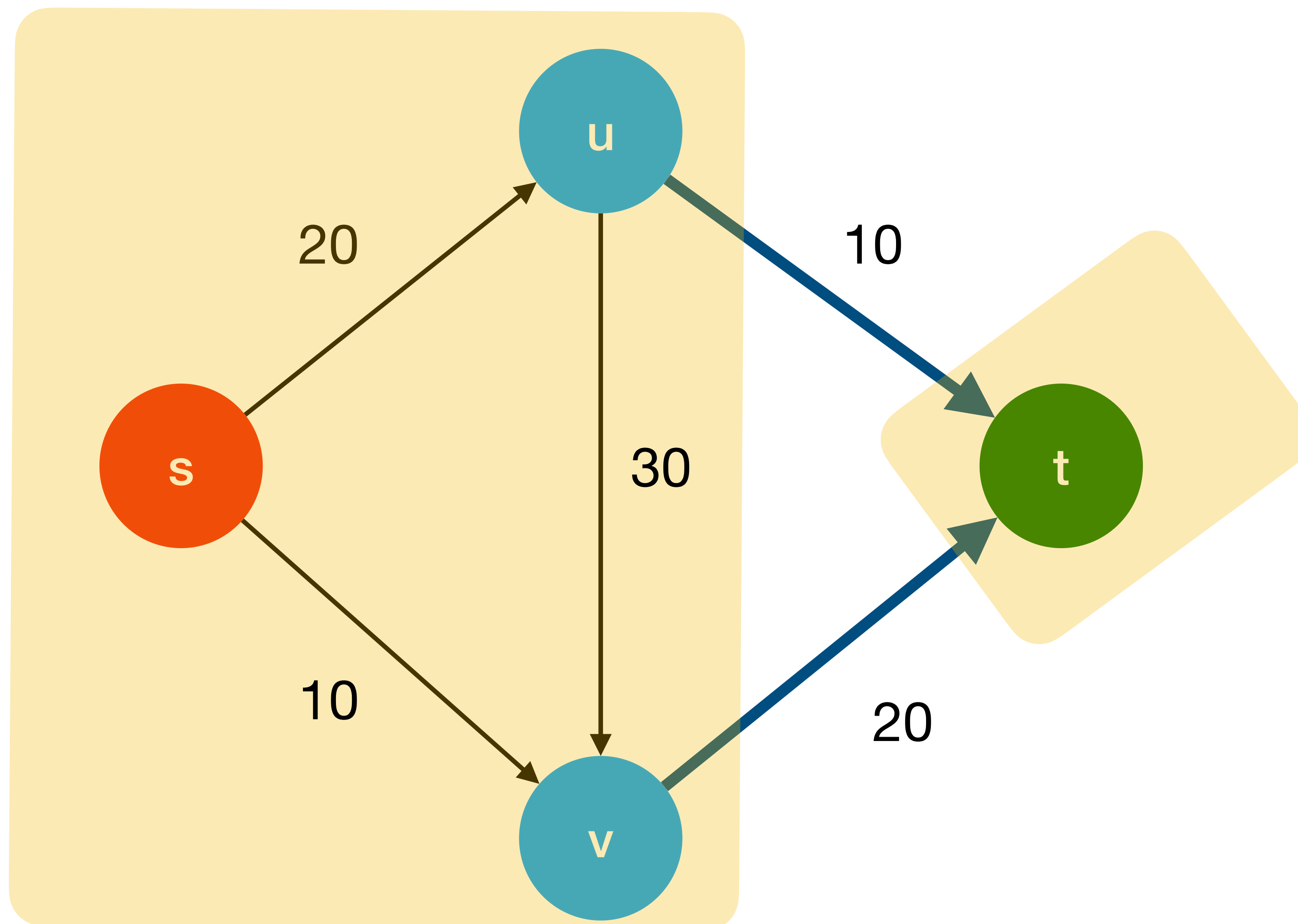

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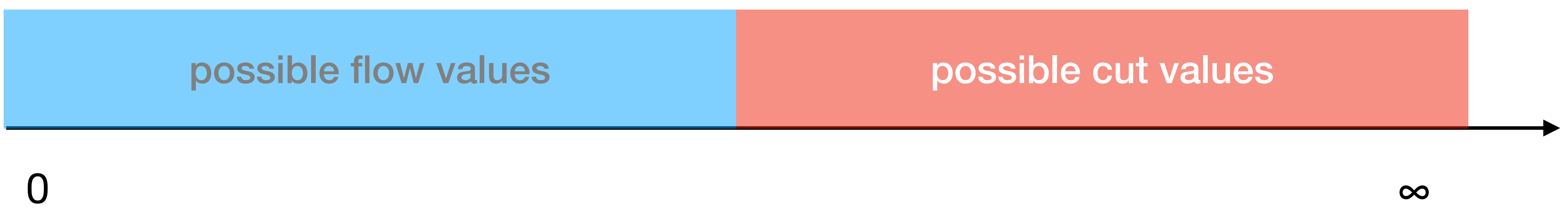
Proof idea



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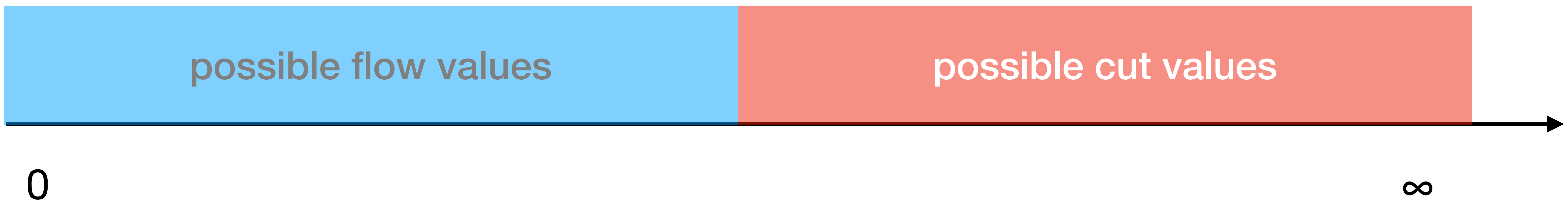

How can we prove that a flow f* is maximum?

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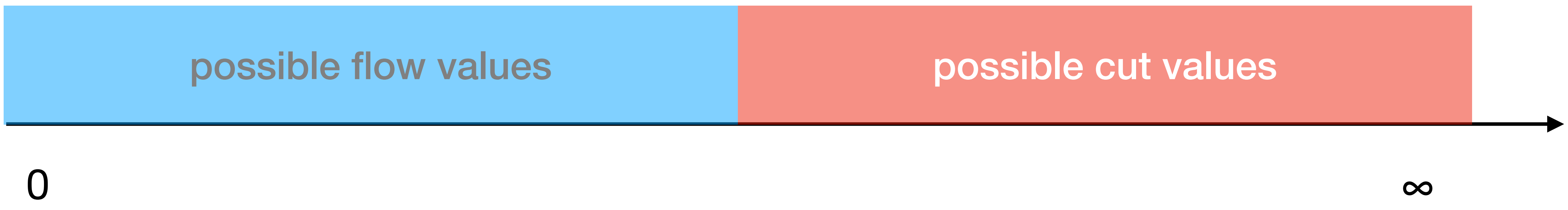

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Find a cut with capacity $c = f^*$.

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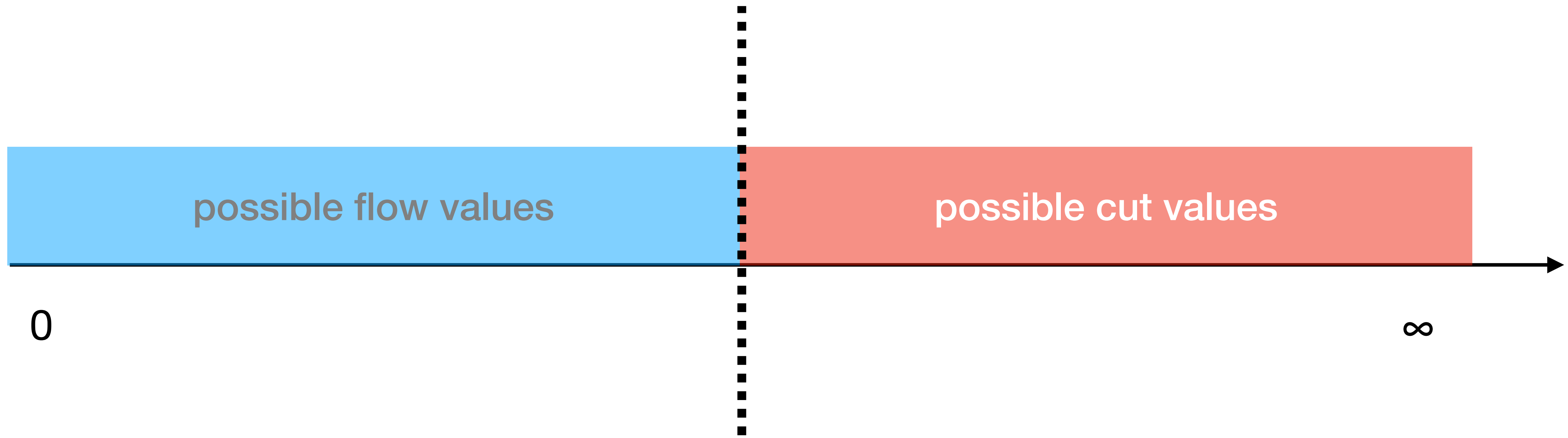

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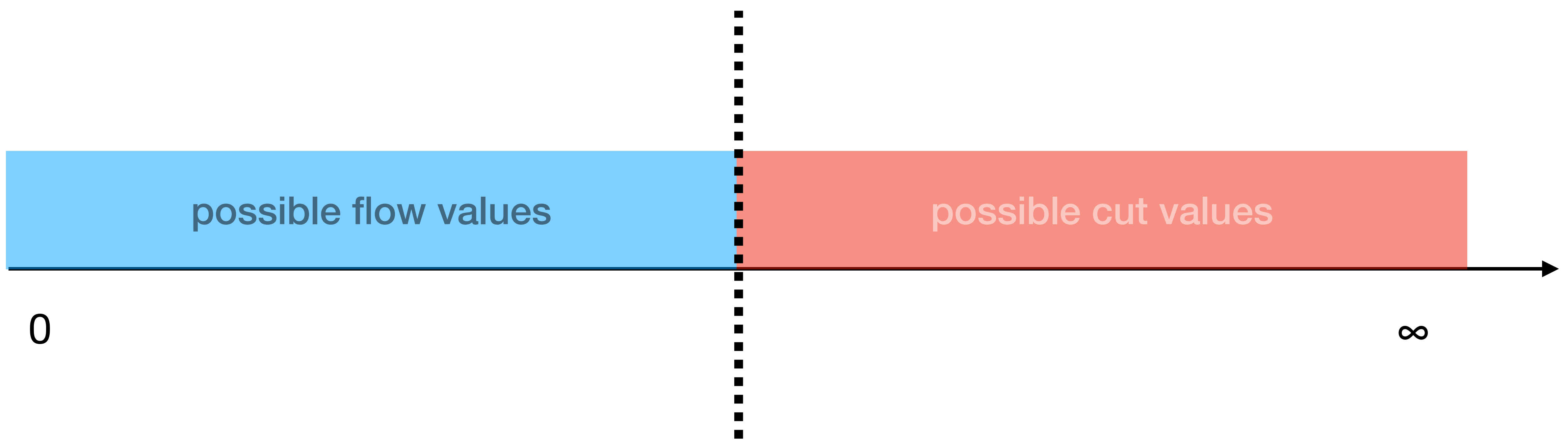

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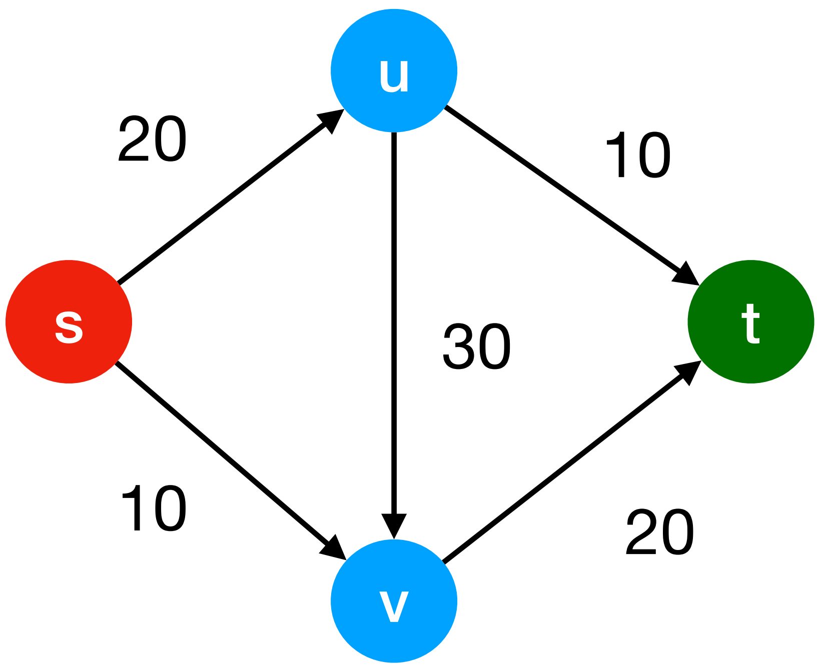

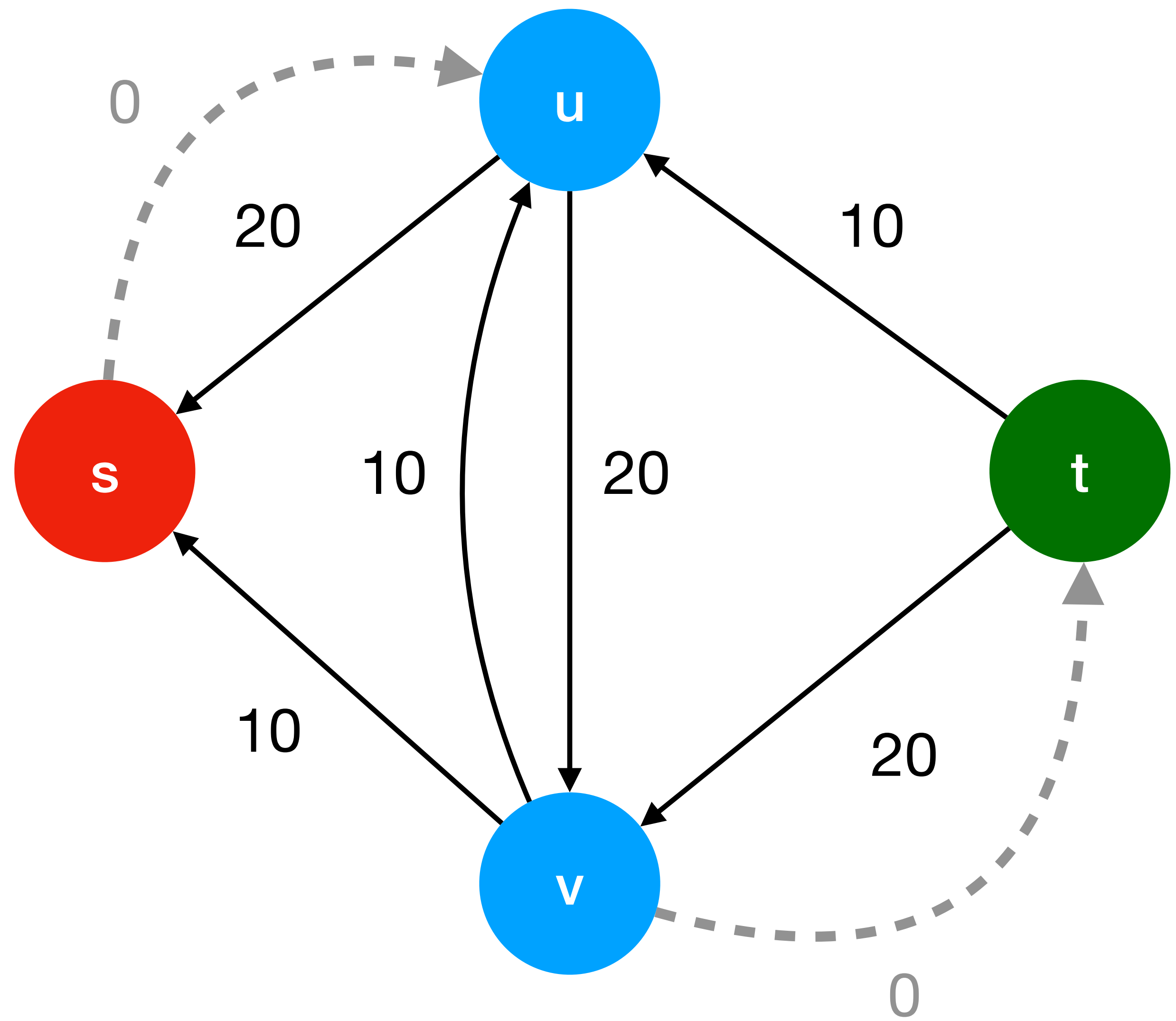

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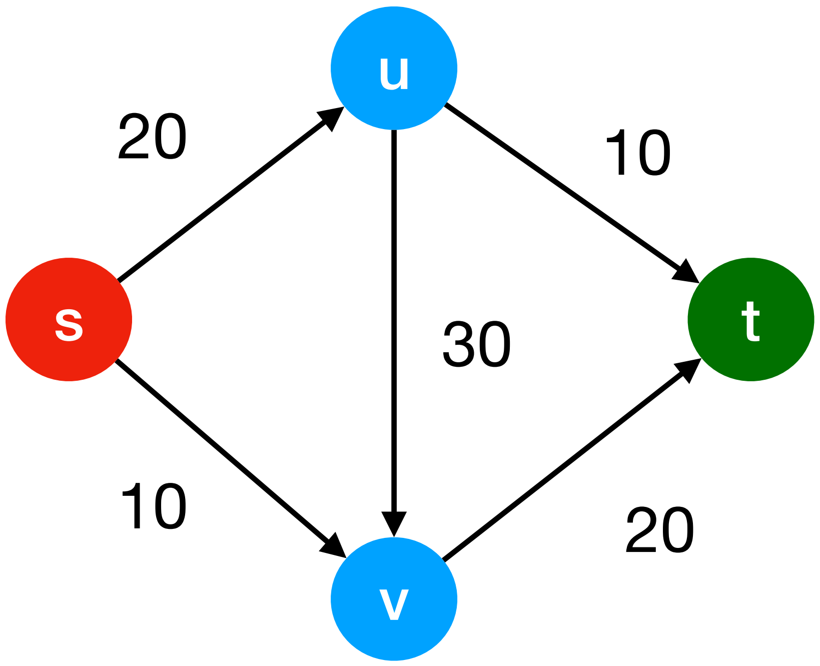

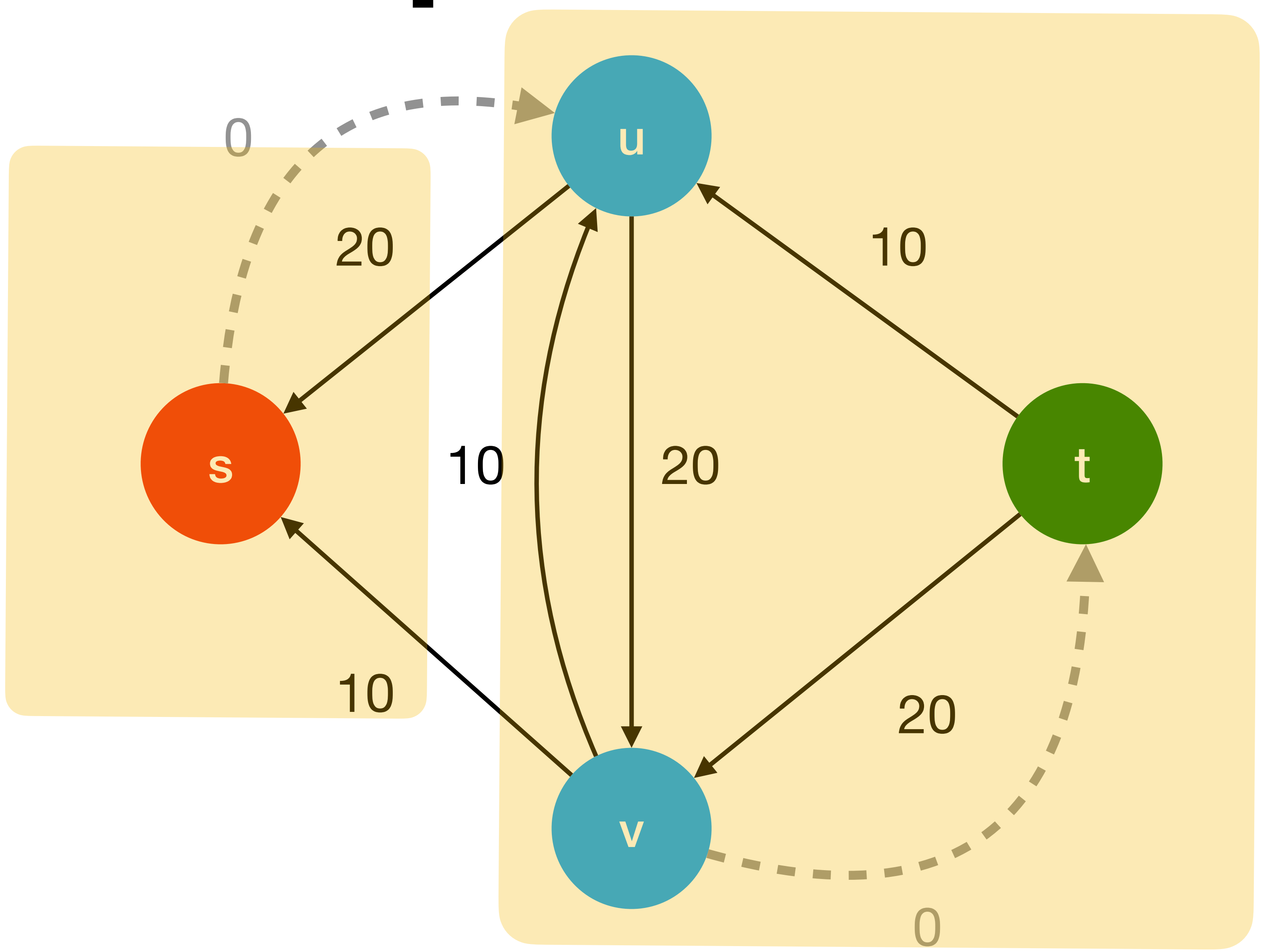

Proving Fact 4

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Proving Fact 4

In the residual graph G_f, identify the nodes that are reachable from the source s.

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Is this a cut?

s is in S*.

t is in T* (why?).

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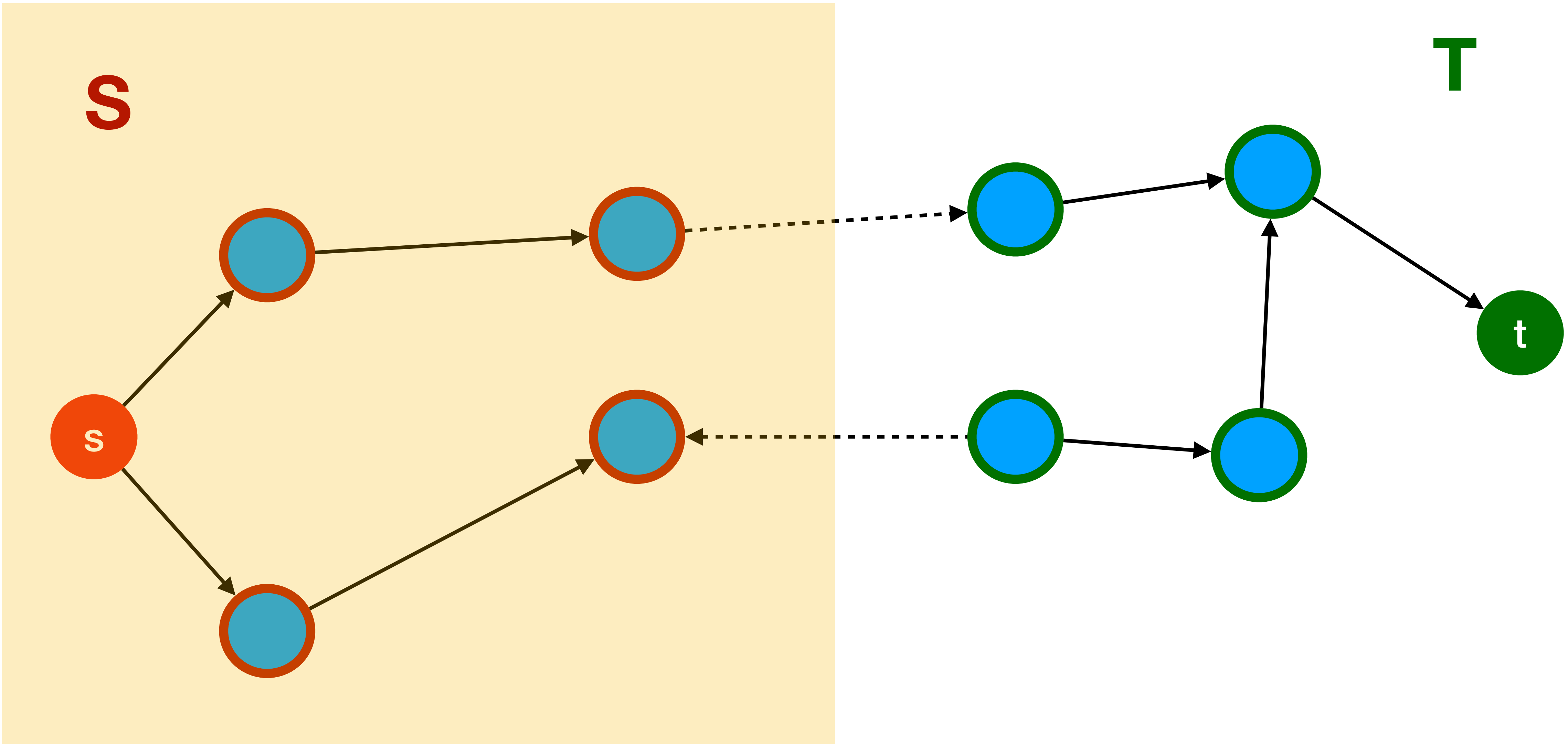

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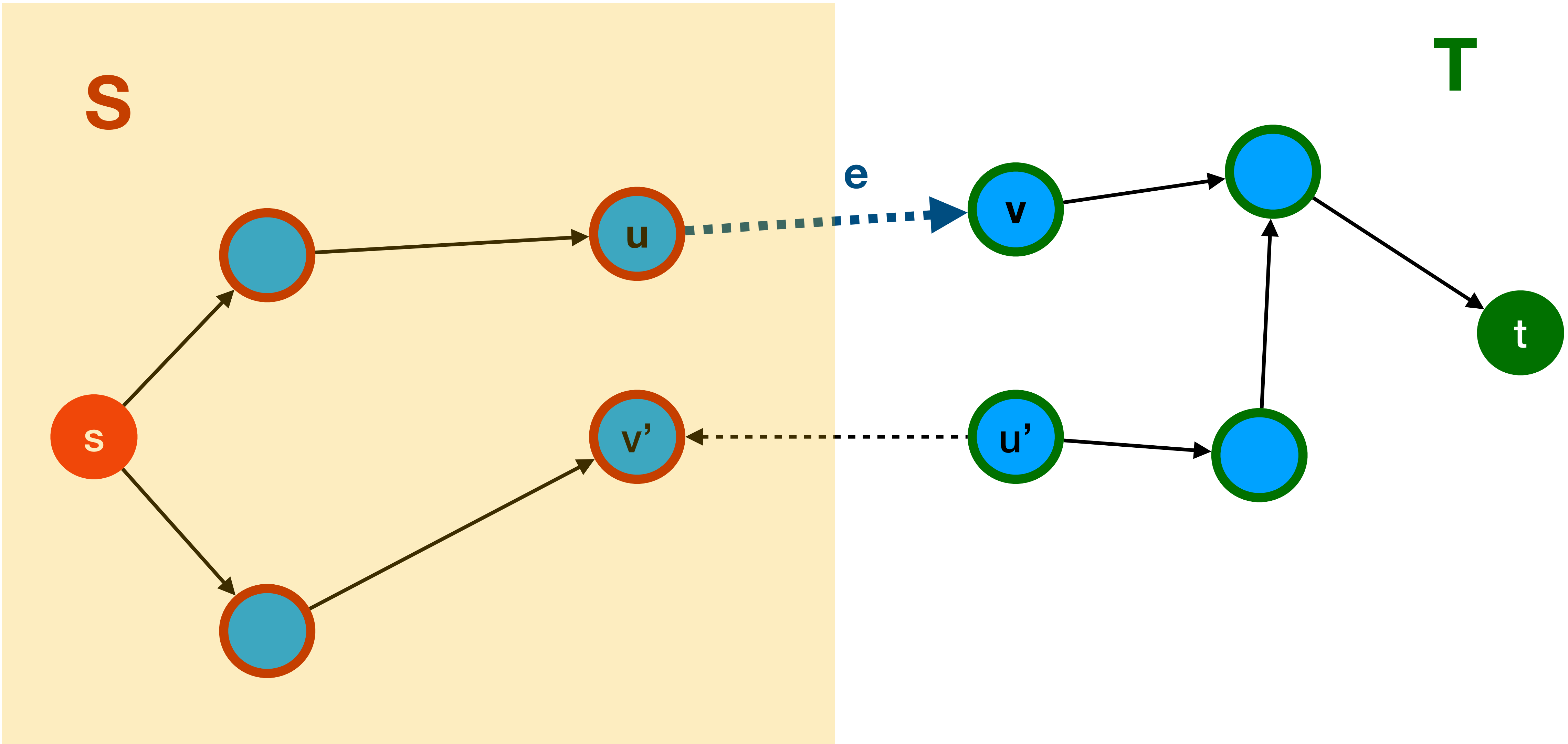

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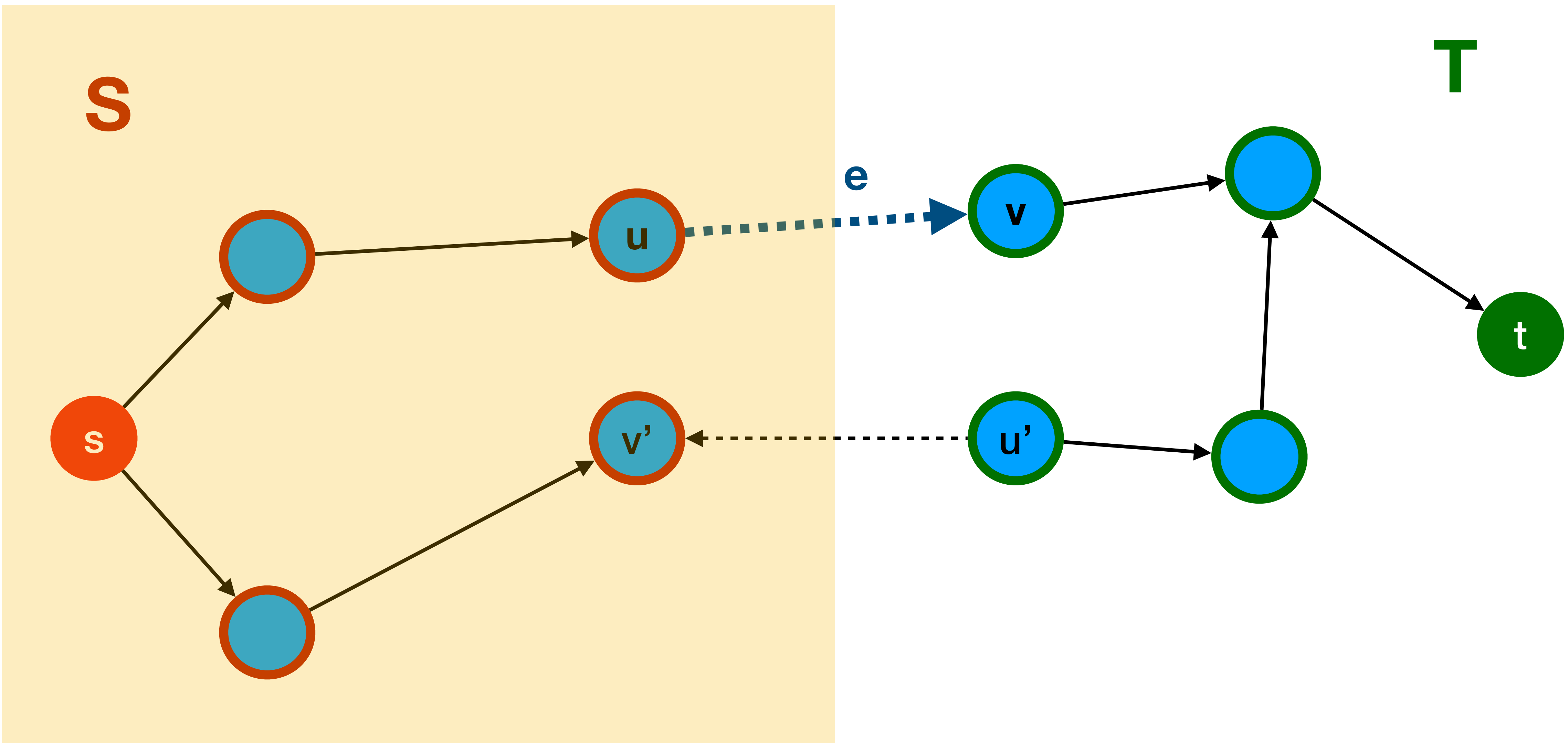

Claim: **in G ,** f(e) = c_e (i.e., e **in G** is *saturated* by the flow f).

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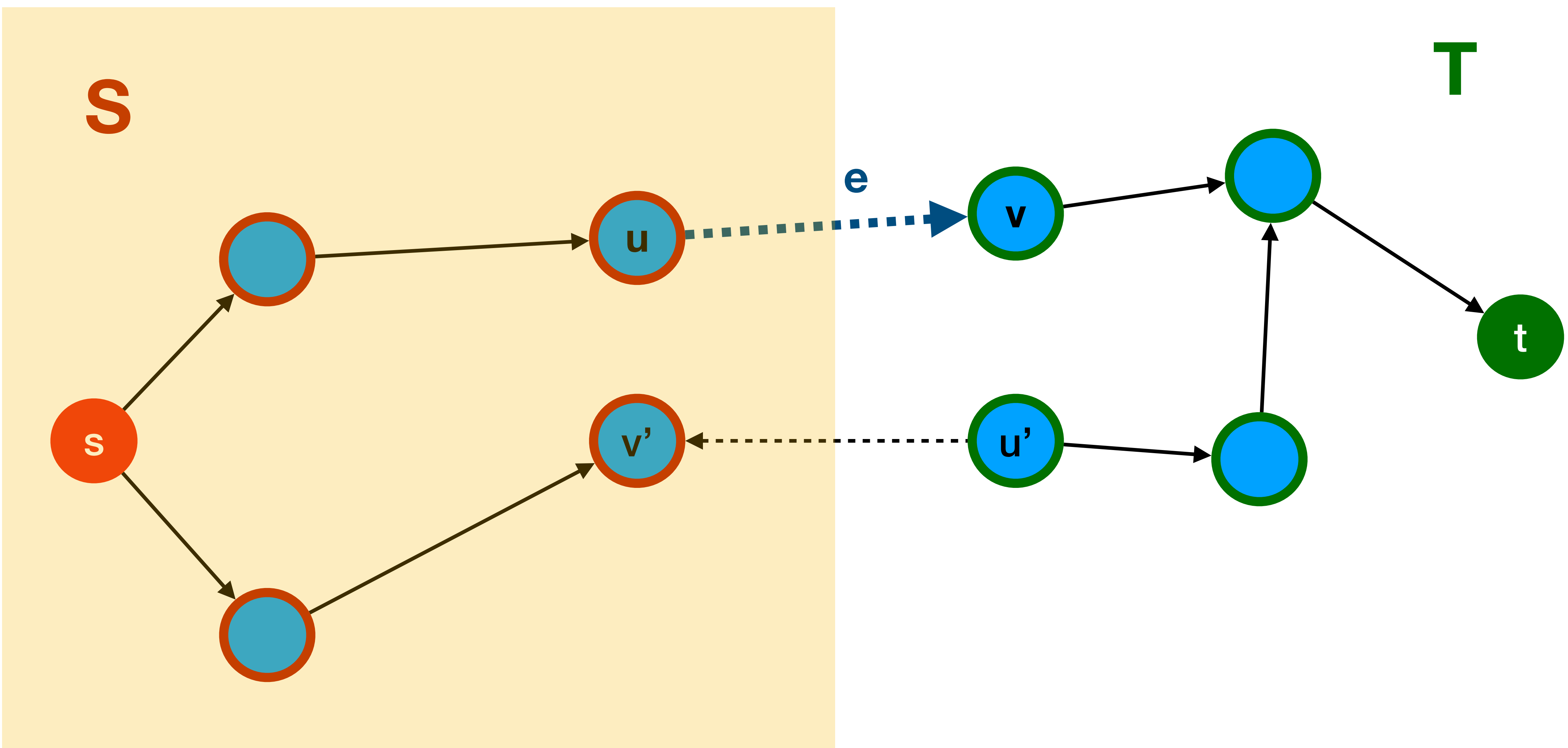

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If not, e would be a *forward edge* in G_f.

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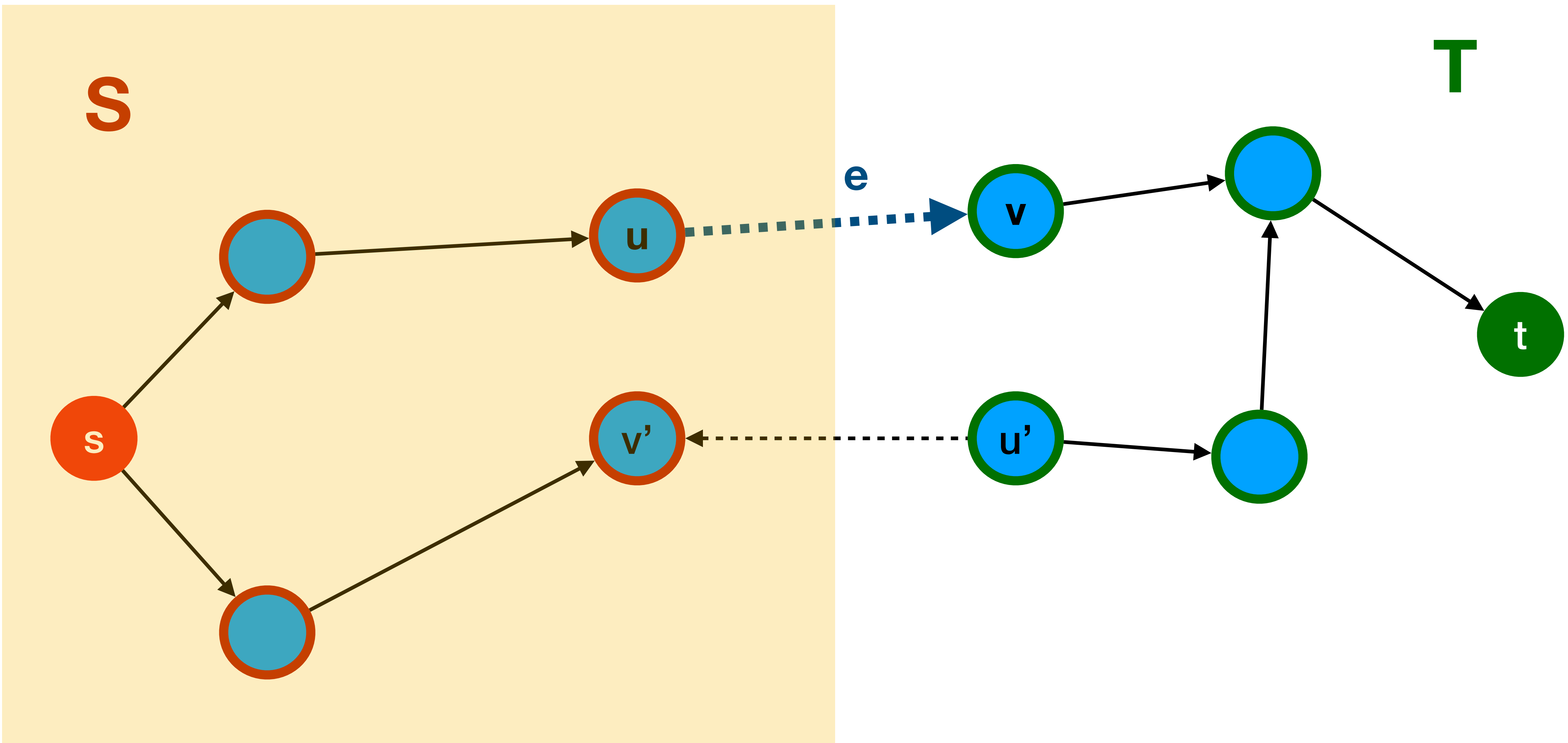

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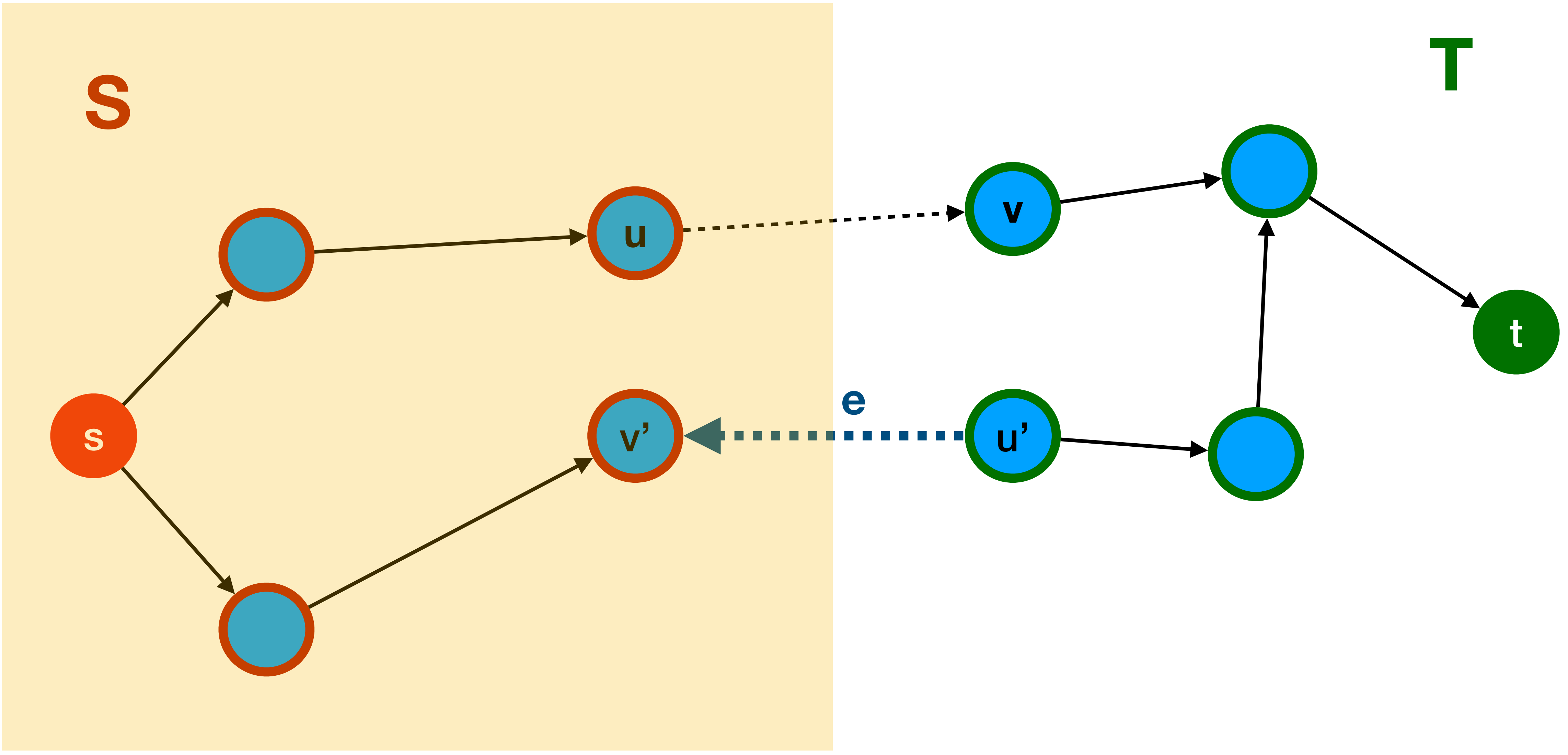

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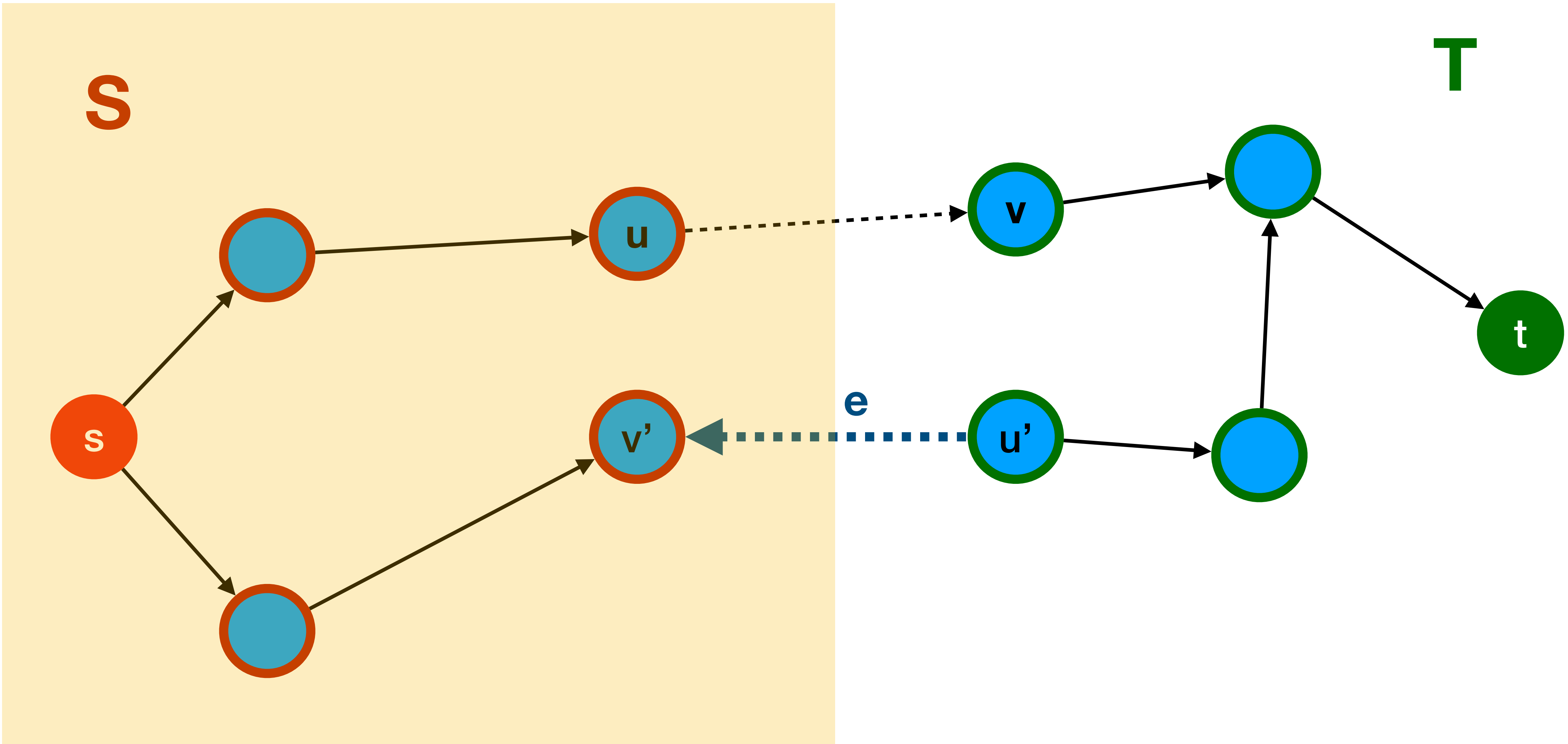

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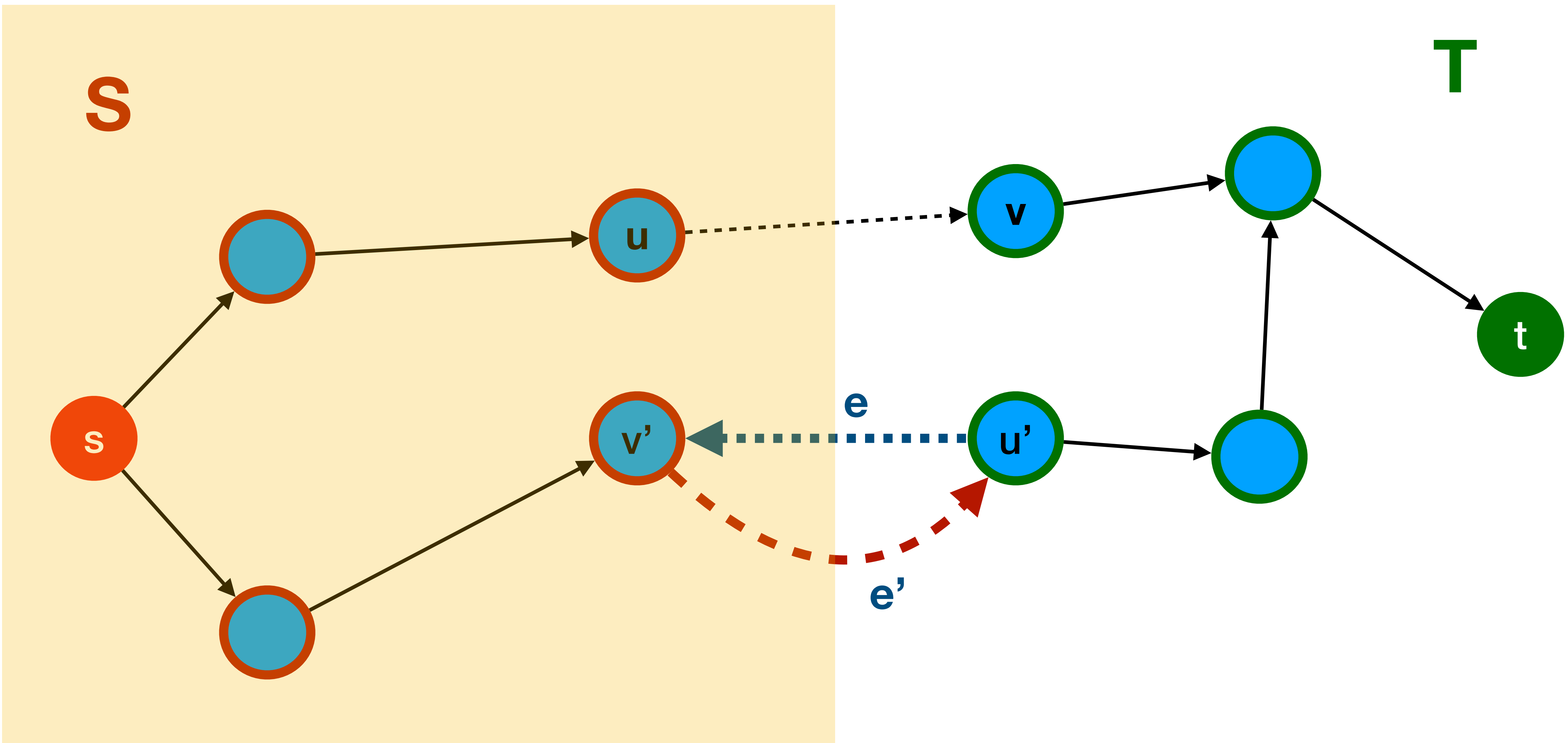

Claim: **in G ,** f(e) = 0.

If not, e would generate a *backward edge e’* in G_f.

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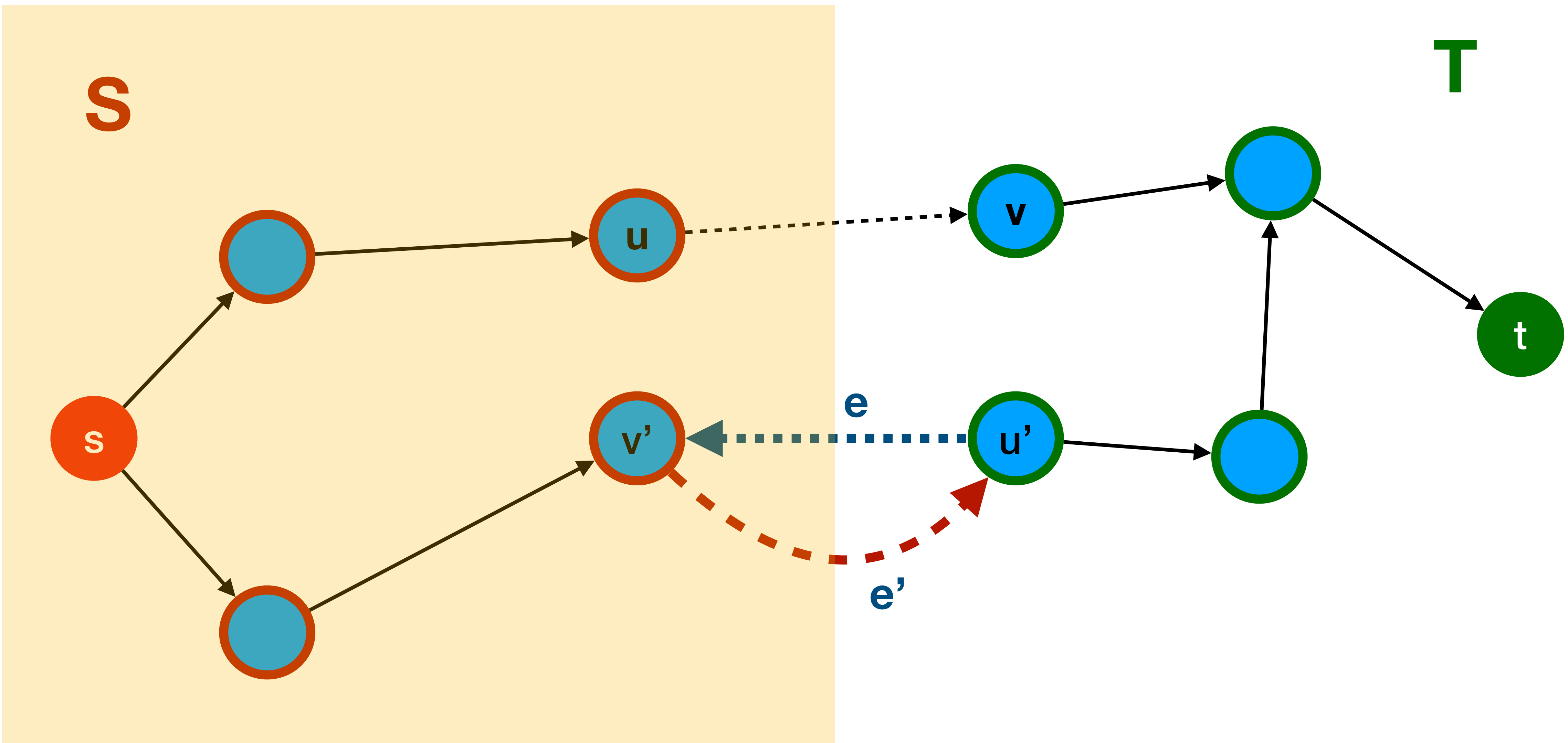

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There would exist a path (s, u').

Proving Fact 4

What do we get from this?

All edges *out of* S* are *saturated* by f.

All edges *into* S* *have 0 flow* in f.

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Putting everything together

Fact 4: Let f by any (s-t) flow in G such that the residual graph G_f has no *augmenting paths*. Then there is an (s-t) cut C(S*, T*) in G such that c(S*, T*) = v(f).

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Ford-Fulkerson stops when there are no augmenting paths in the residual network.

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Ford-Fulkerson stops when there are no augmenting paths in the residual network.

The value of the flow is equal to the capacity of *some* cut.

This means that the value of the flow is maximum.

Related question

Related question

How do we find *the value of the minimum cut* in a flow network?

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Run Ford-Fulkerson and output the value of the computed flow.

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How do we find *the value of the minimum cut* in a flow network?

Run Ford-Fulkerson and output the value of the computed flow.

How do we find *a minimum cut* in a flow network?

Related question

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Put the nodes reachable from s to S and the remaining nodes to T.

The Max-Flow Min-Cut Theorem

Theorem: In every flow network, the value of the maximum flow is *equal* to the capacity of the minimum cut.

The proof of the theorem follows from the proof of optimality for Ford-Fulkerson!