Algorithms and Data Structures

Modelling with Flows

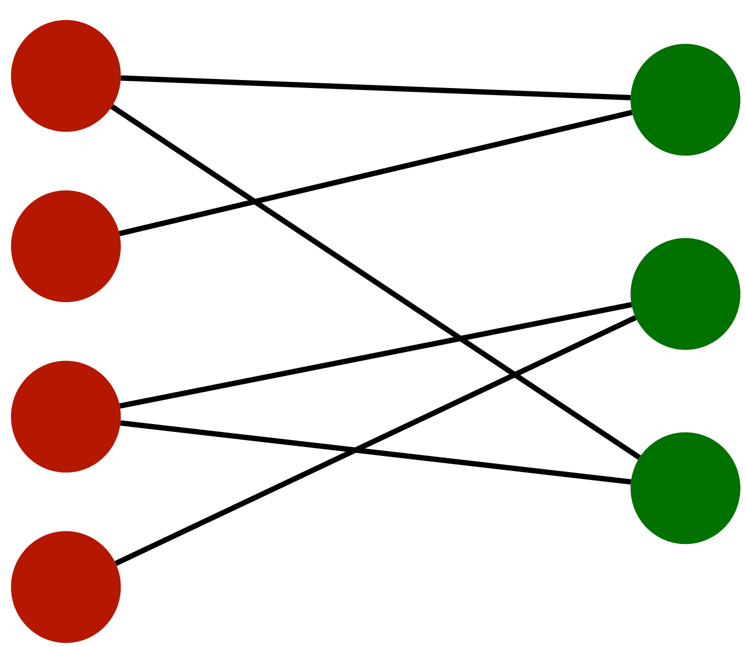
Bipartite Matching

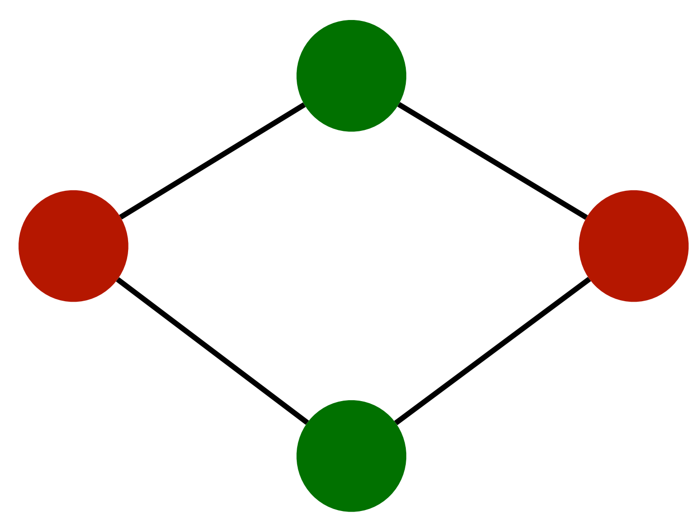
Maximum Bipartite Matching or Maximum matching on a bipartite graph G.

Bipartite graphs

A graph G=(V,E) is bipartite *if any only if* it can be partitioned into sets A and B such that each edge has one endpoint in A and one endpoint in B.

Often, we write G=(L,R,E).





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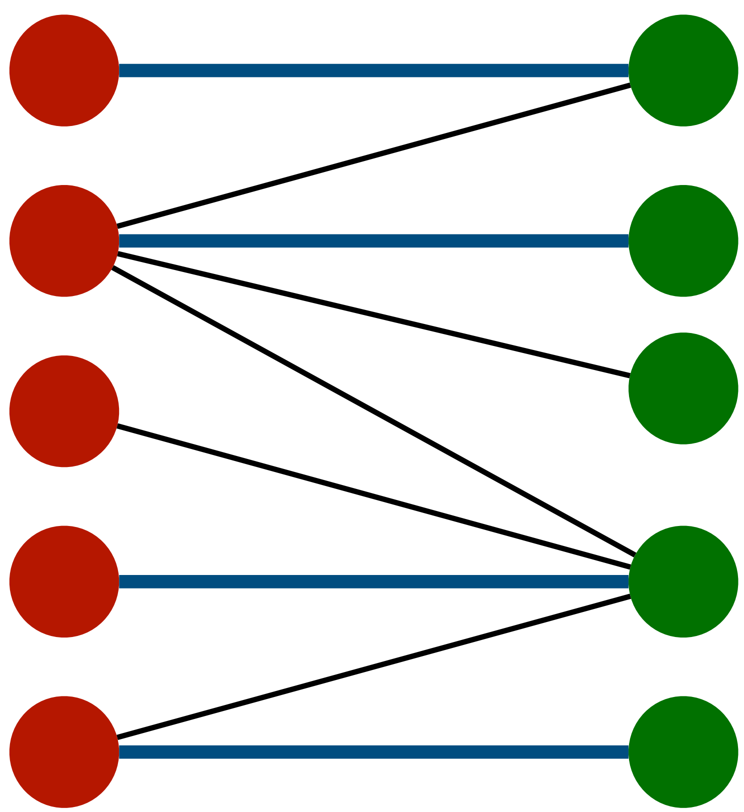
Maximum Bipartite Matching or Maximum matching on a bipartite graph G.

Matching: A subset M of the edges E such that each node v of V appears in at most one edge e in E.

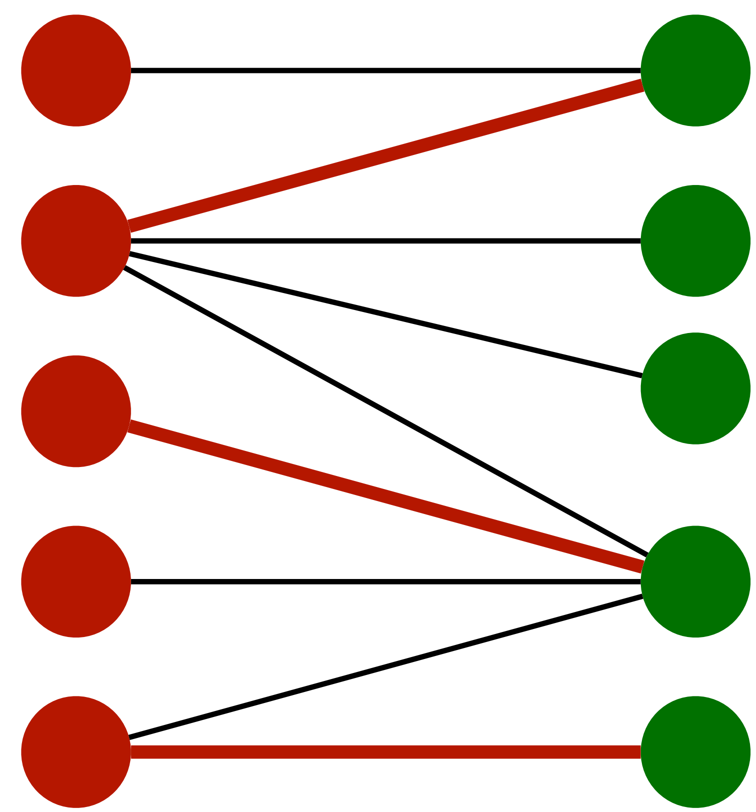
Maximum matching: A matching with maximum cardinality. (i.e., |M| is maximised).

Example

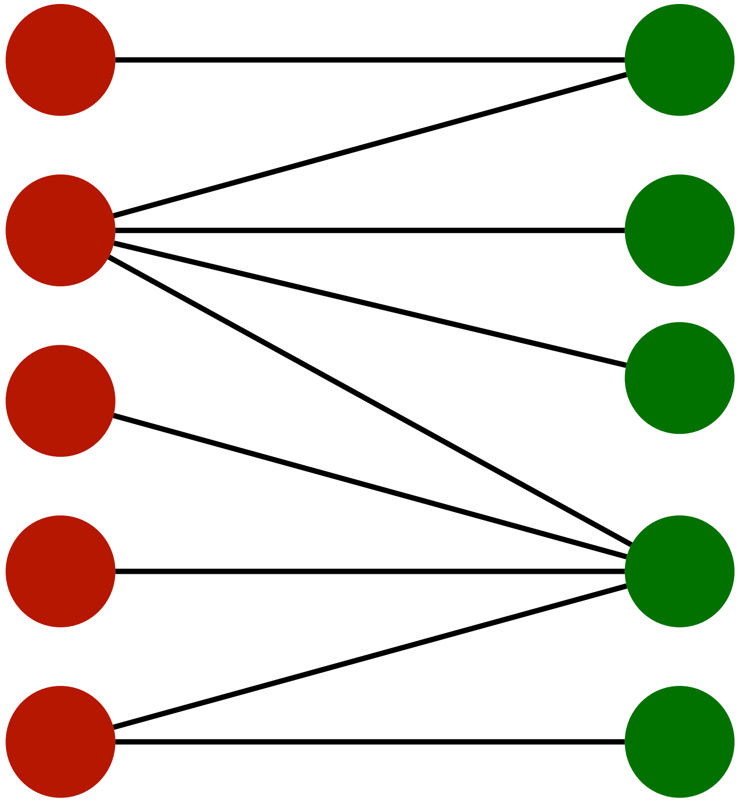
A maximum matching



A maximal matching



From matchings to flows



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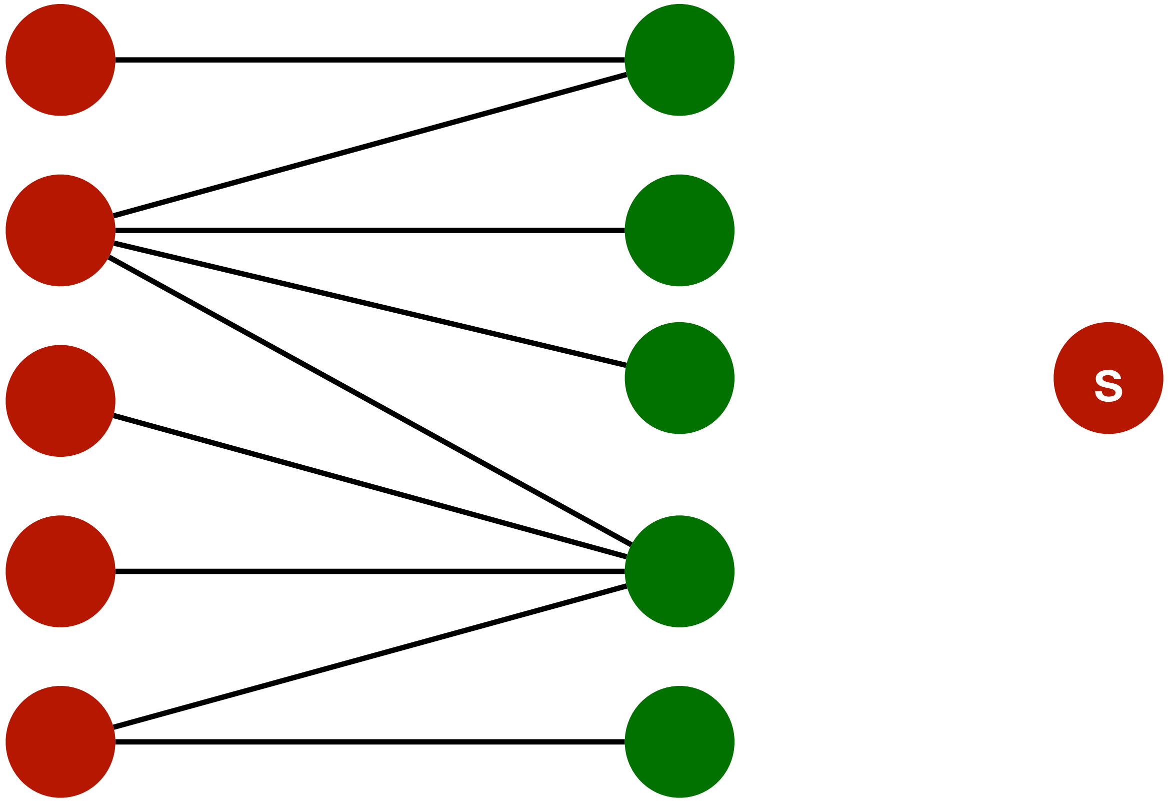

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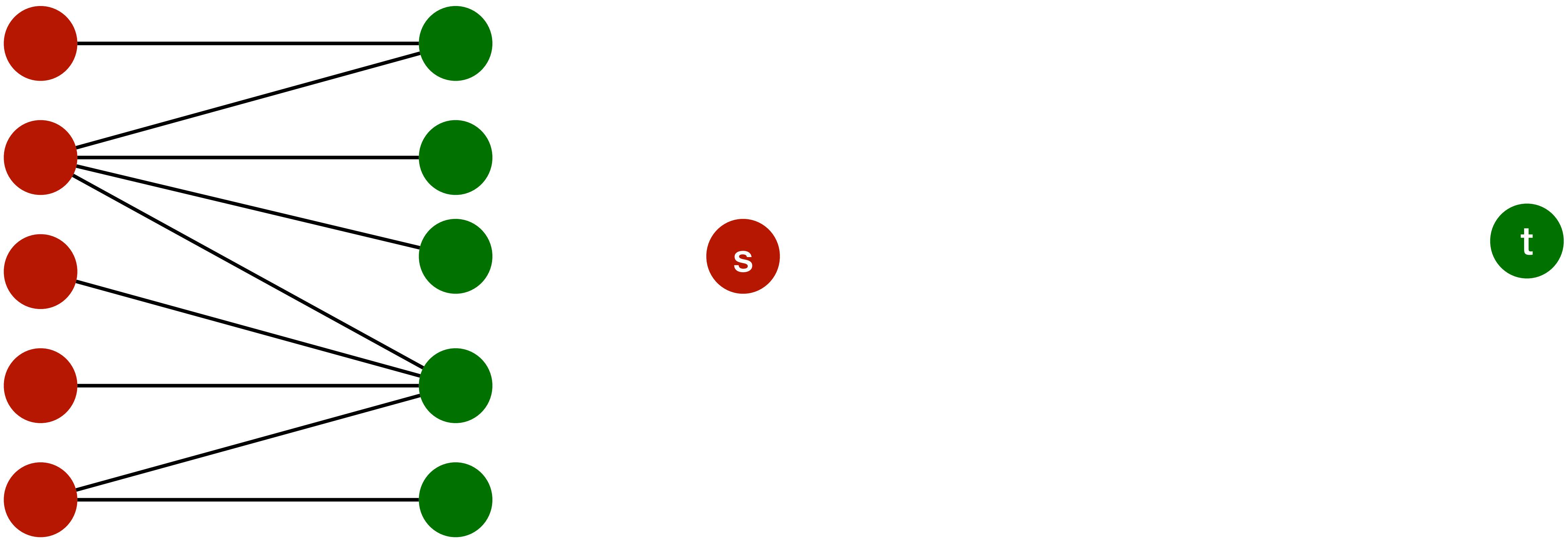

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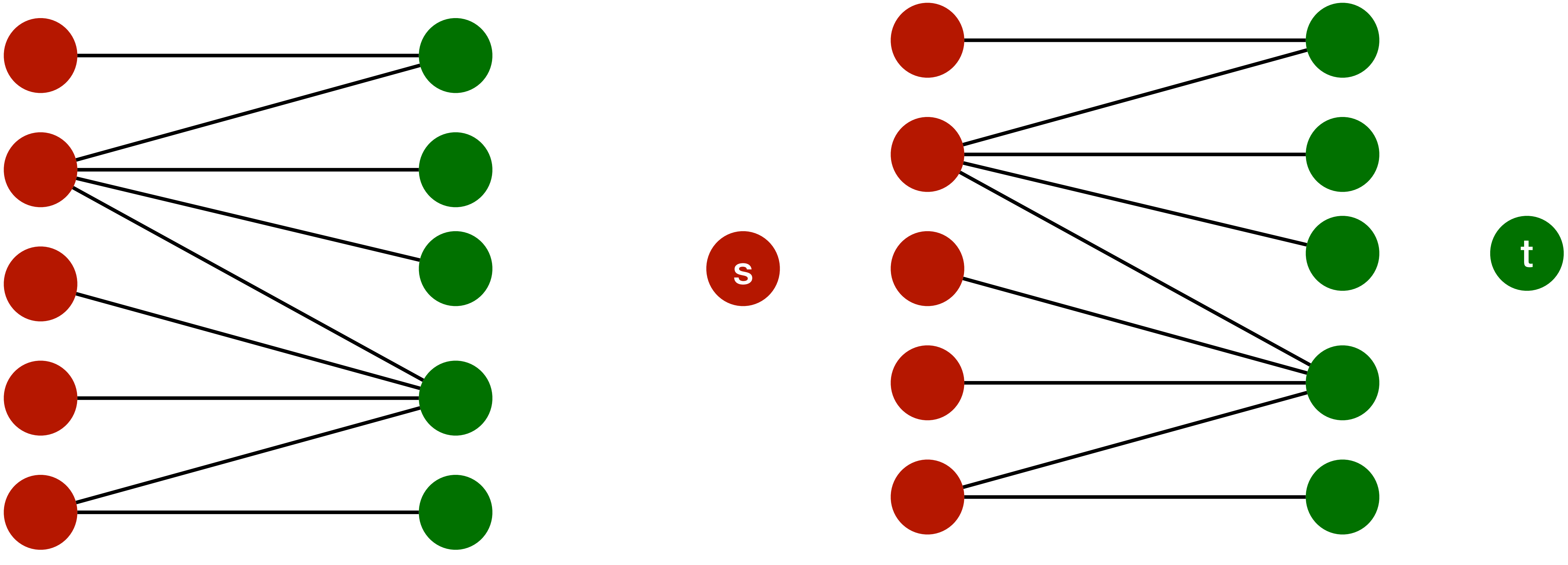

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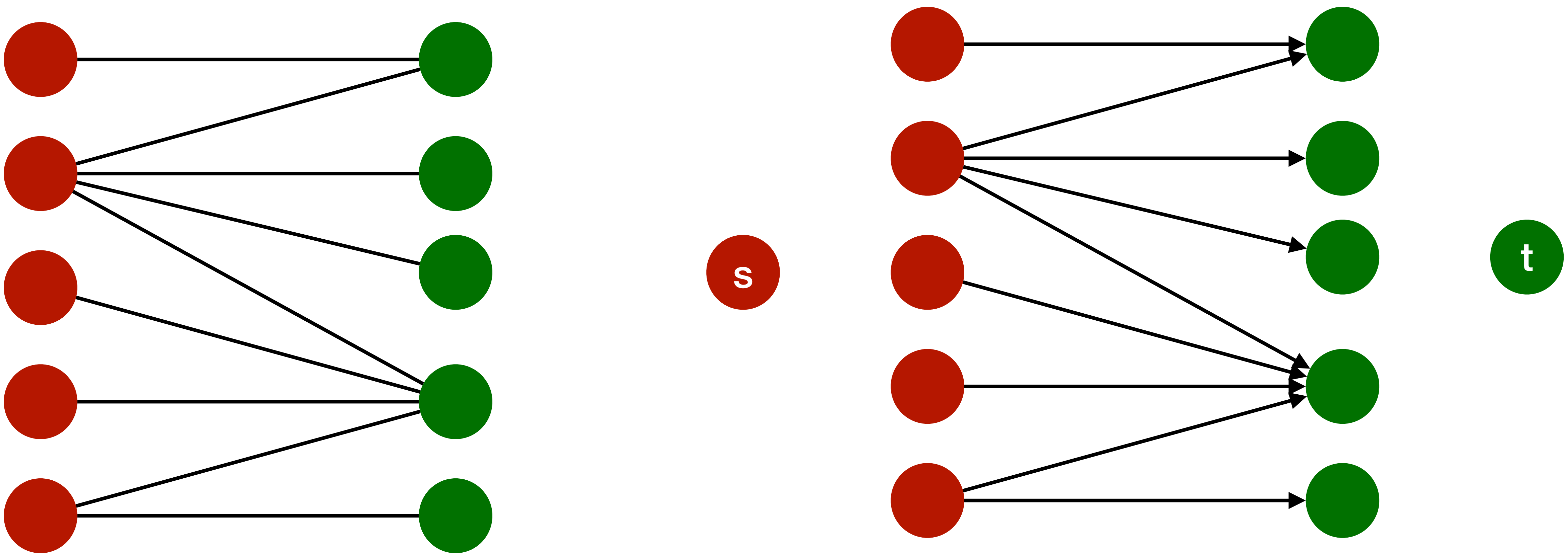

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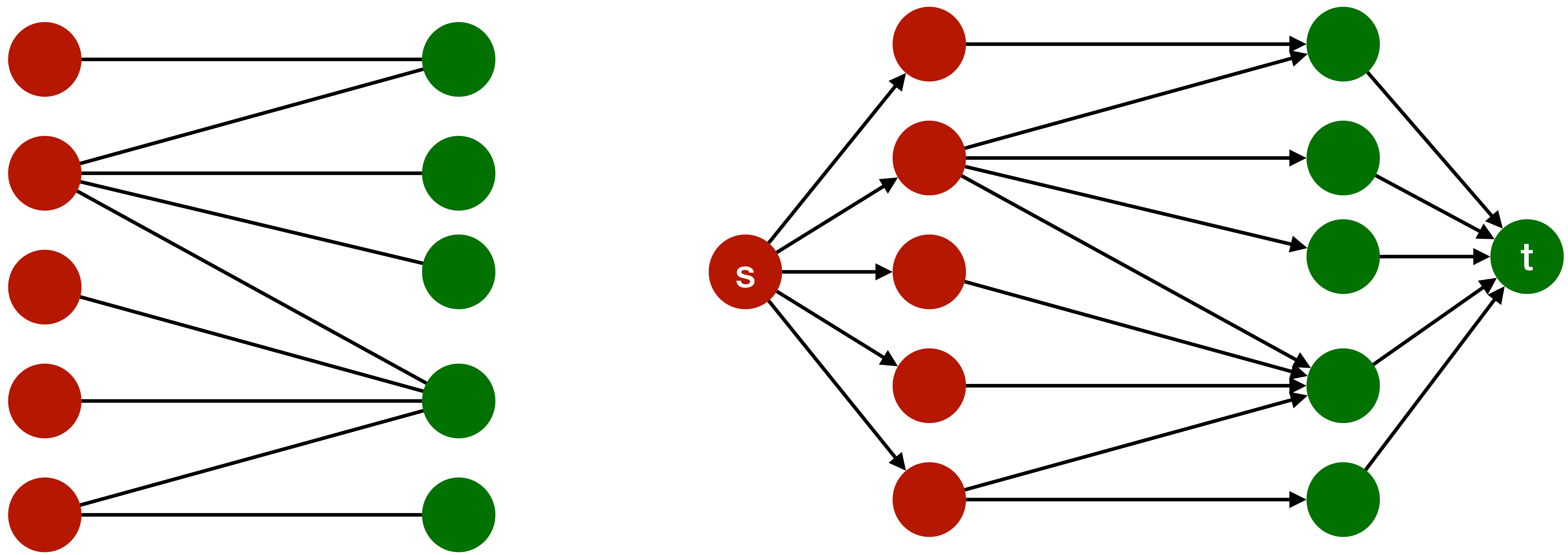

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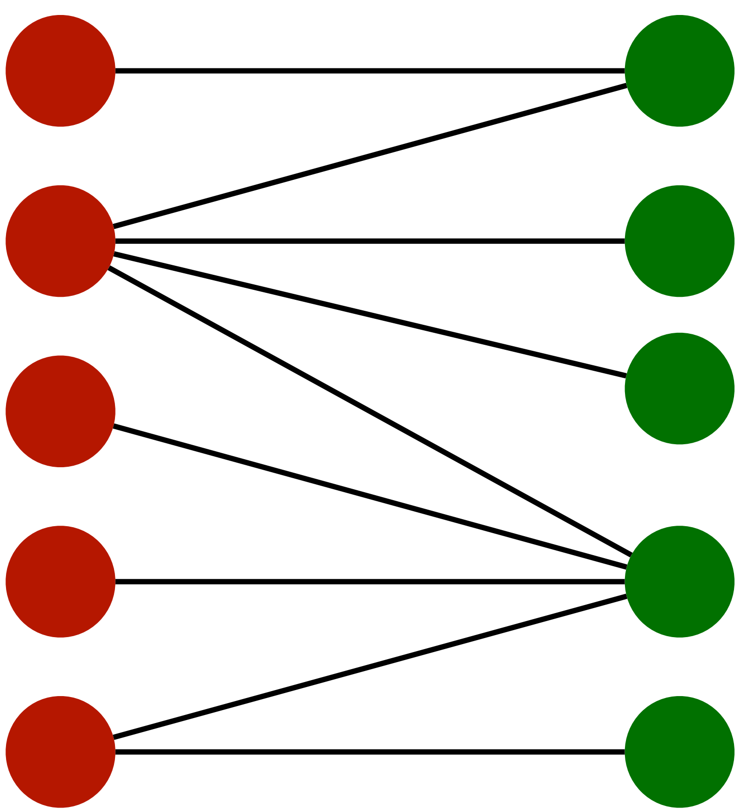

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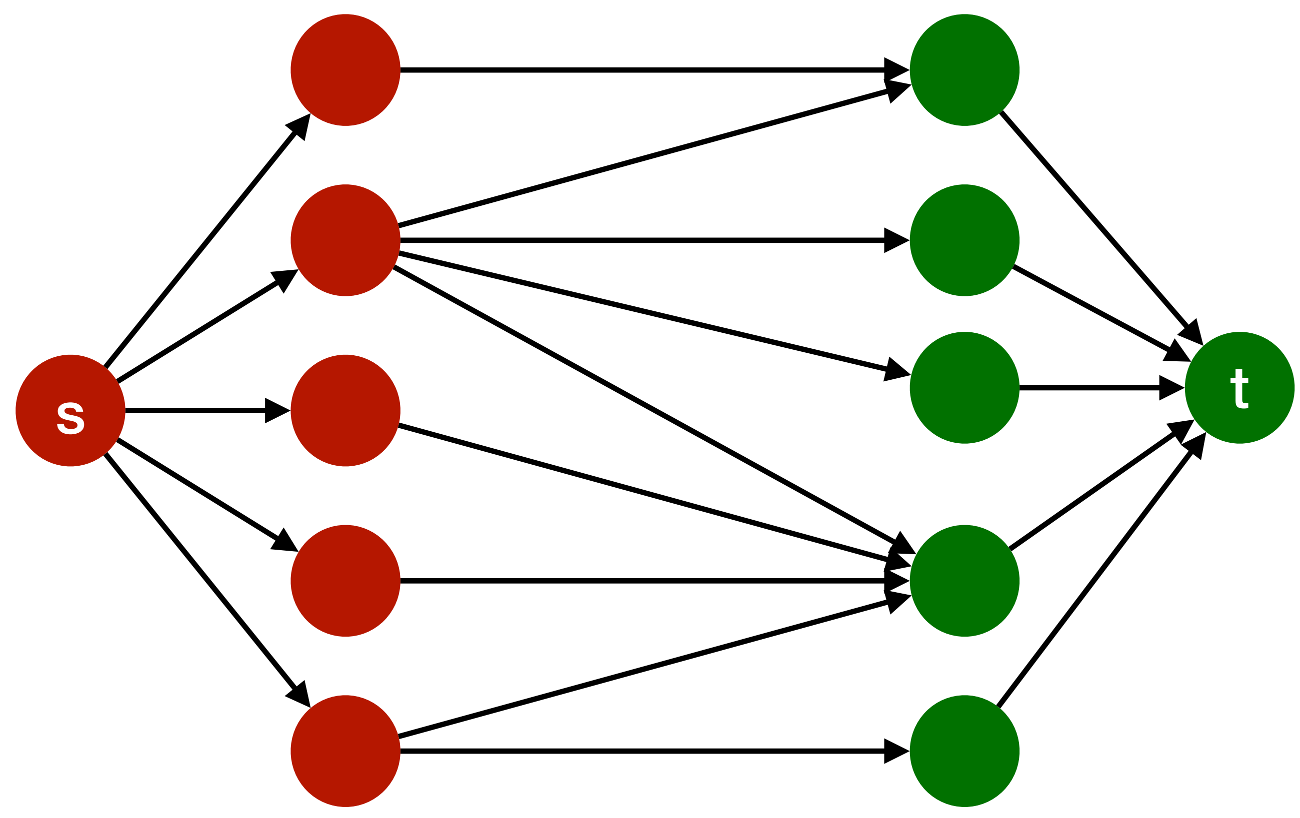

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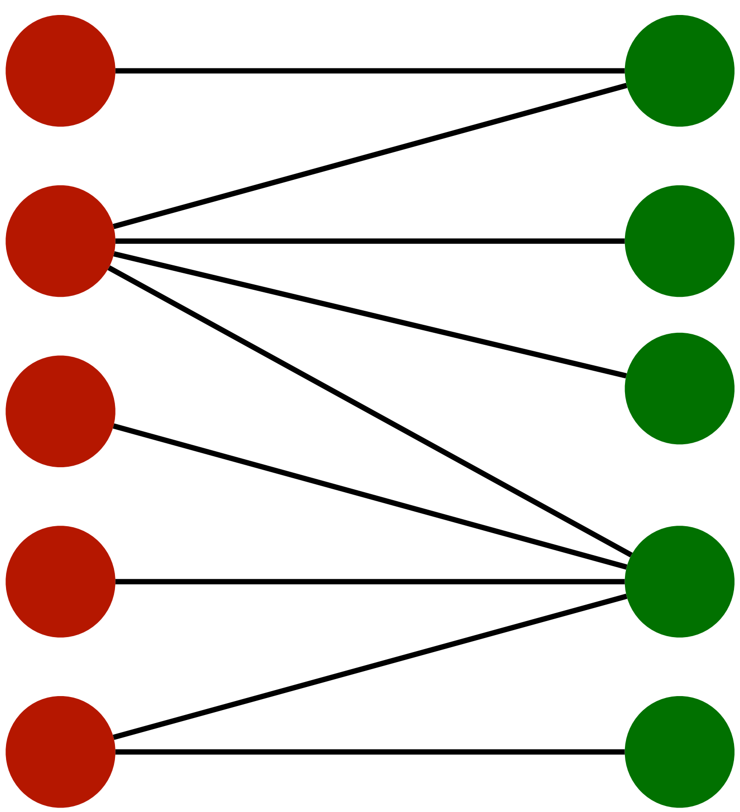

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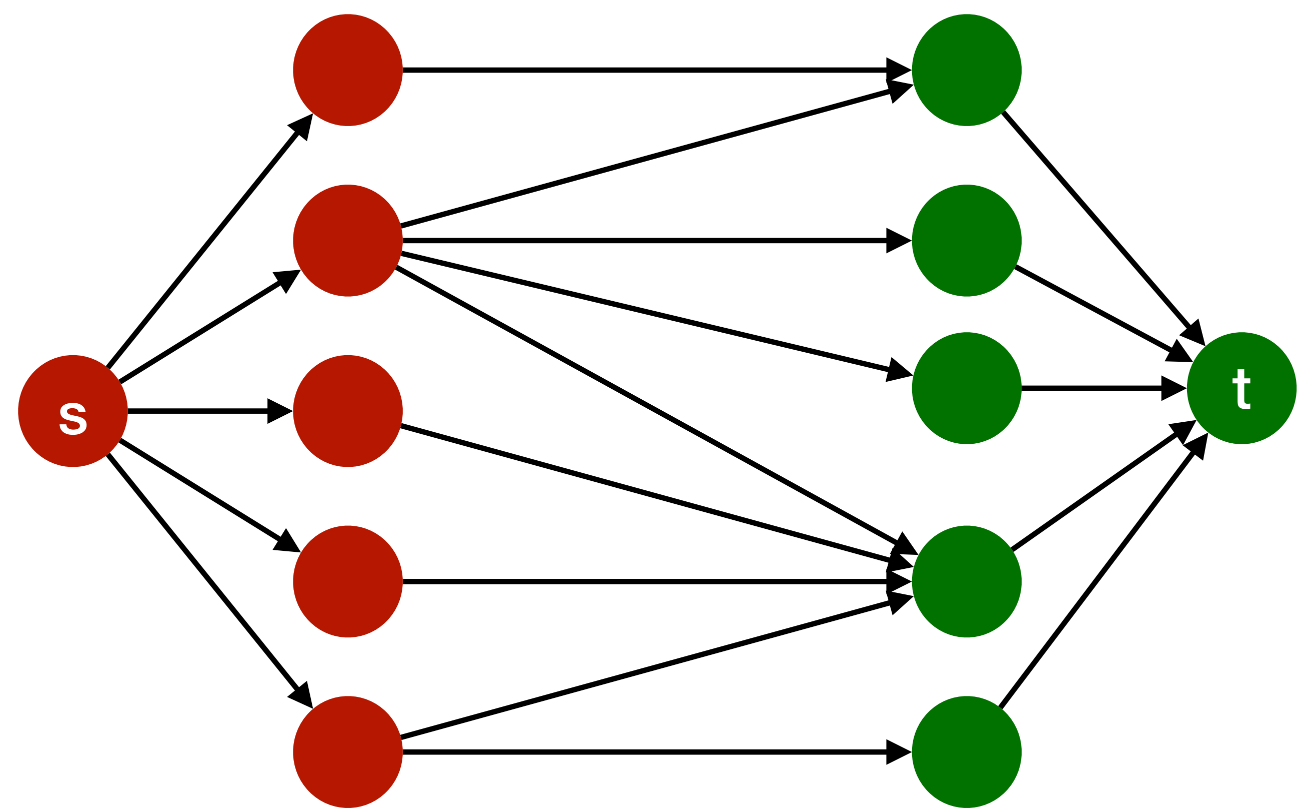

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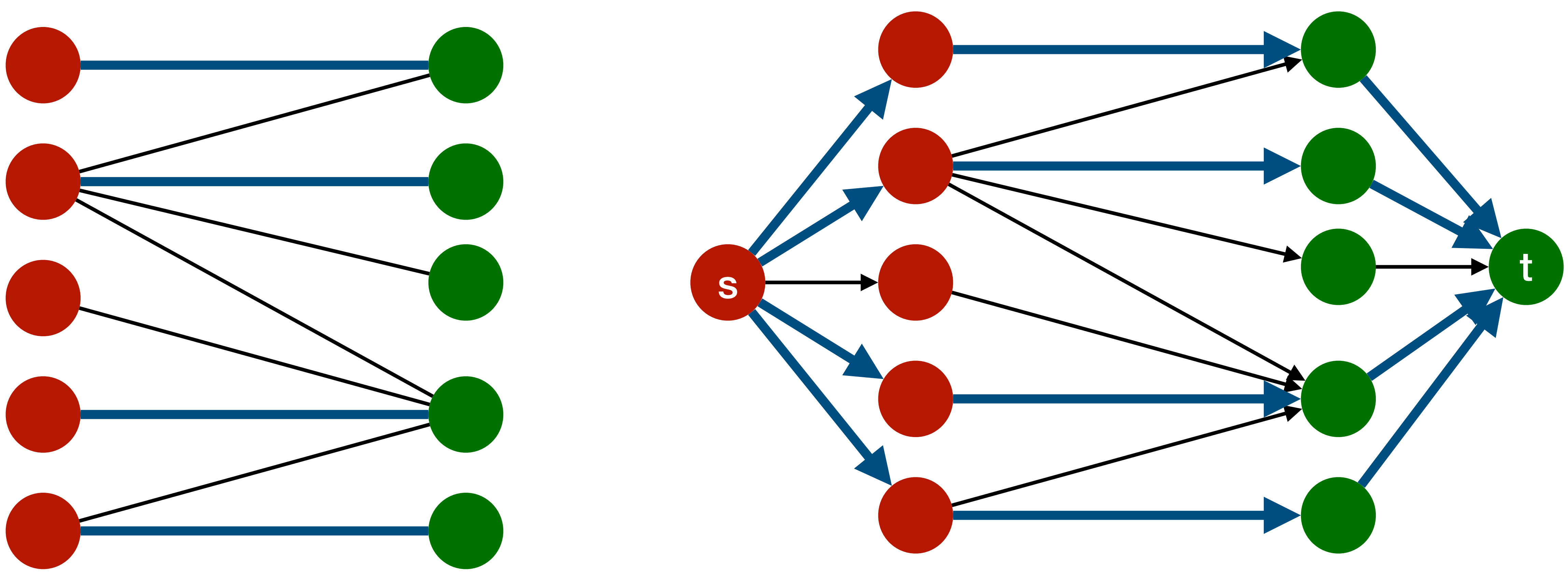

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Consider the set M' of edges (of the middle level) with f(e) = *1*.

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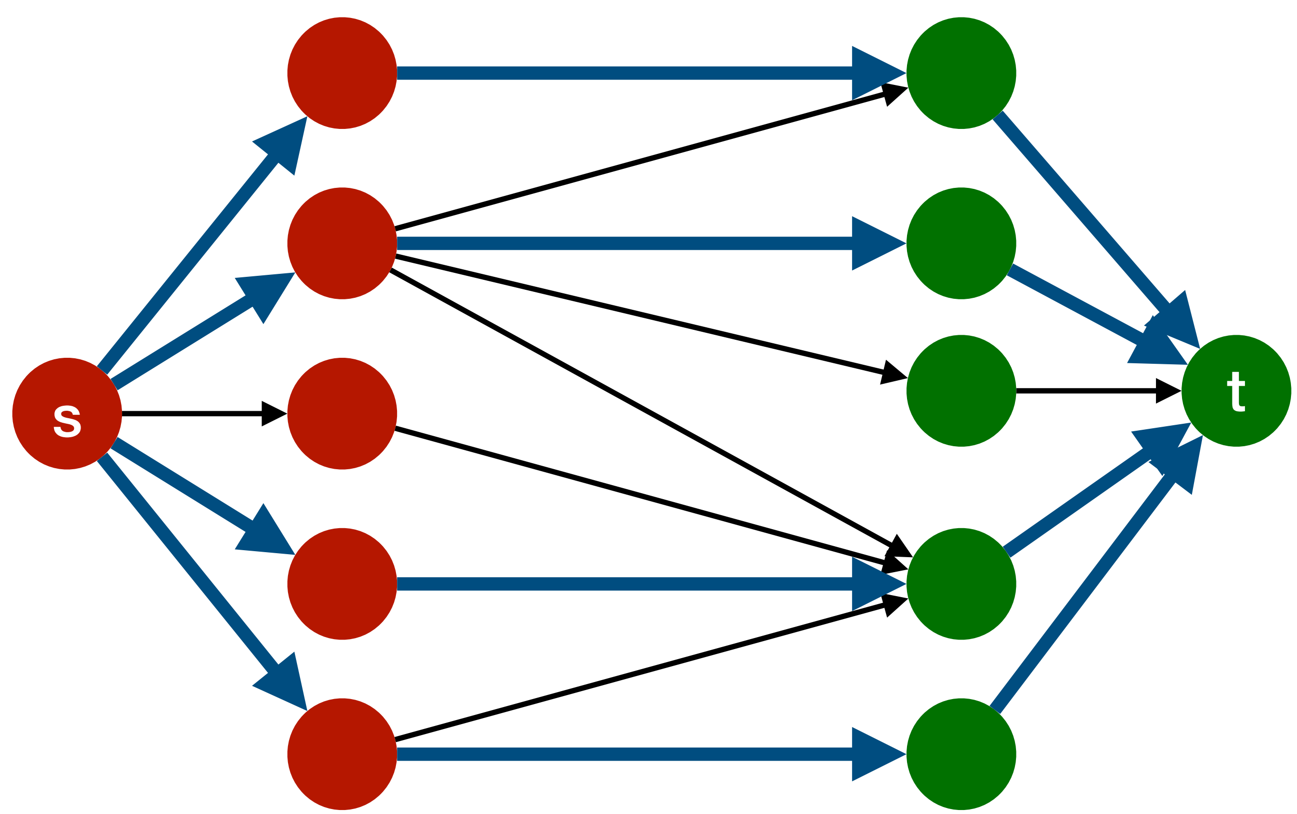

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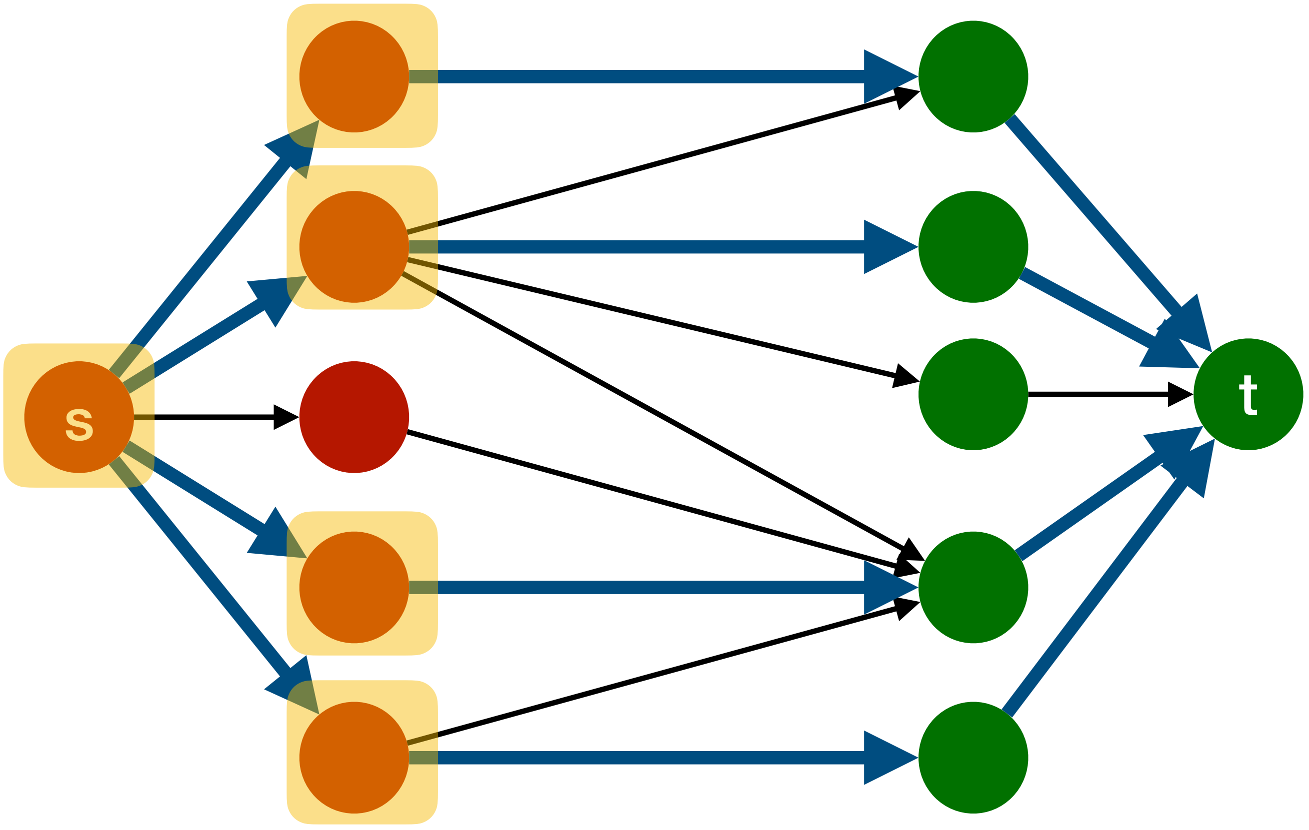

From flows to matchings

Consider the set M’ of edges with $f(e) = 1$.

Claim: M’ is a matching.

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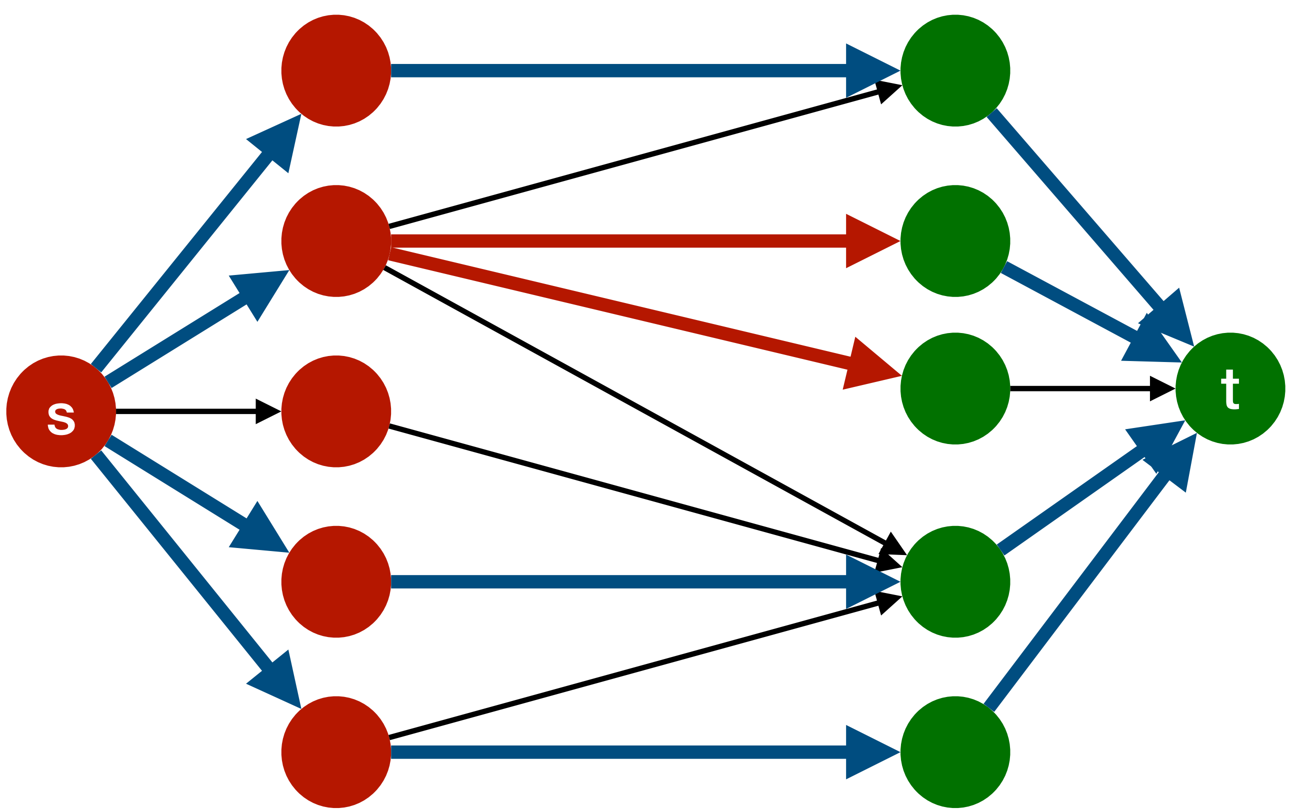

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What was the crucial part, that allows us to establish this?

The integrality theorem.

Running time

What is the running time of the algorithm?

By Edmonds - Karp, we get **O(**nm^2**).**

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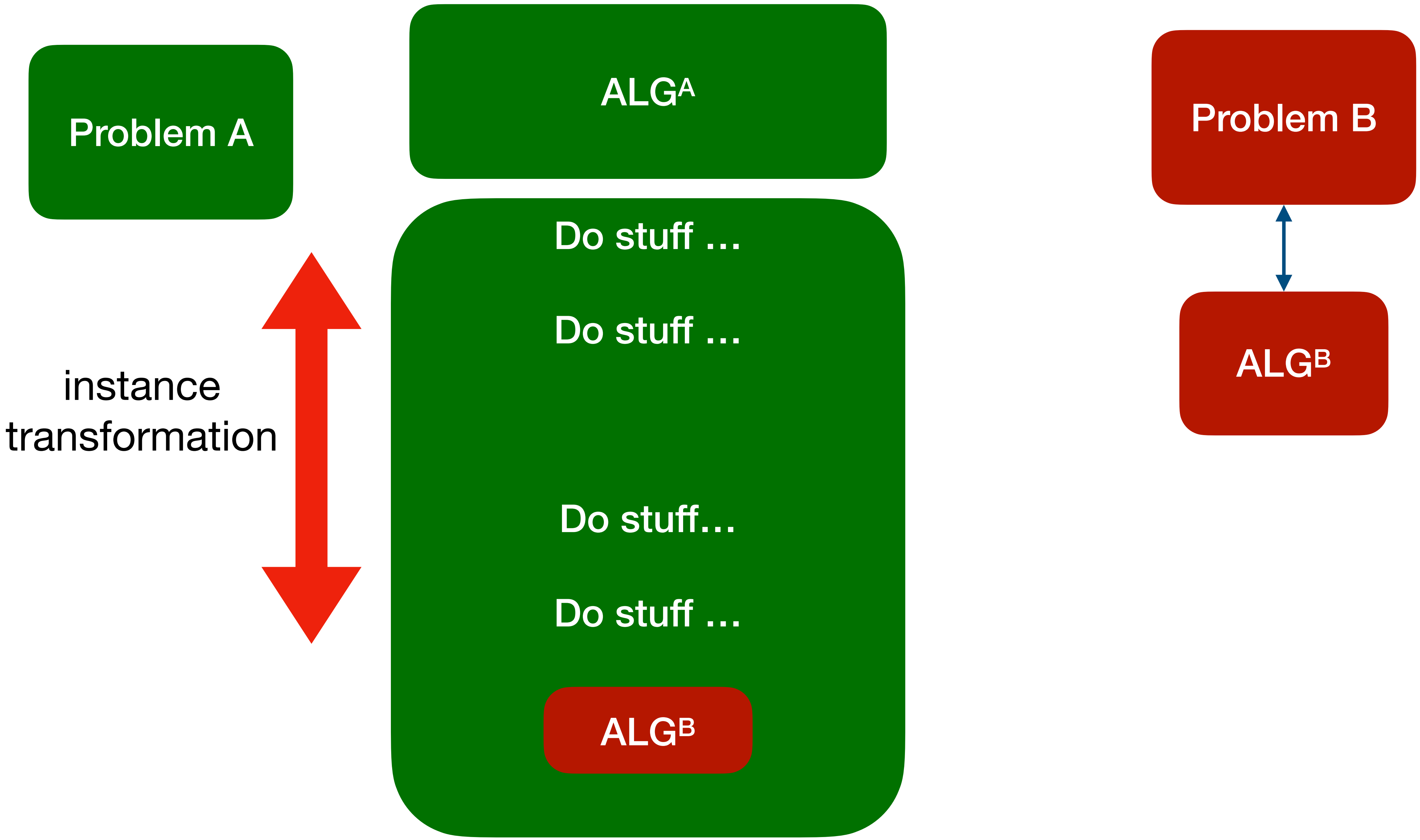
Pictorially



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Baseball Elimination

In the baseball league, there are 4 teams with the following number of wins:

New York	92
Baltimore	91
Toronto	91
Boston	90

There are five games left in the season.

NY vs BLT, NY vs TOR, BLT vs TOR, BLT vs BOS, TOR vs BOS

Question: Can Boston finish (possibly tied for) first?

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The answer is no.

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New York	90
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Toronto	87
Boston	79

These are the games left in the season:

NY vs BLT

NY vs TOR ***6 games***

BLT vs TOR

BOS vs ANY ***4 games (12 games total)***

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Can z win the tournament (possibly in a tie?)

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Is there an allocation of all the remaining g* games (between the other teams) such that no team ends up with more than m wins?

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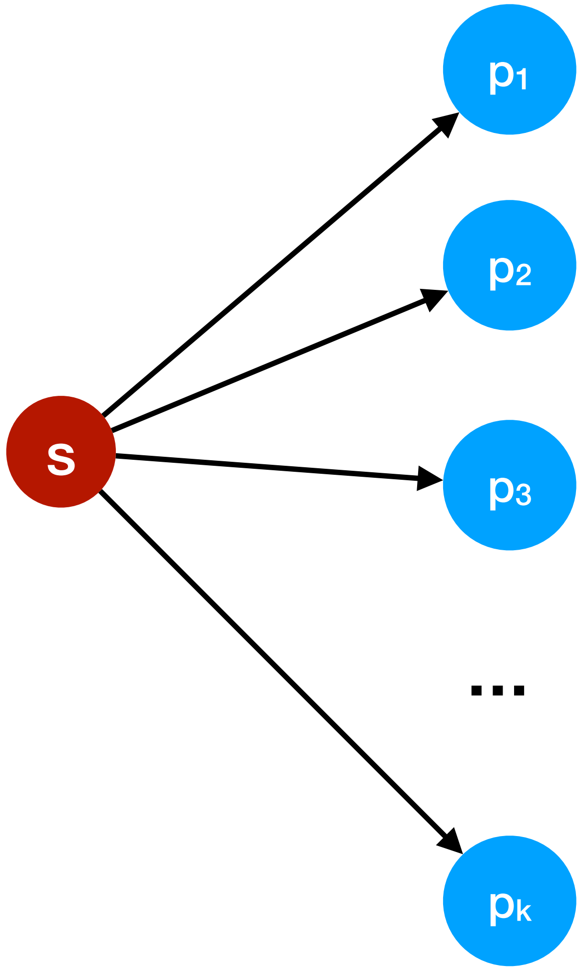

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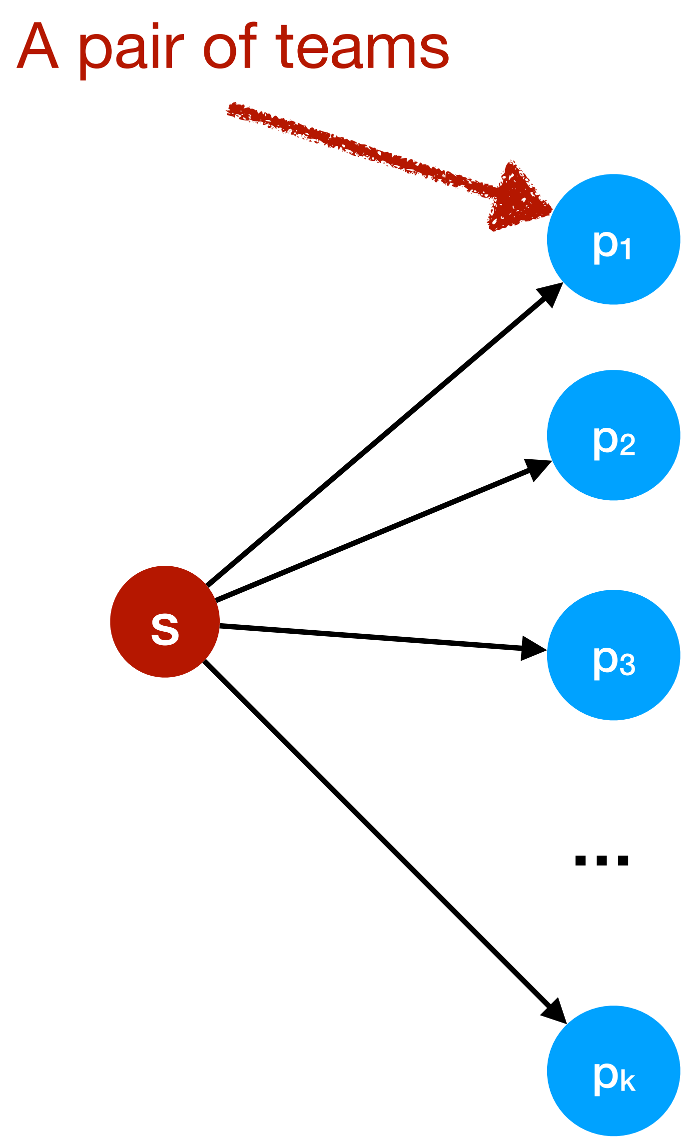

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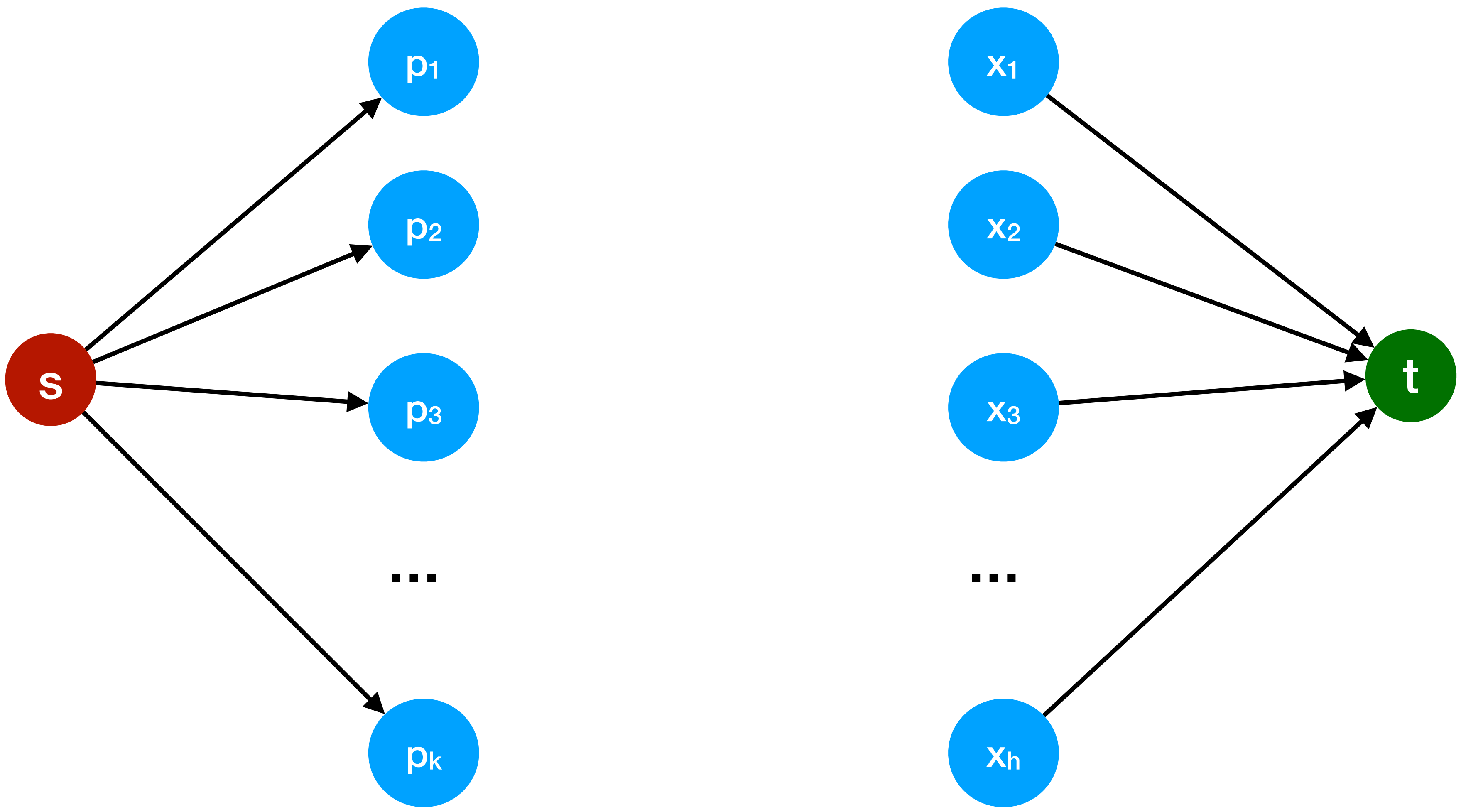

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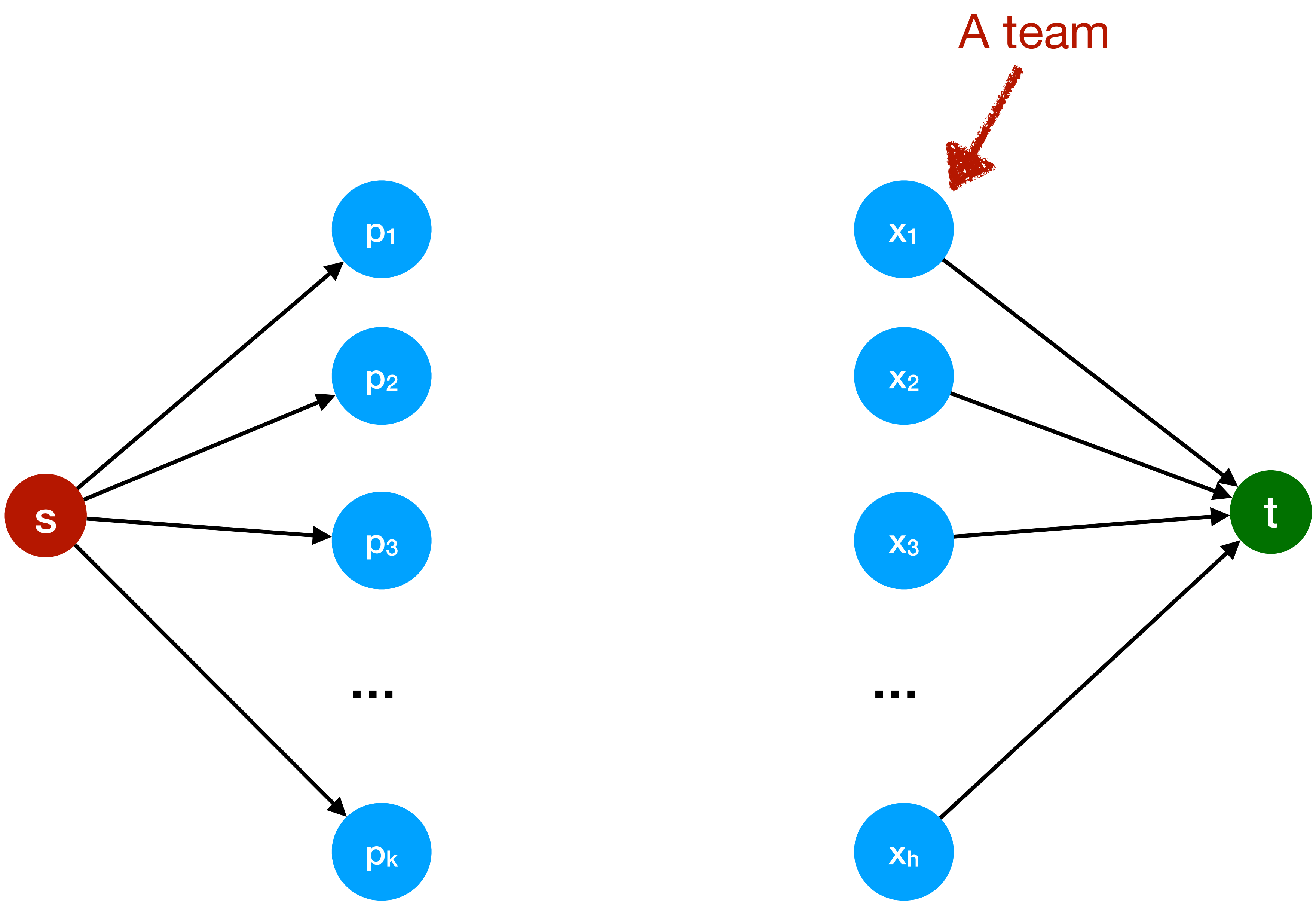

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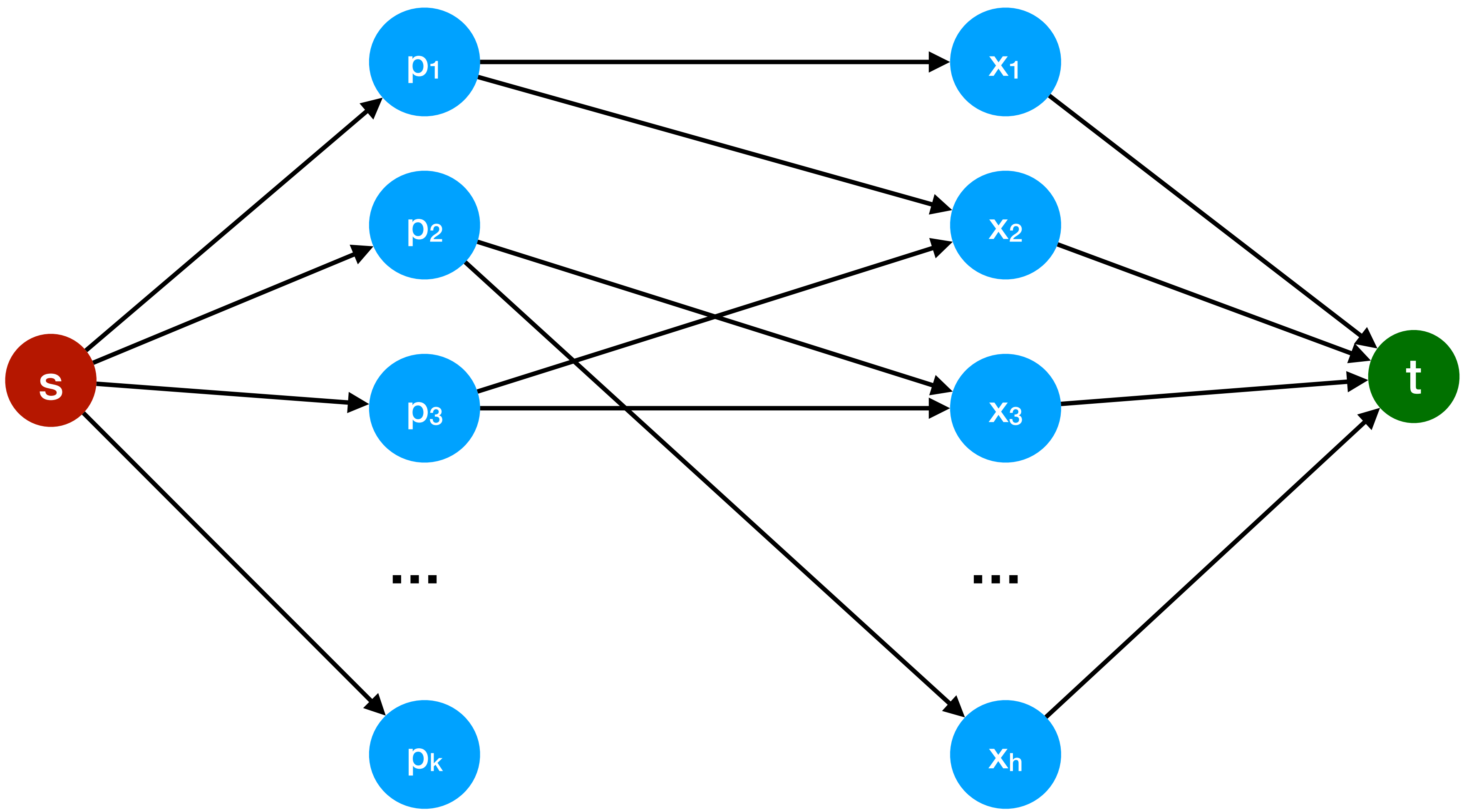

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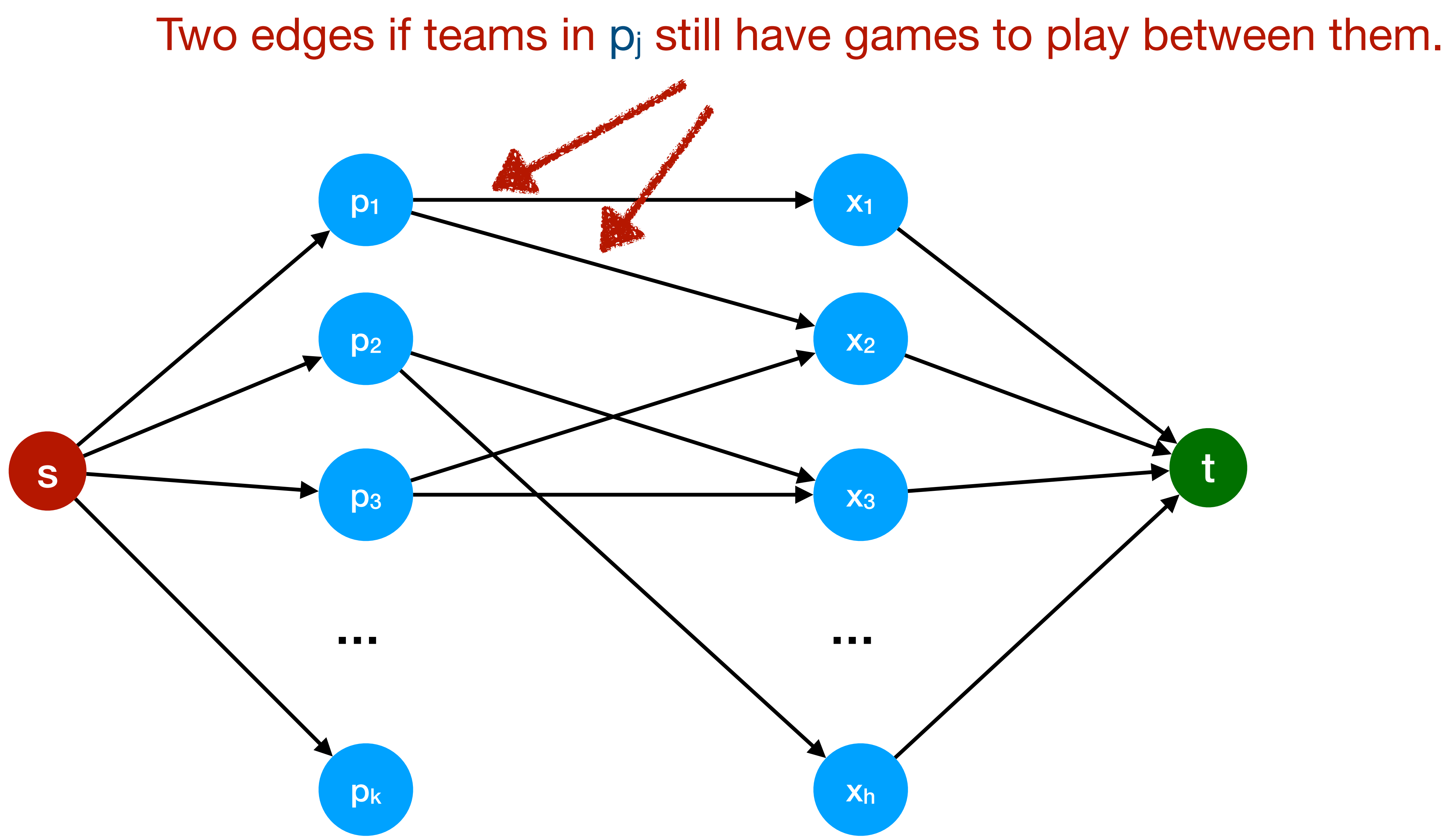

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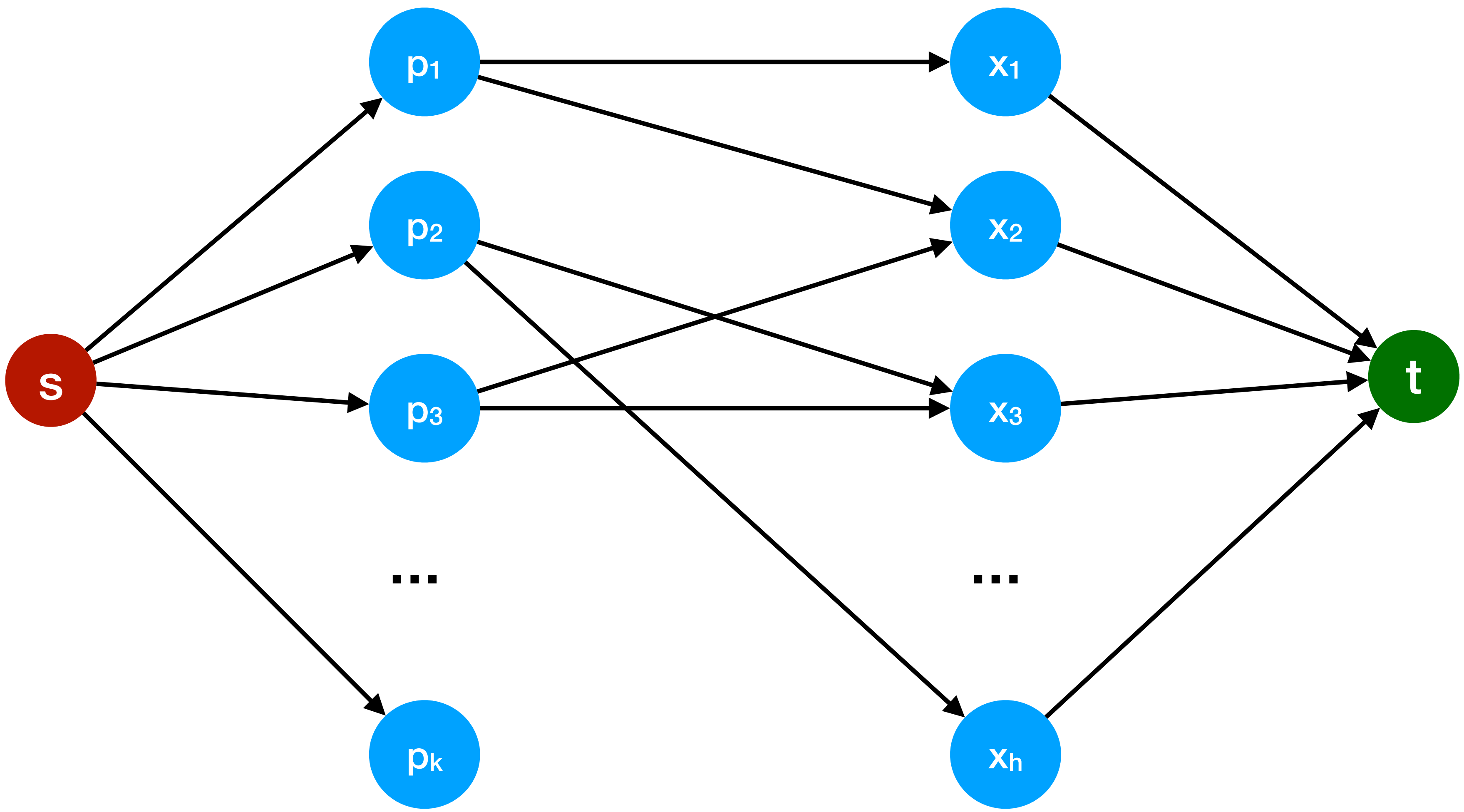
Let $p_j = (x, y)$



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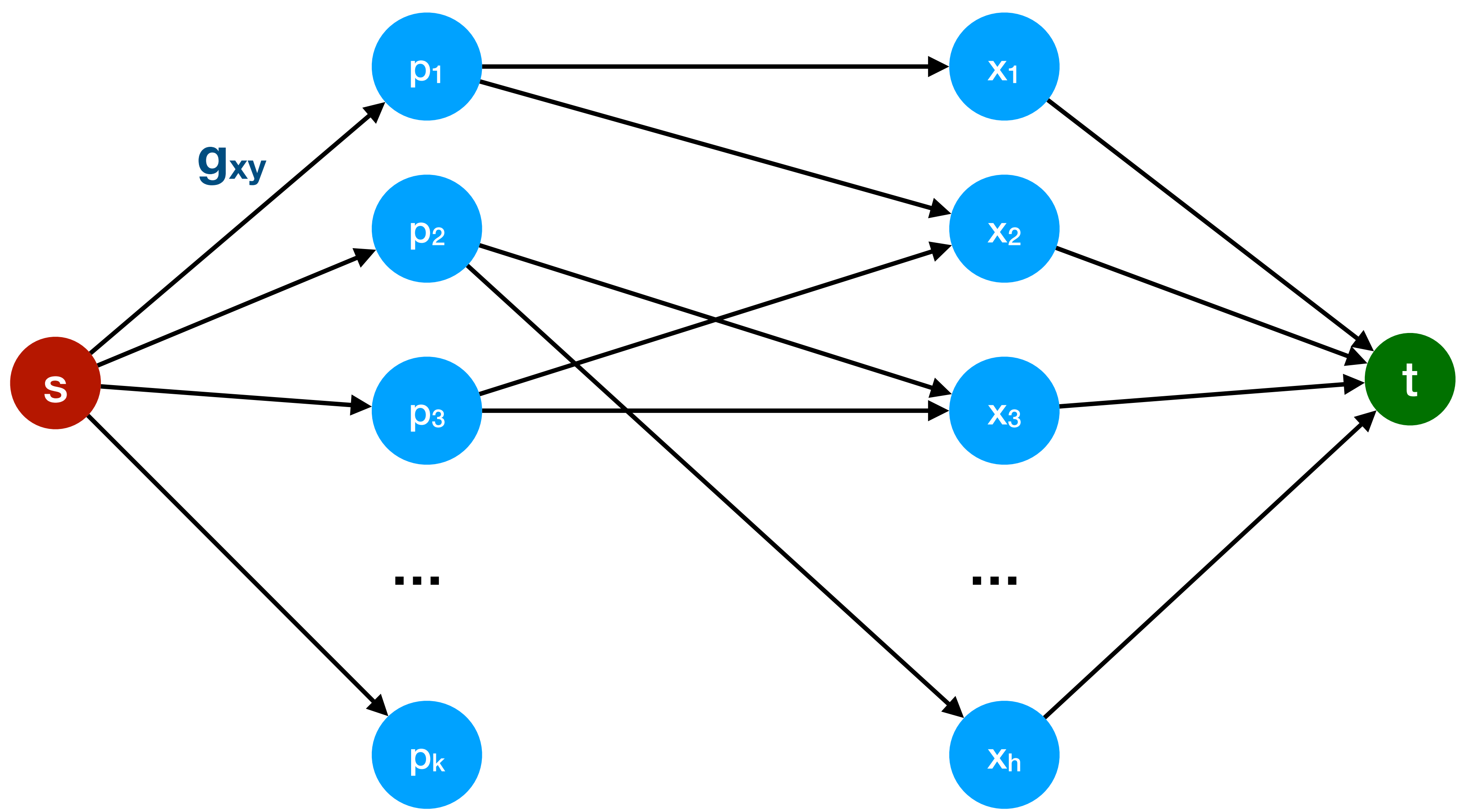

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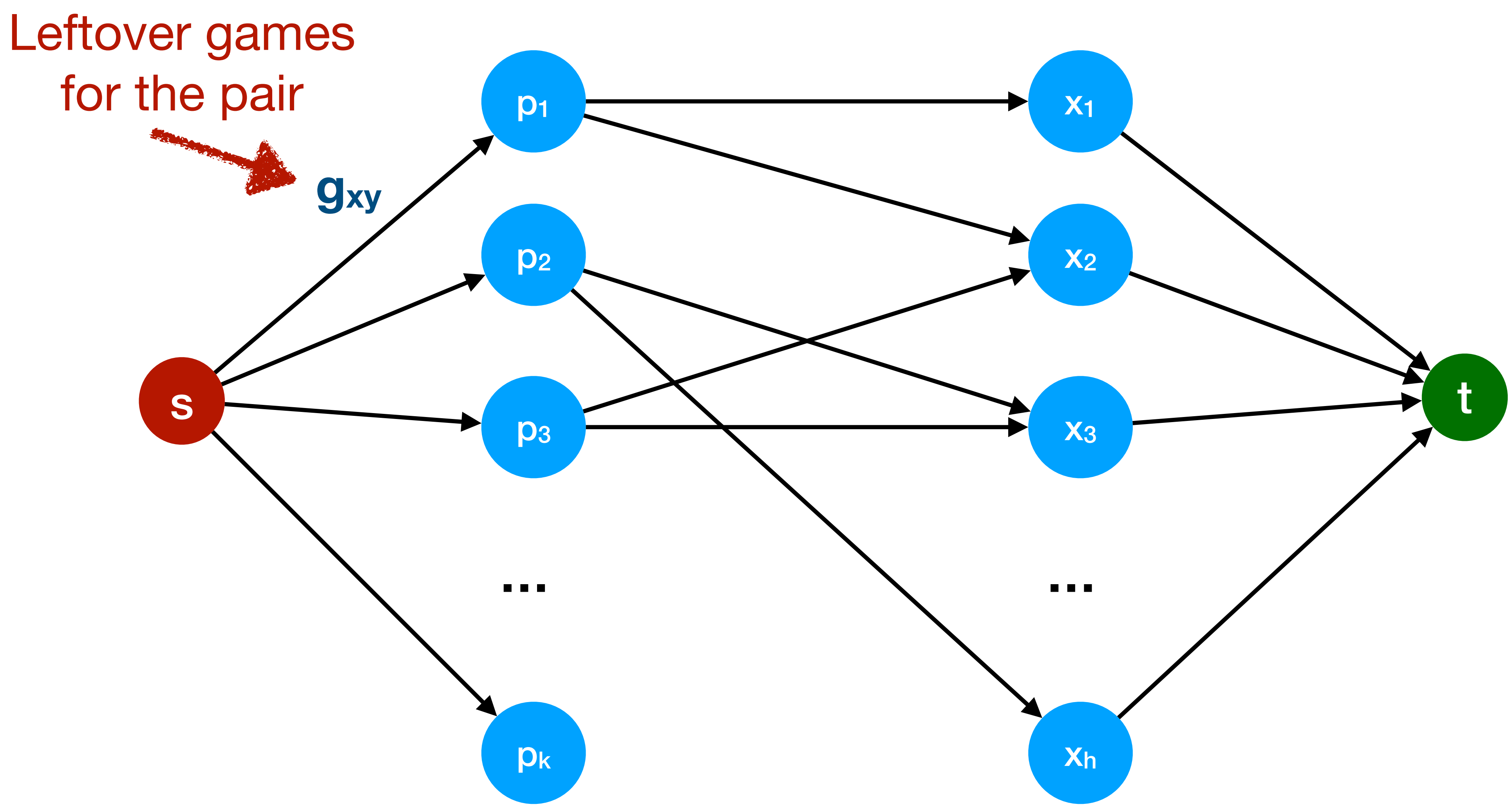

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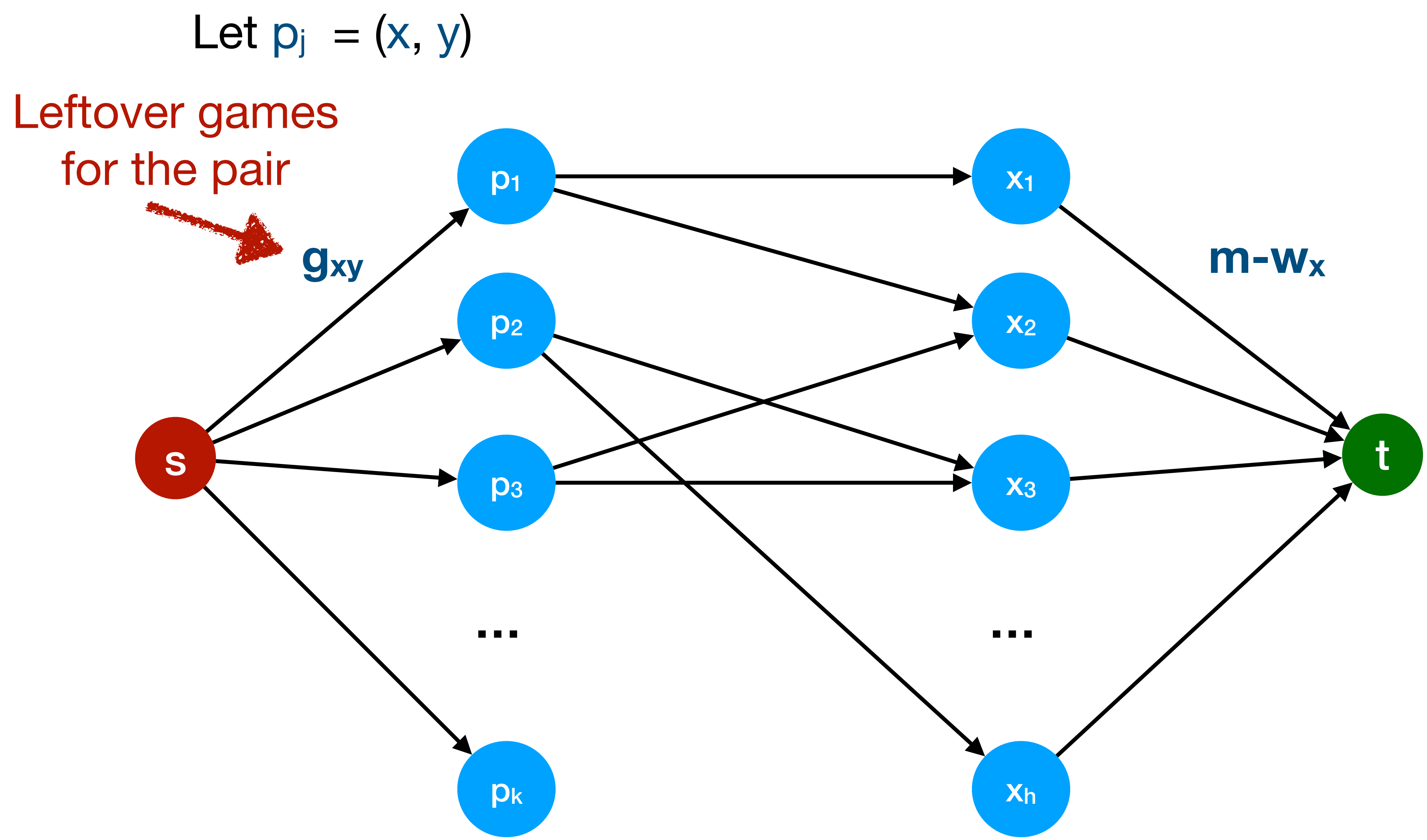

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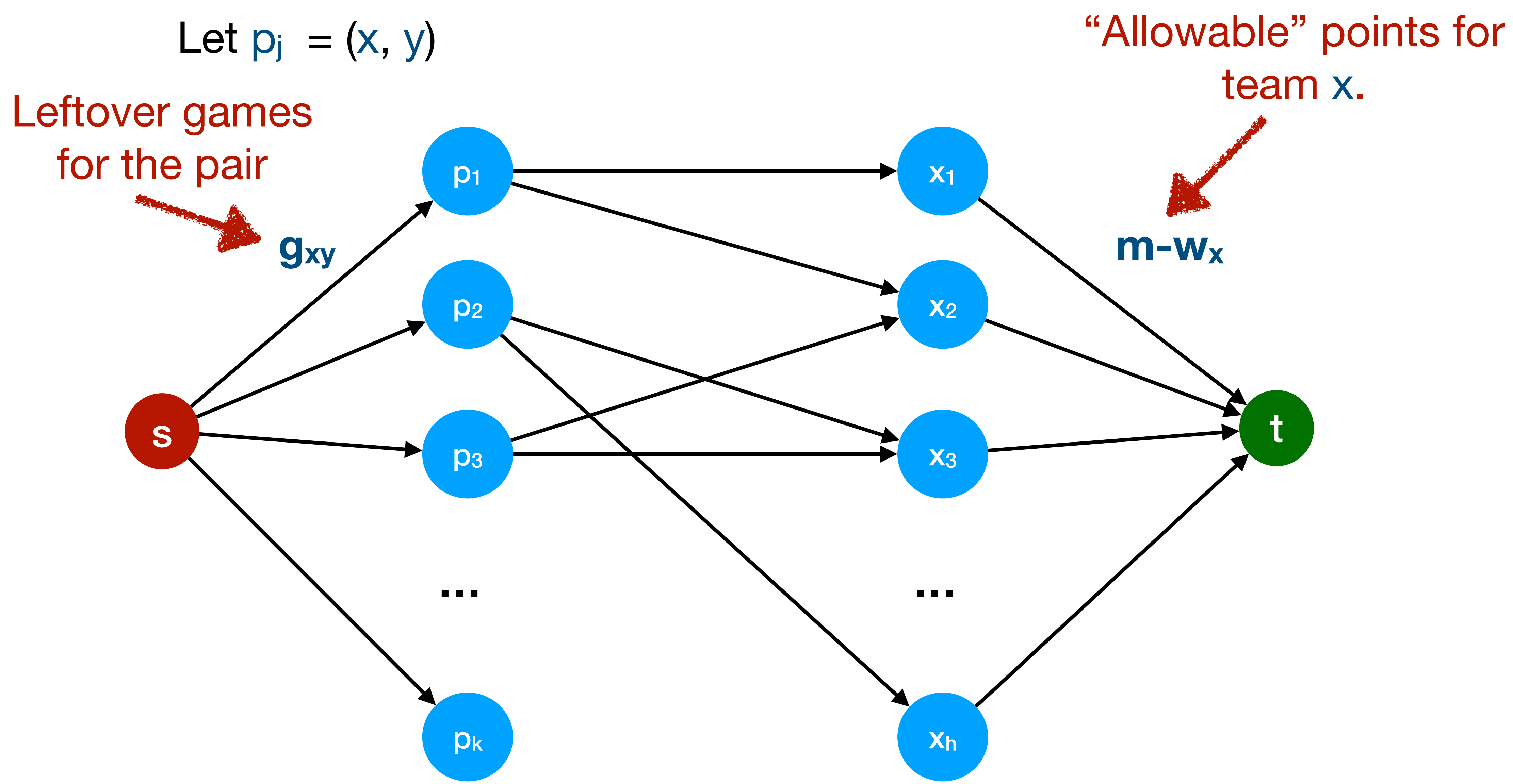

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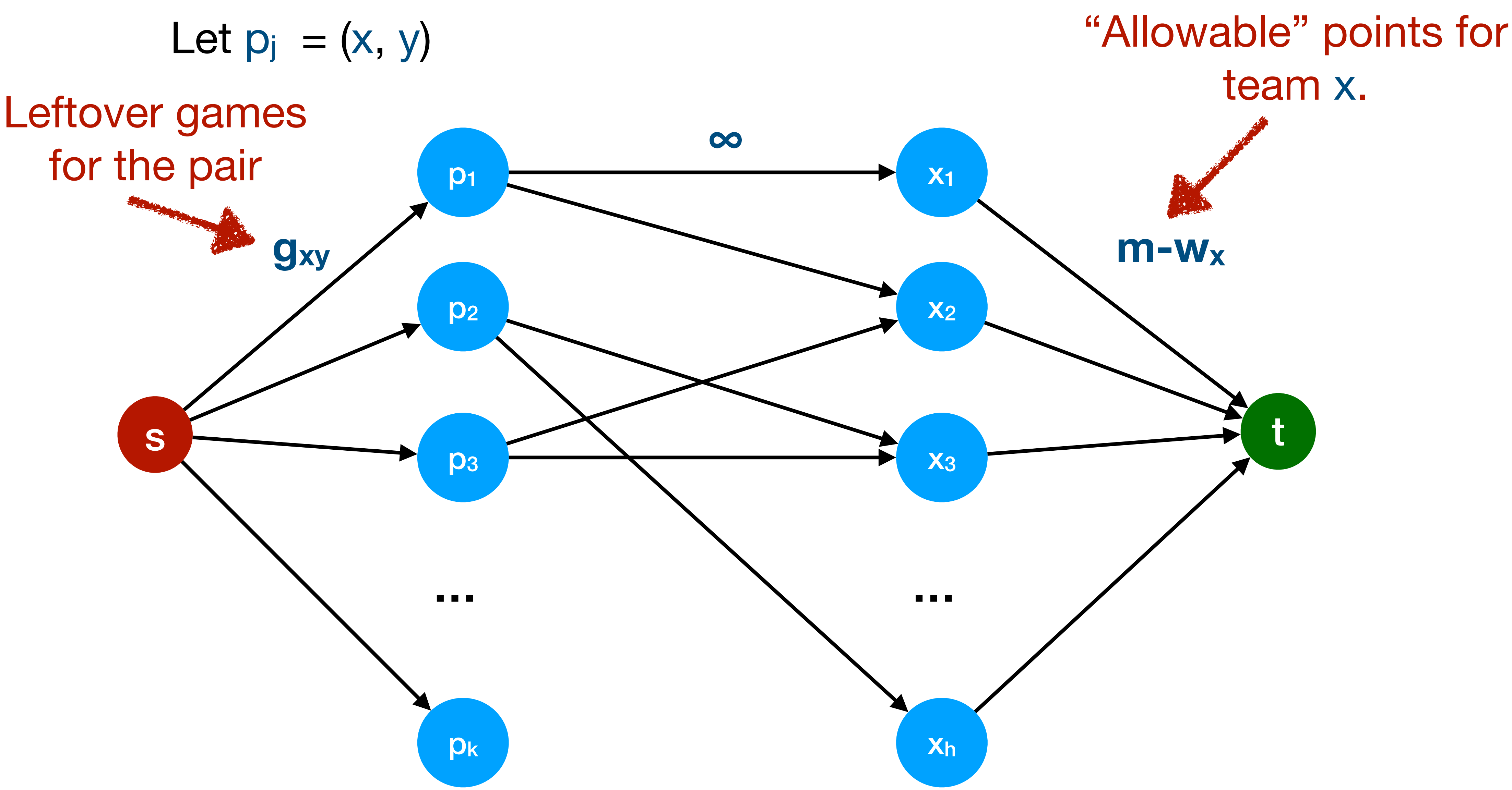

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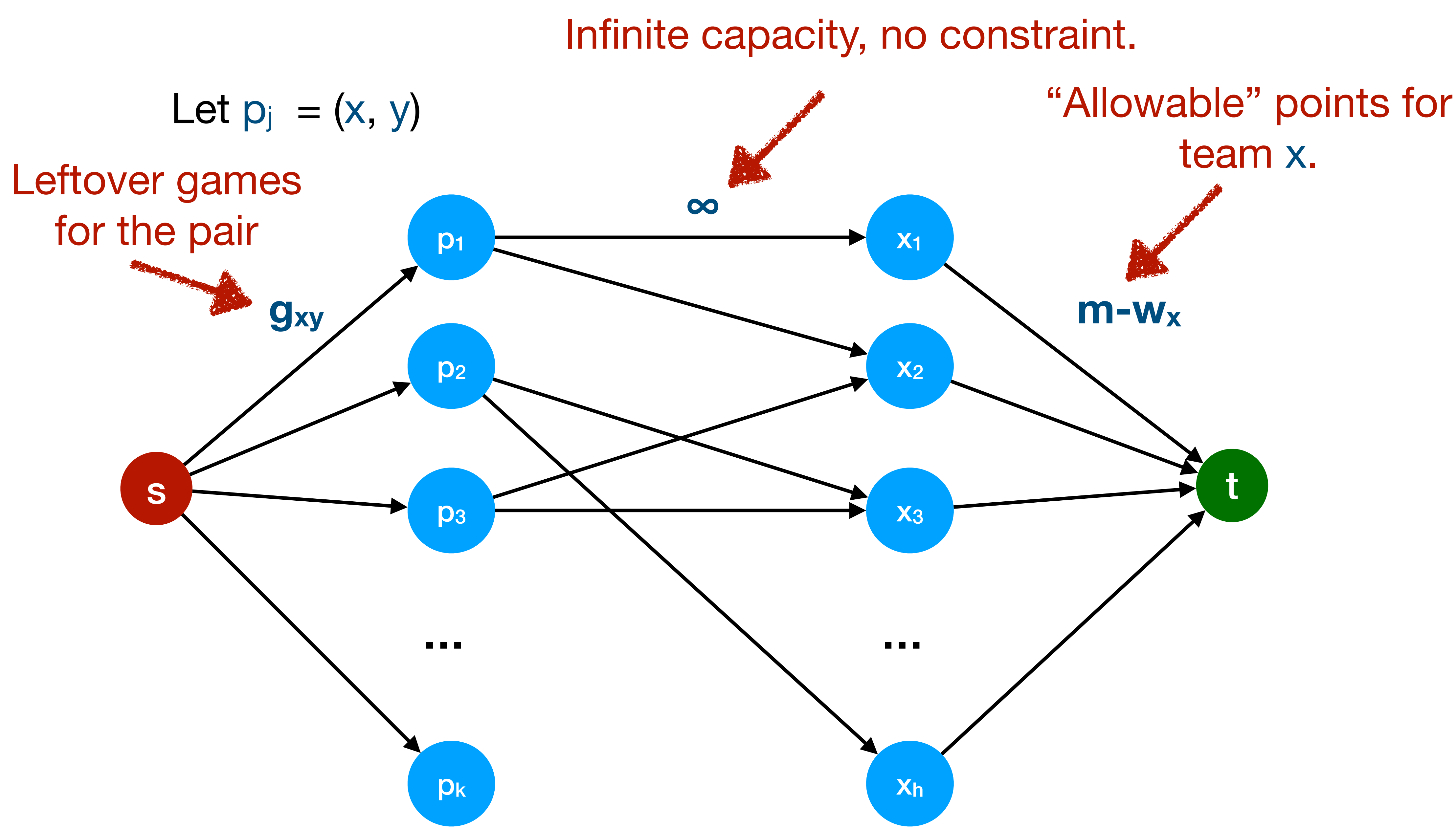

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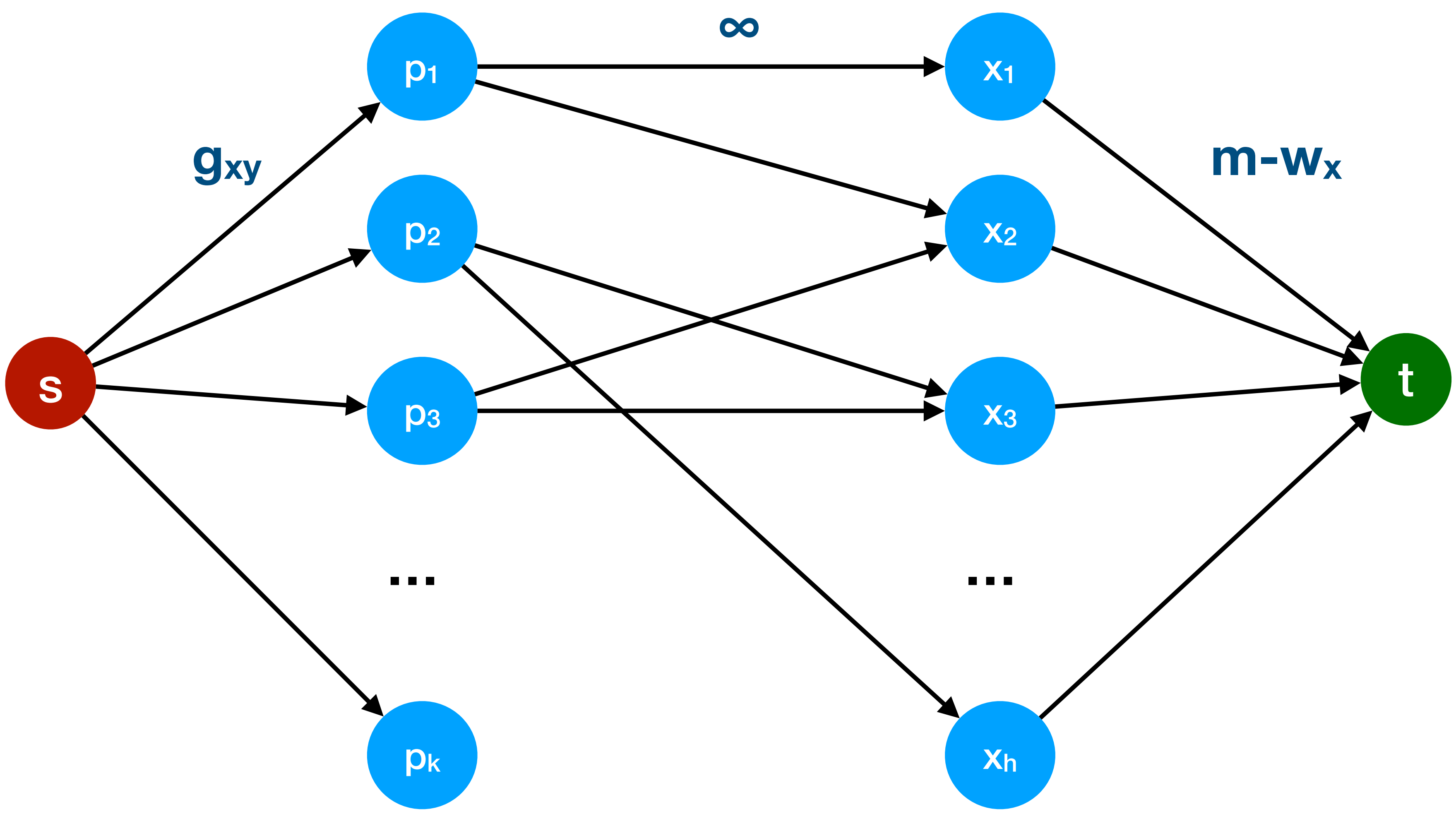

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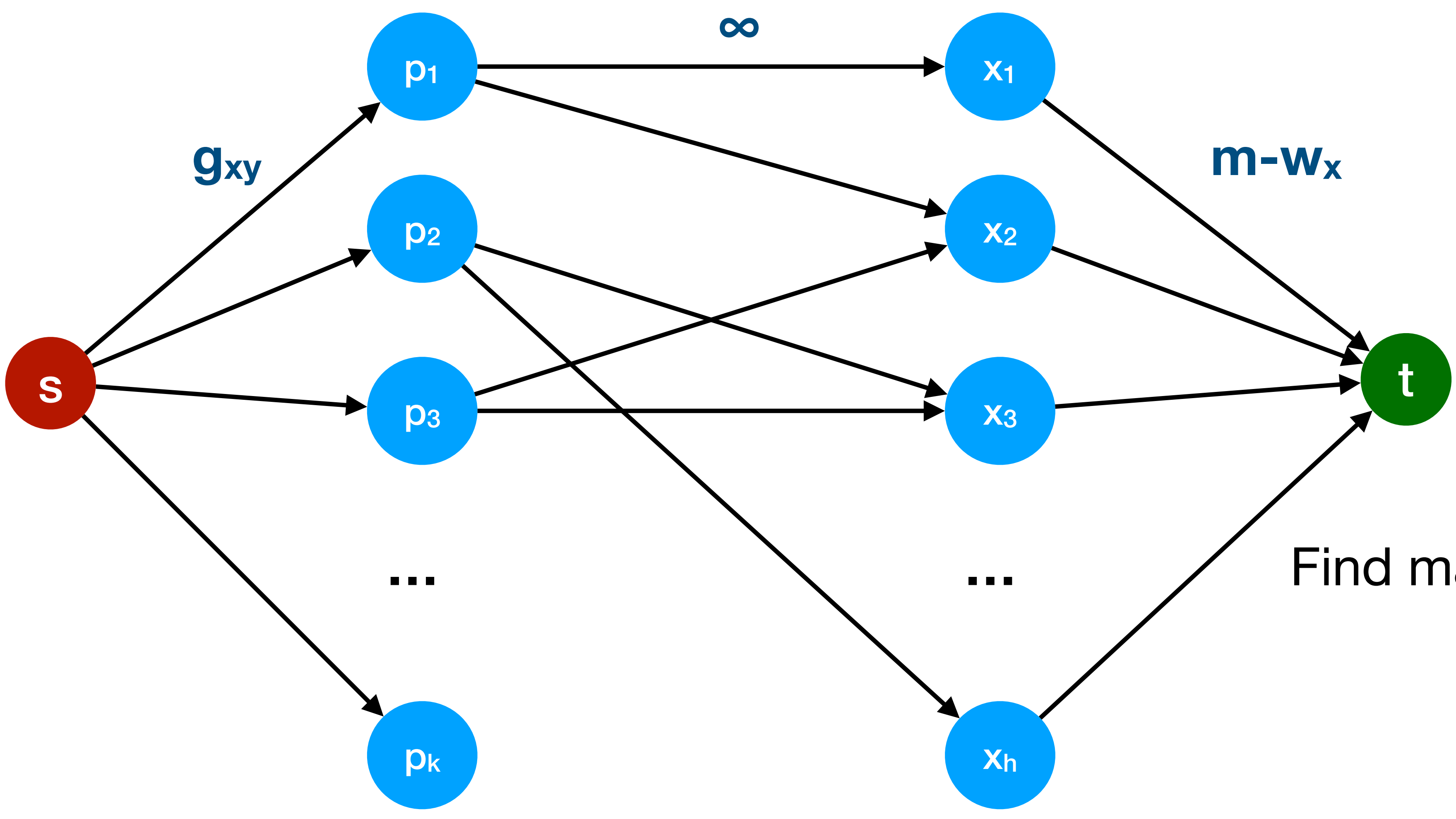
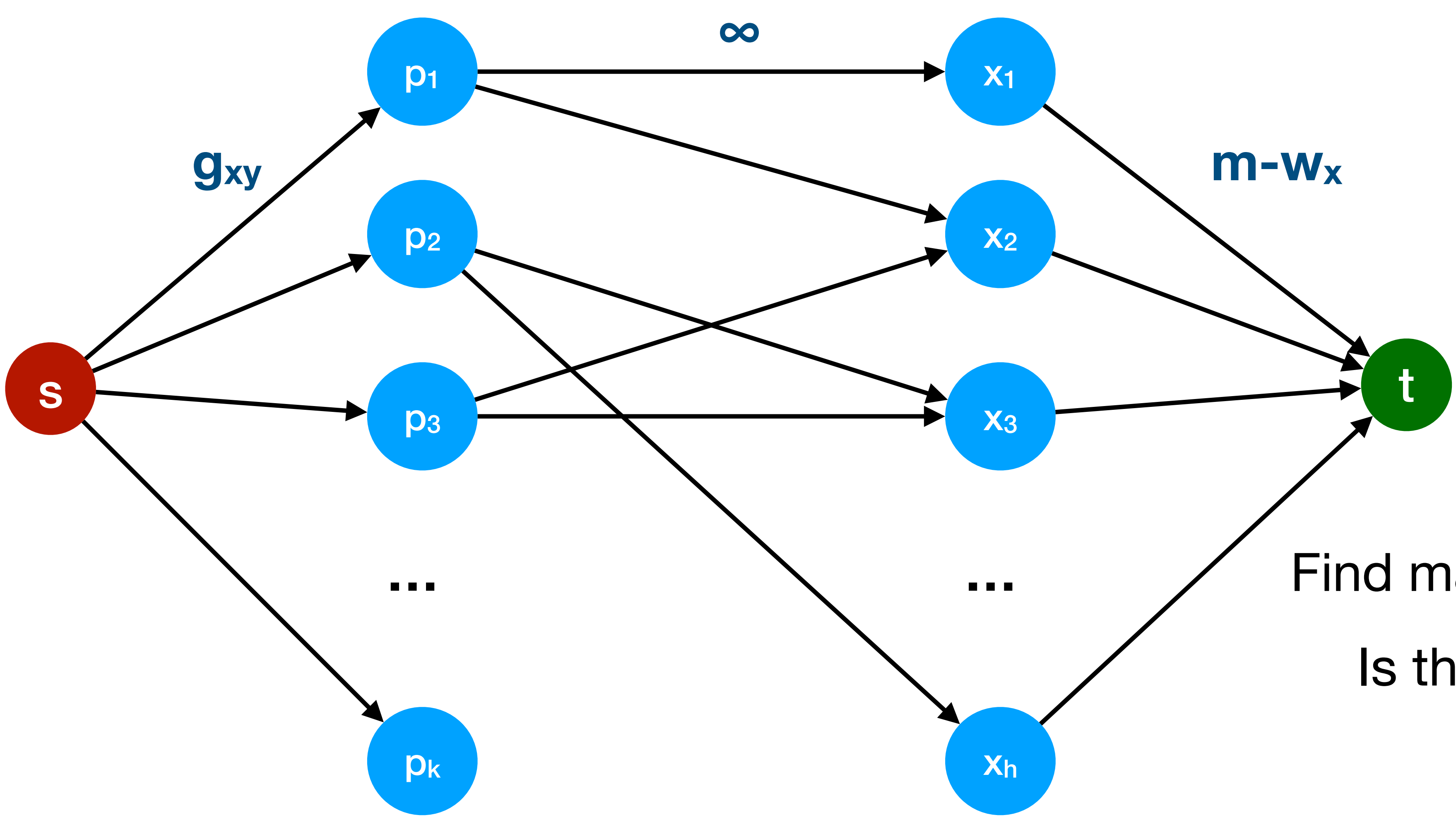


Find max flow in the network

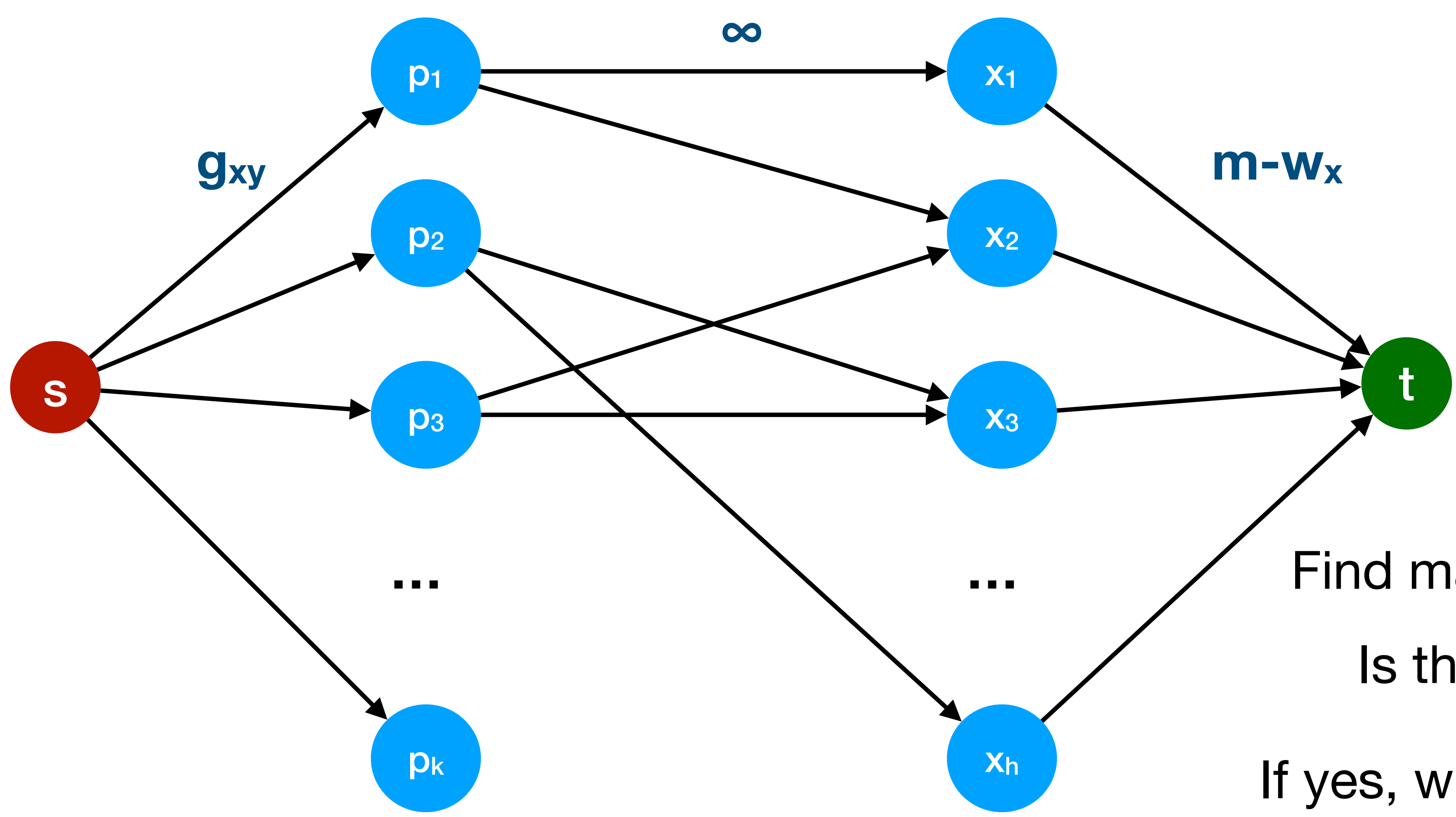
From baseball to flows



Find max flow in the network

Is the value at least g^*?

From baseball to flows



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Is the value at least g^*?

If yes, winning is still possible

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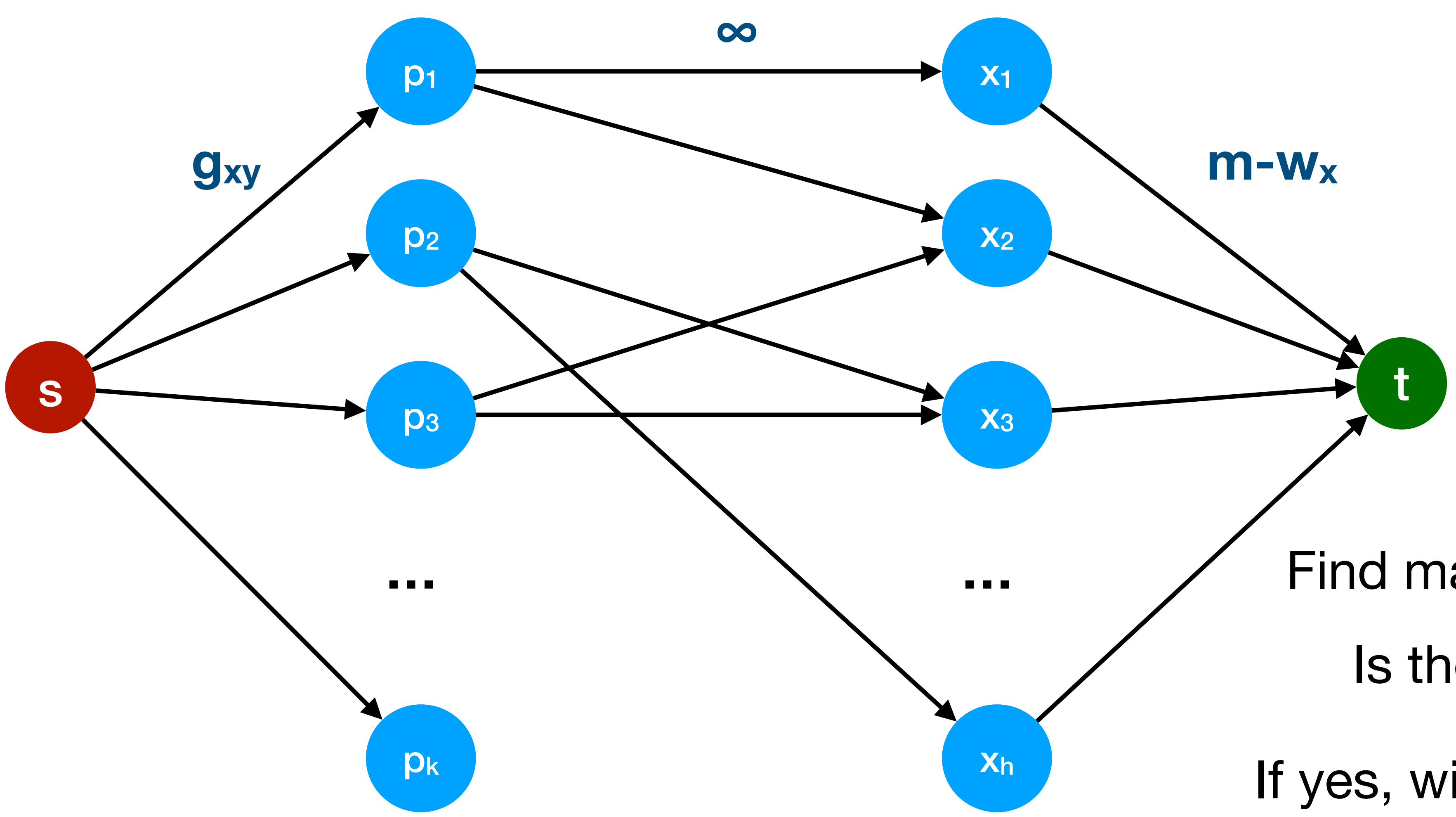

Find max flow in the network

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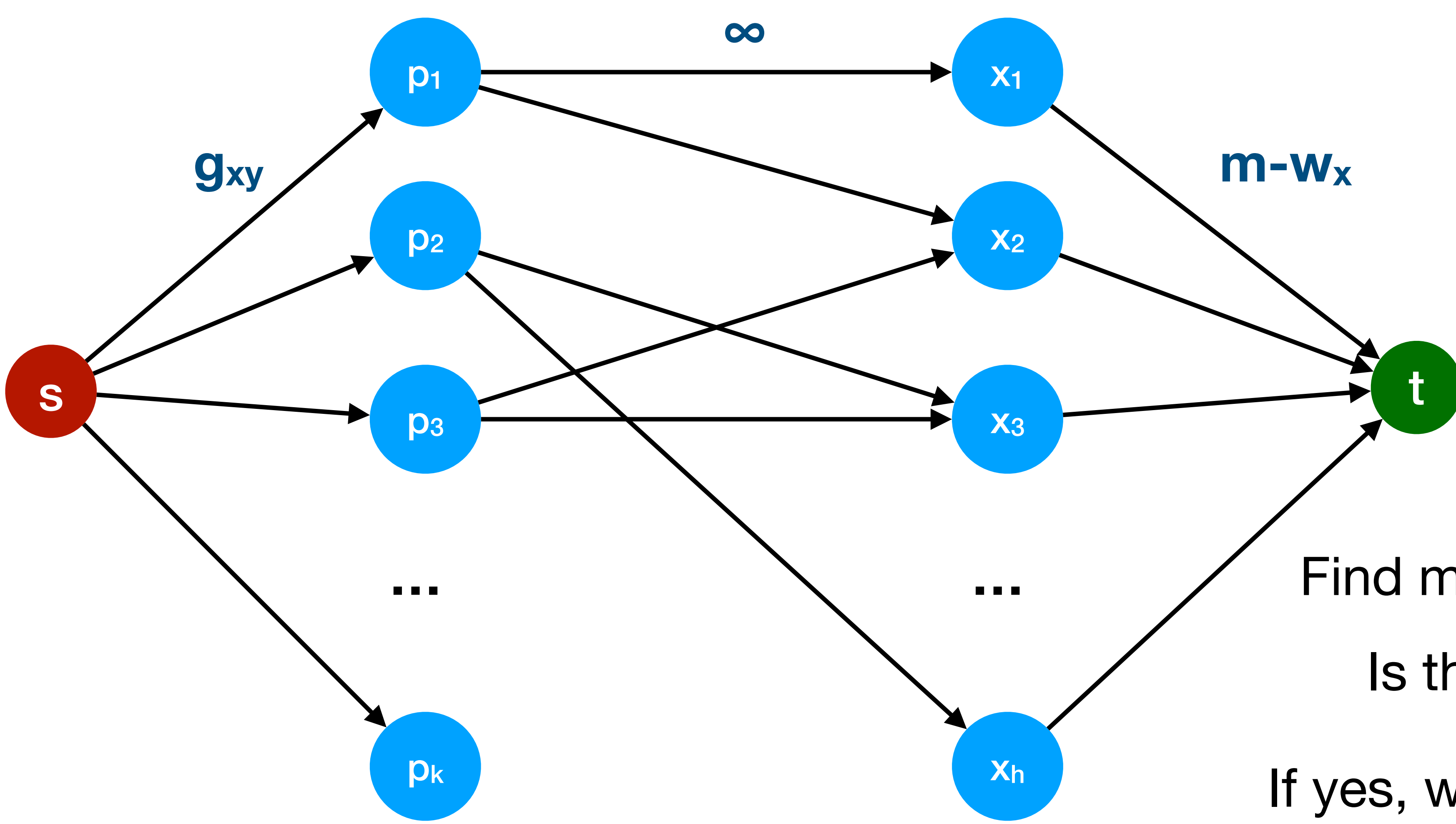
If yes, winning is still possible
If no, winning is not possible

Why does this work?

Assume that the algorithm says yes.

The value of the flow is equal to the number of remaining games. (why?)

From baseball to flows



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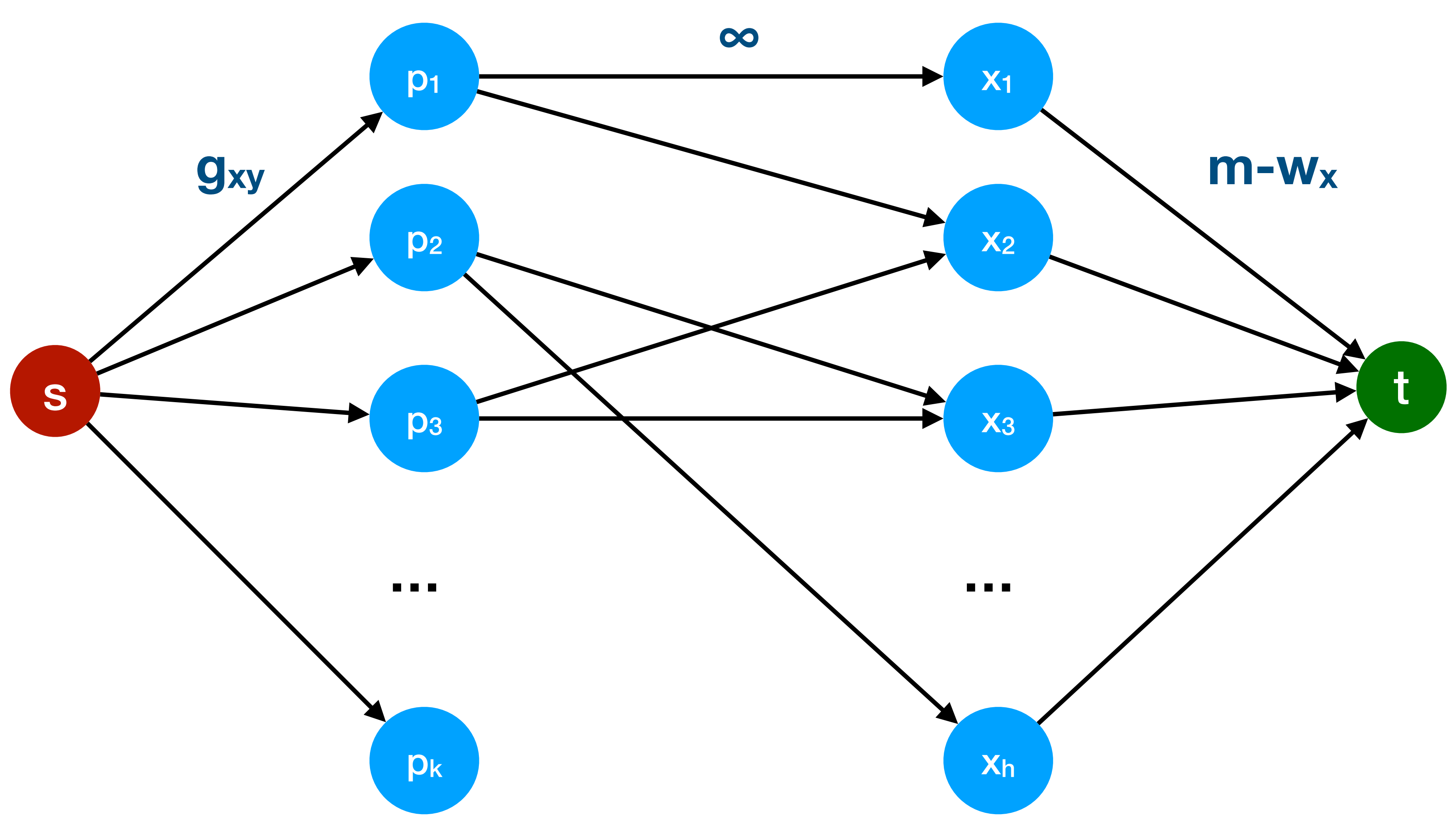
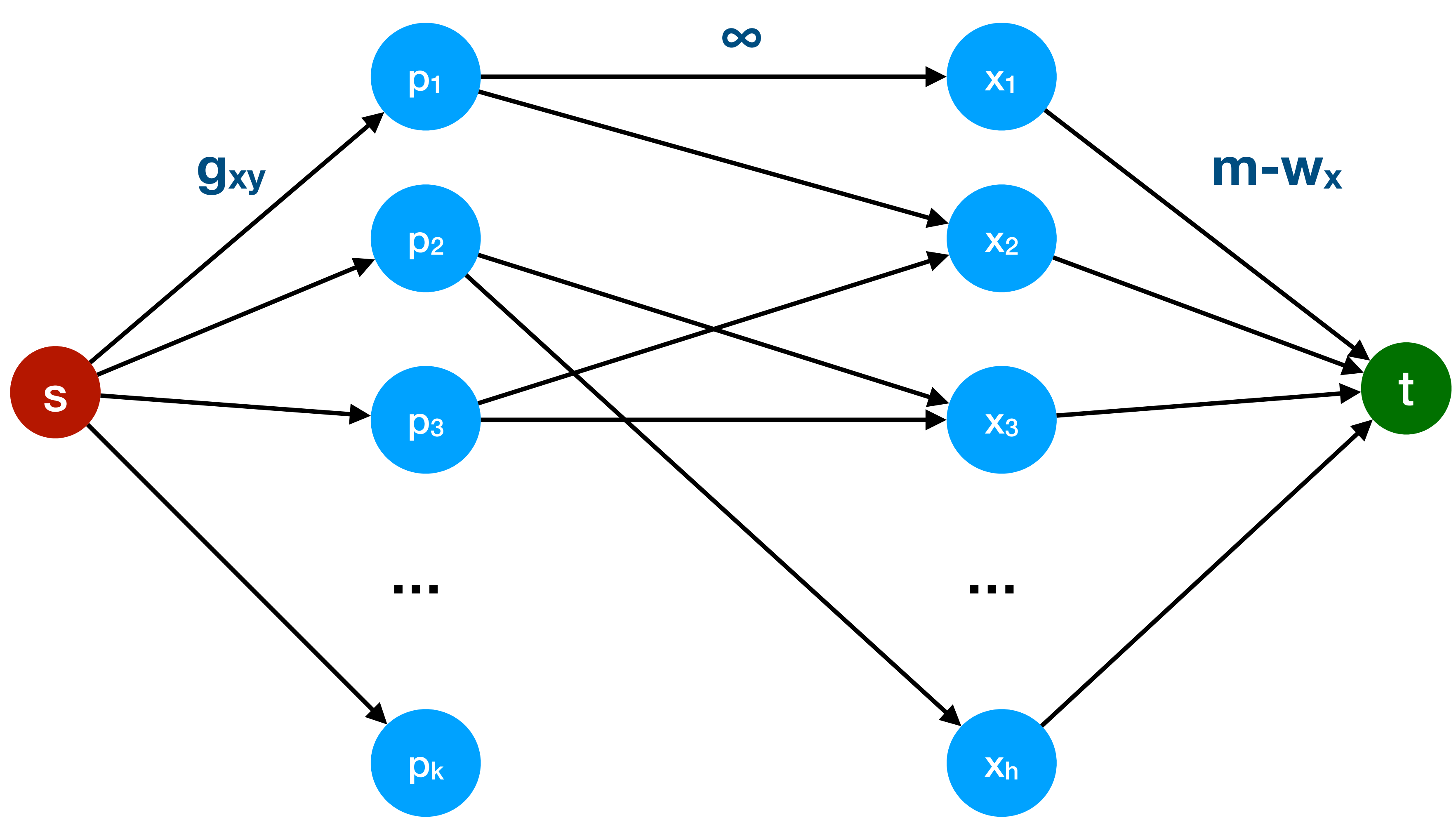


Another way to think about it

Let's look at the final residual graph. Why do we have no augmenting paths?



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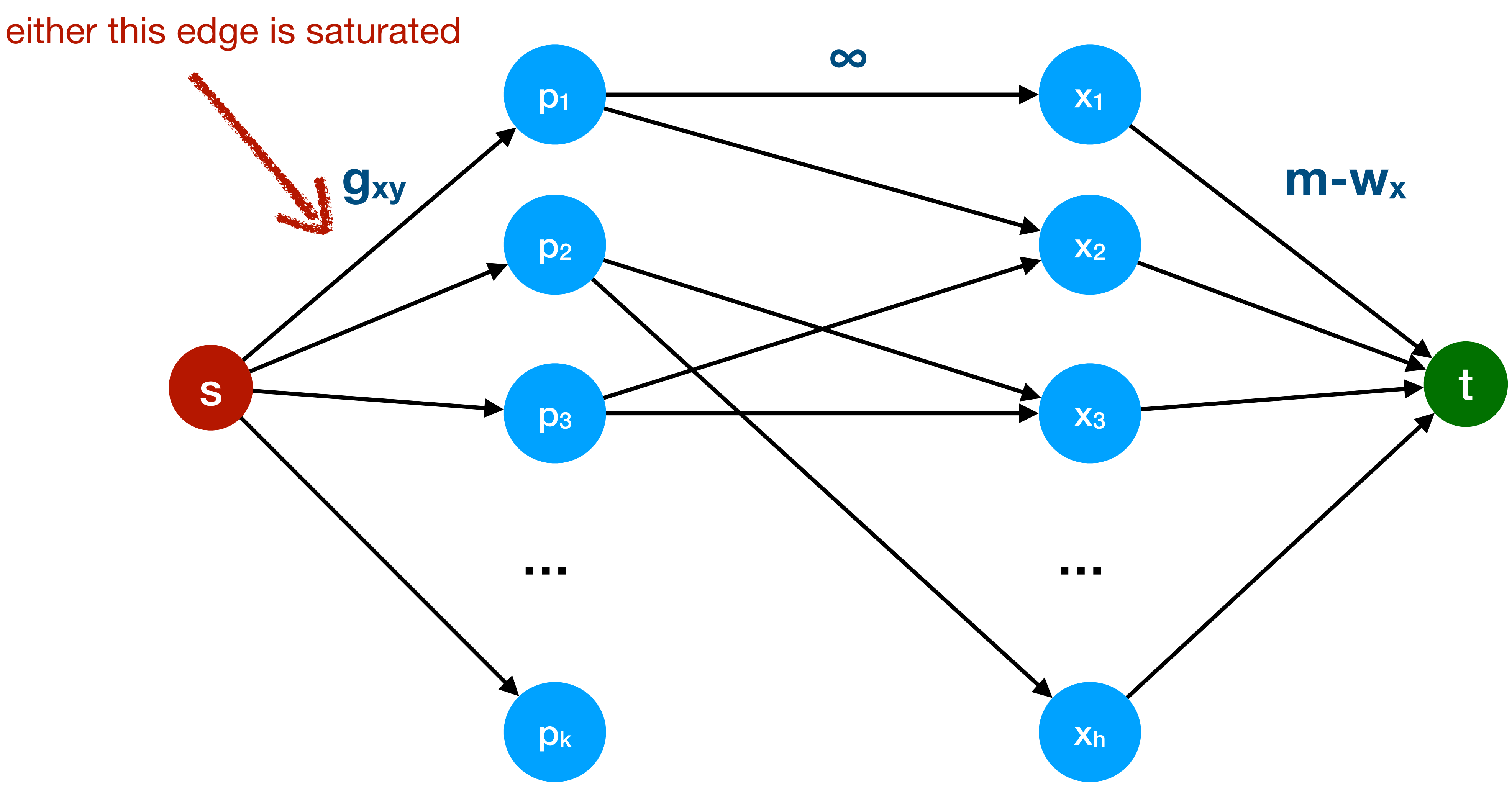

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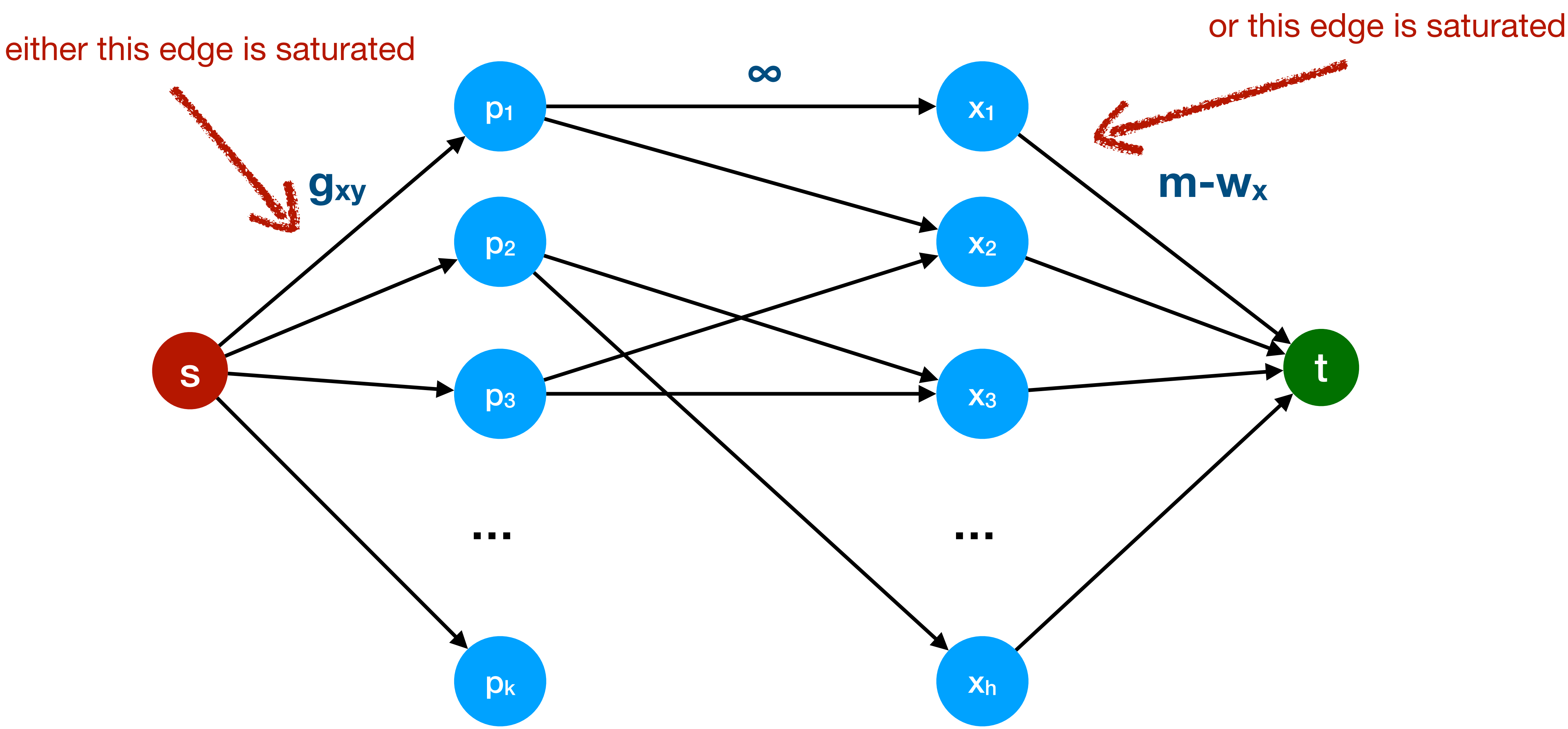

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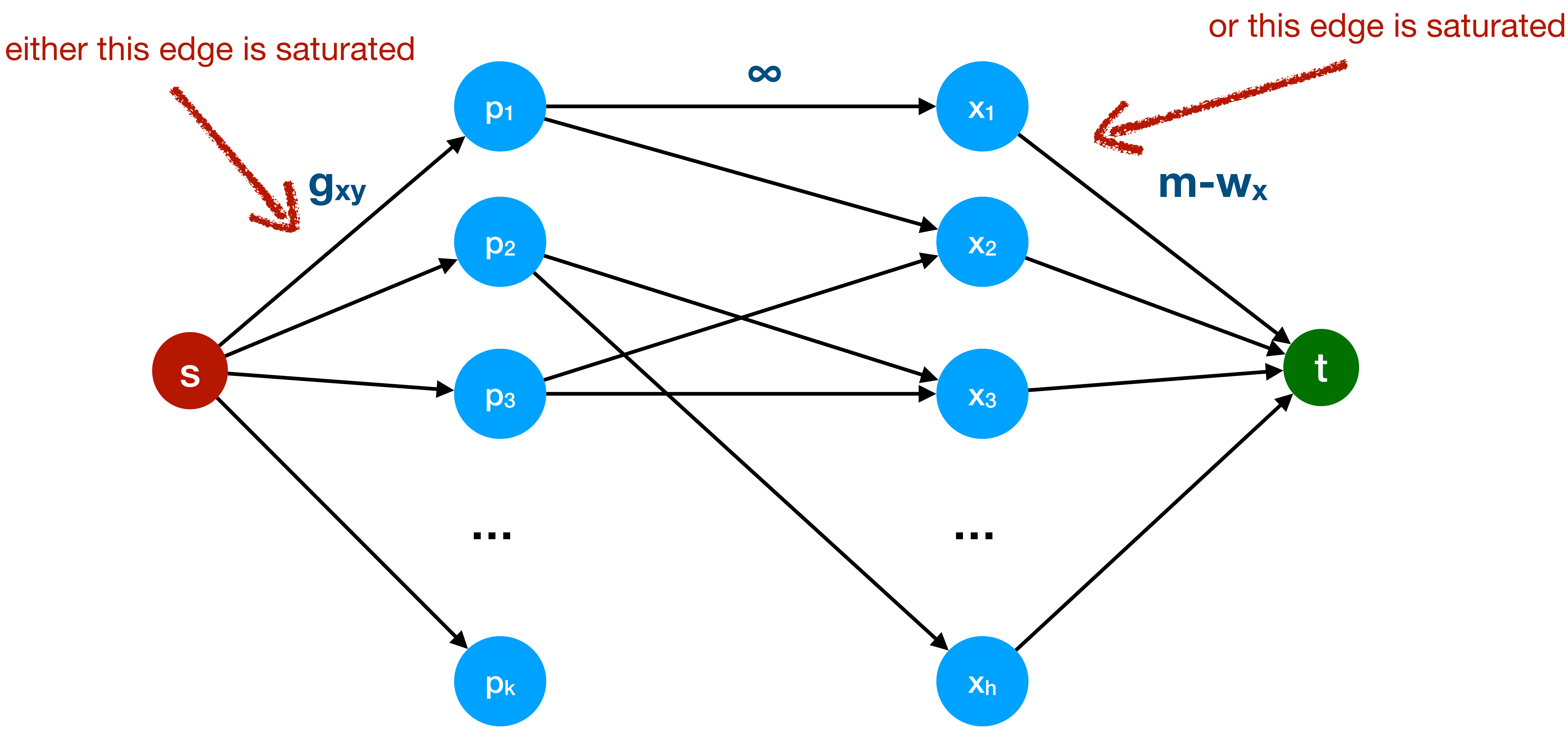


Either all the games have been played, or some team cannot win any more games.

Example

In the baseball league, there are 4 teams with the following number of wins:`

New York	90
Baltimore	88
Toronto	87
Boston	79

There are five games left in the season.

NY vs BLT

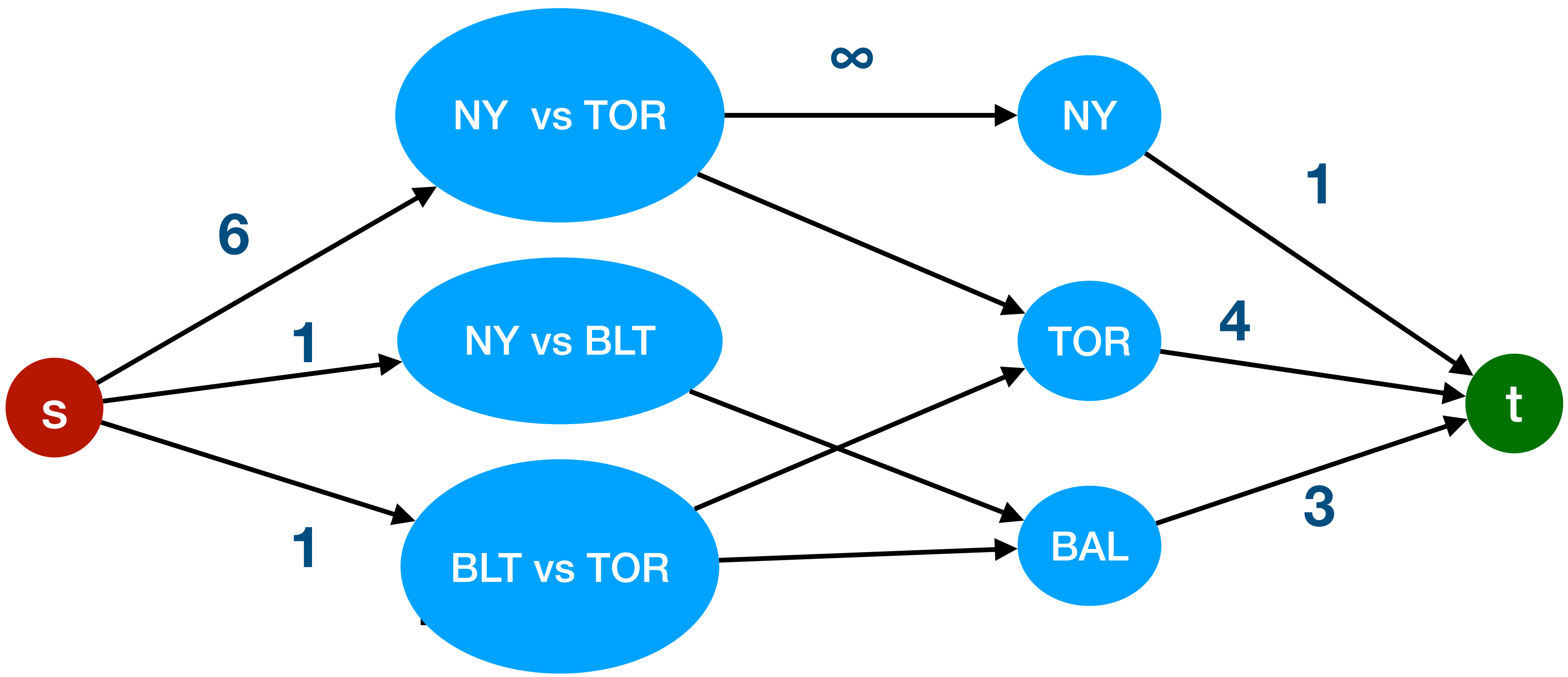
NY vs TOR ***6 games***

BLT vs TOR

BOS vs ANY ***4 games (12 games total)***

Question: Can Boston finish (possibly tied for) first?

Example



m = 91

Open pit mining

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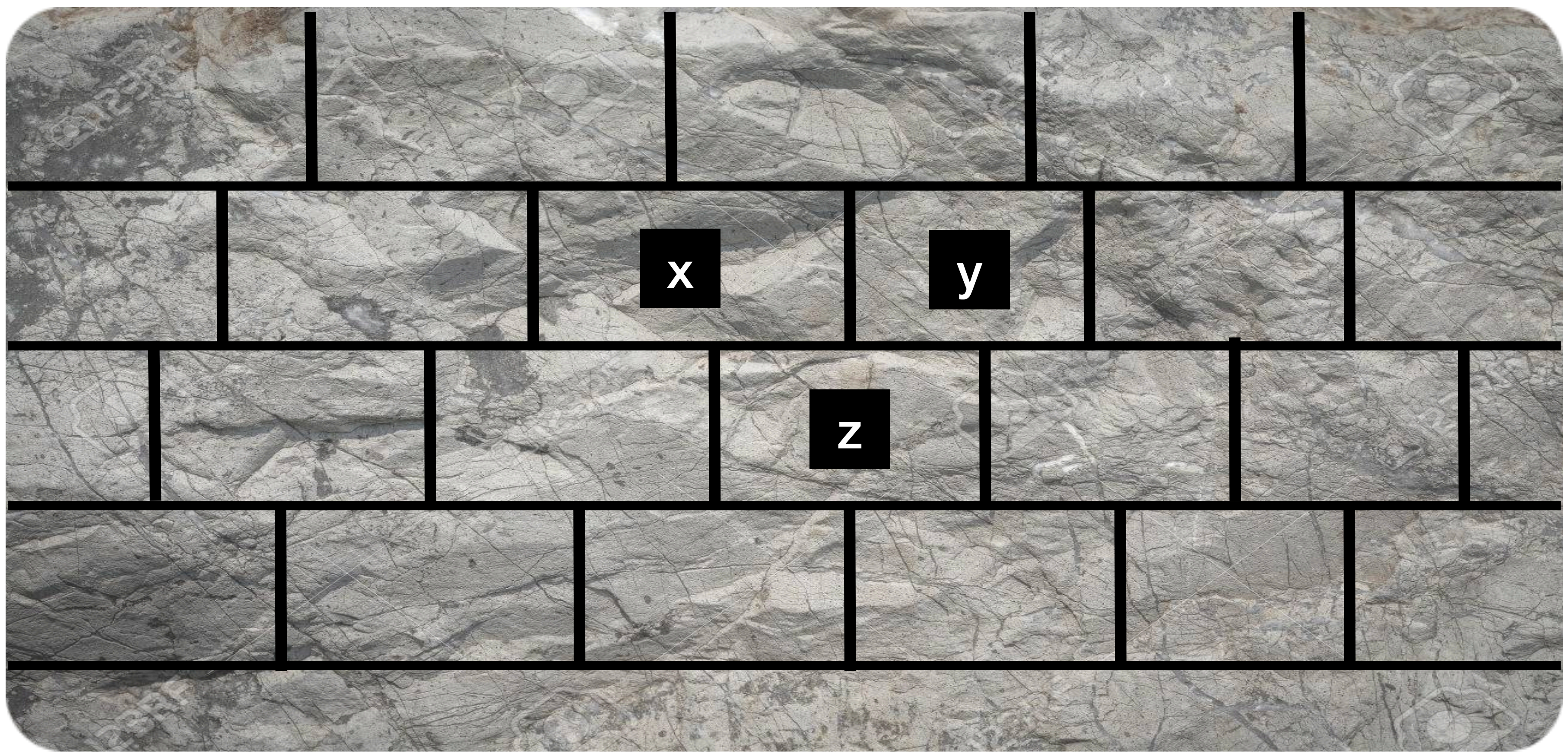
We want to earn as much money as possible.

Open pit mining



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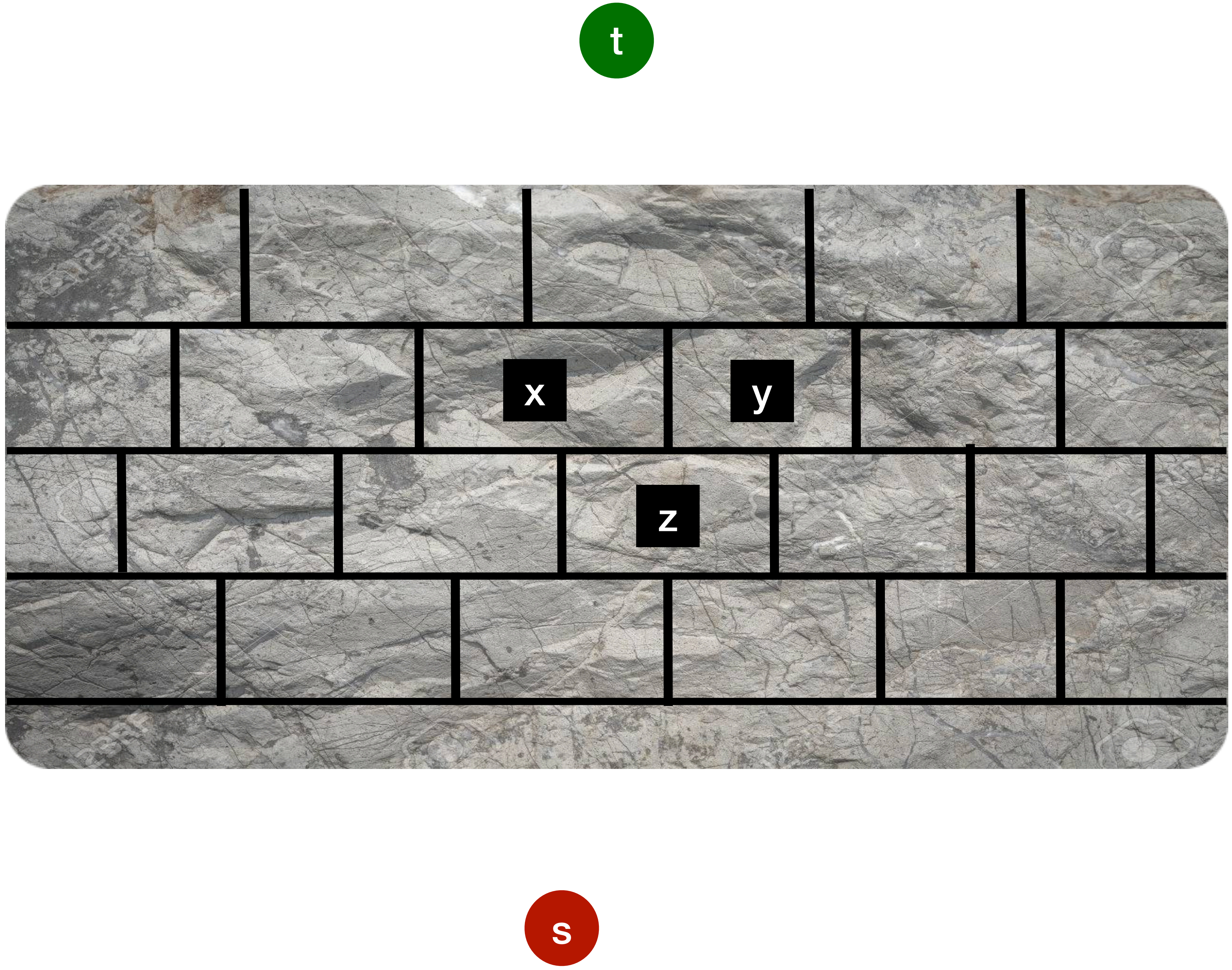

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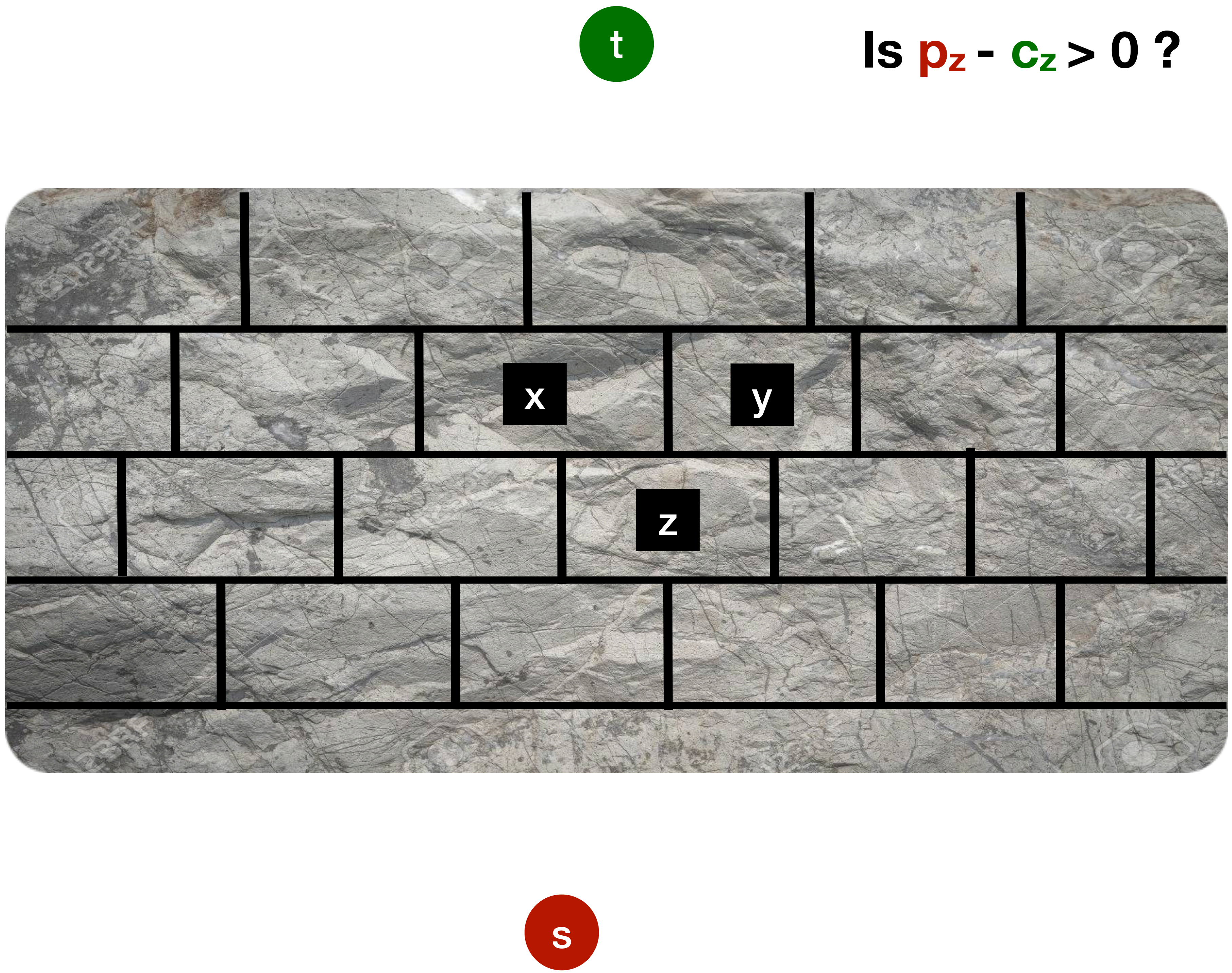

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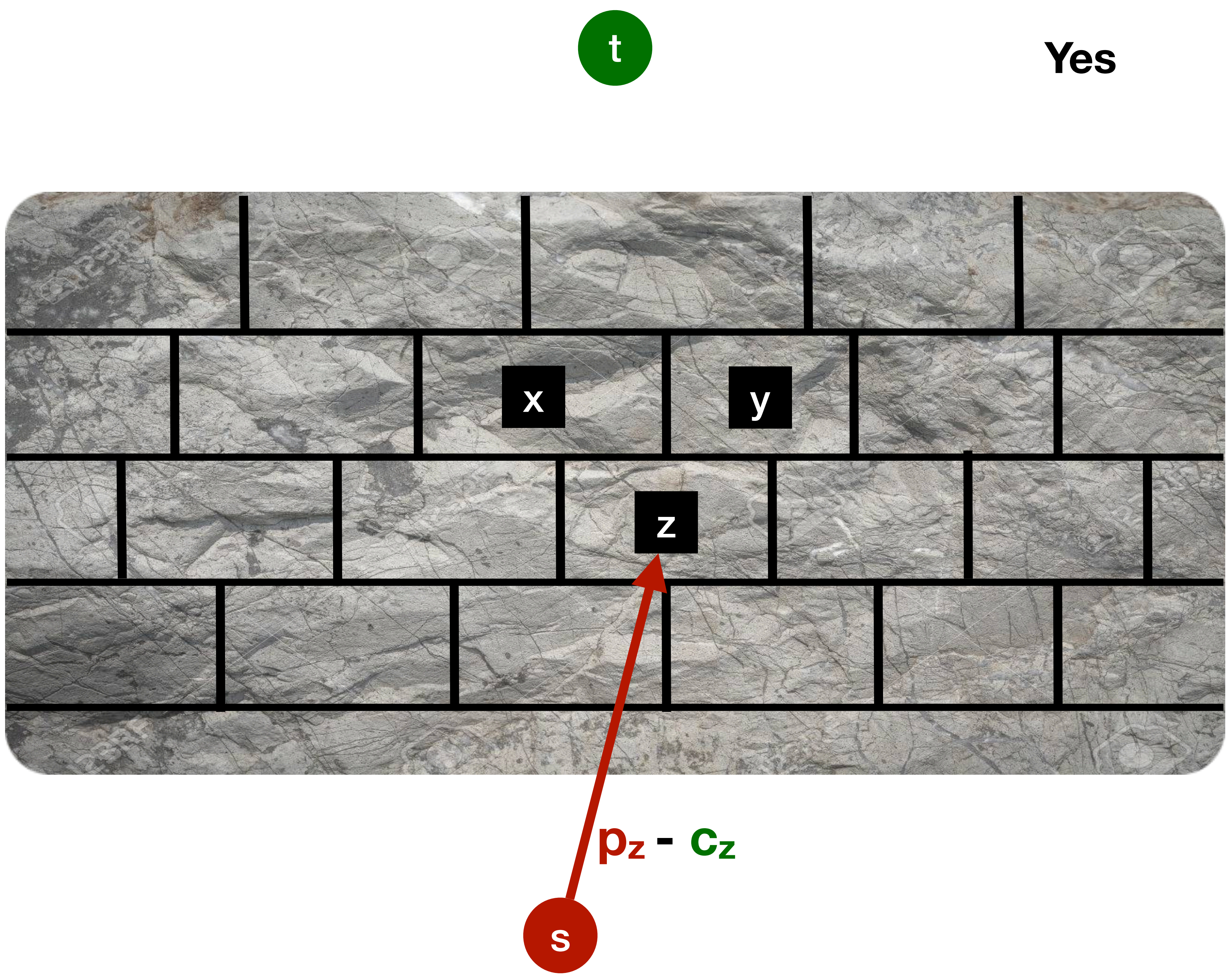

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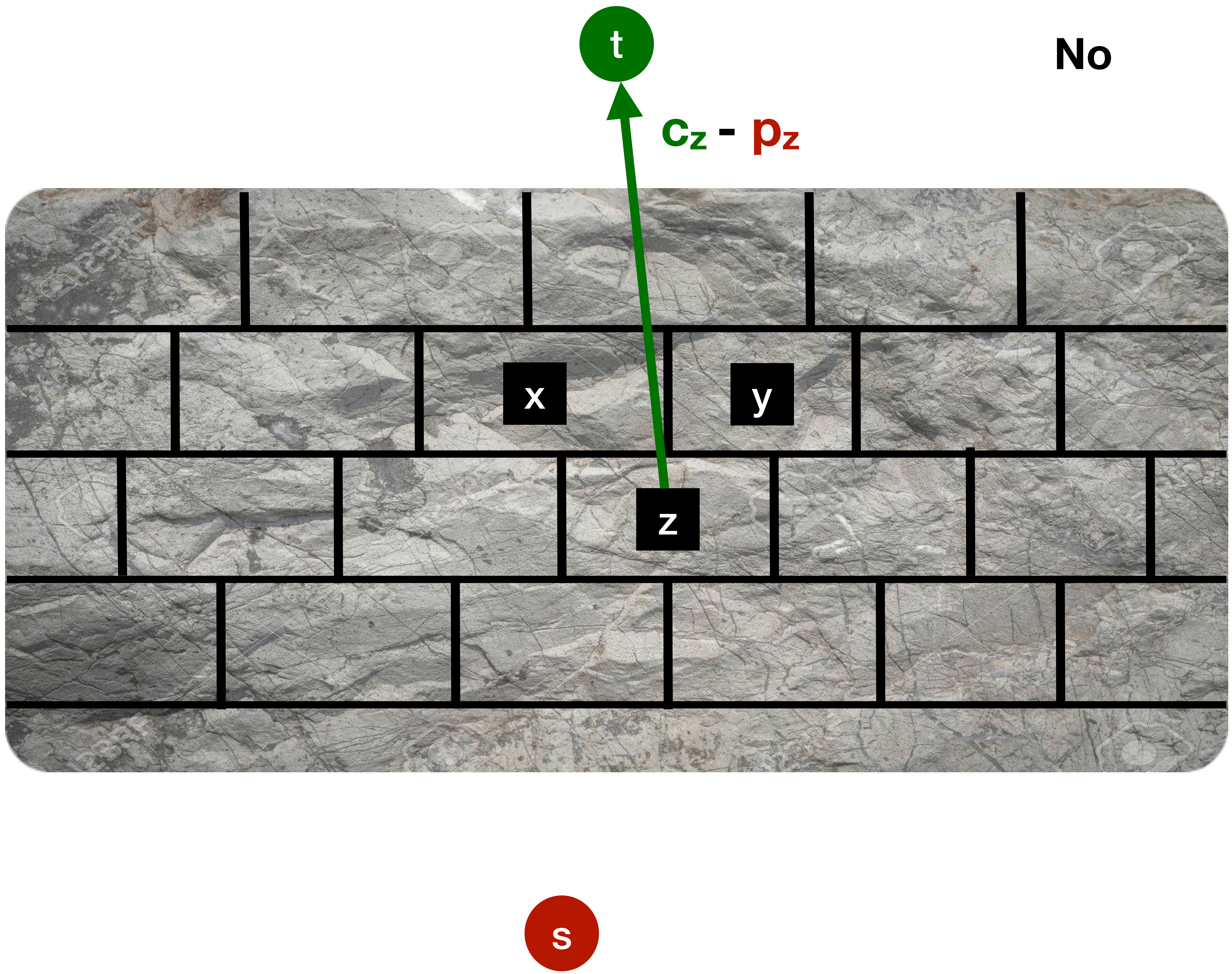
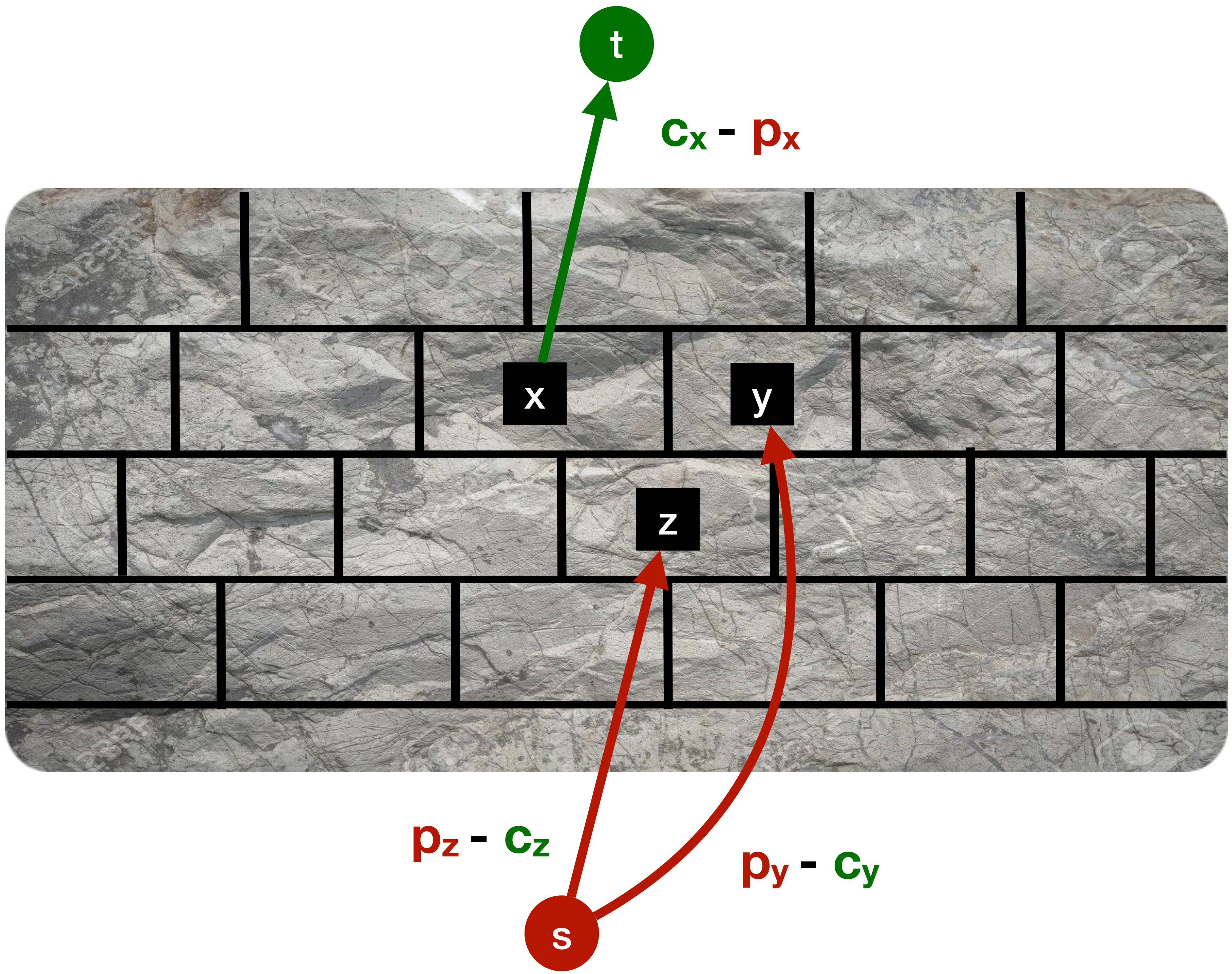


No

From pits to flows



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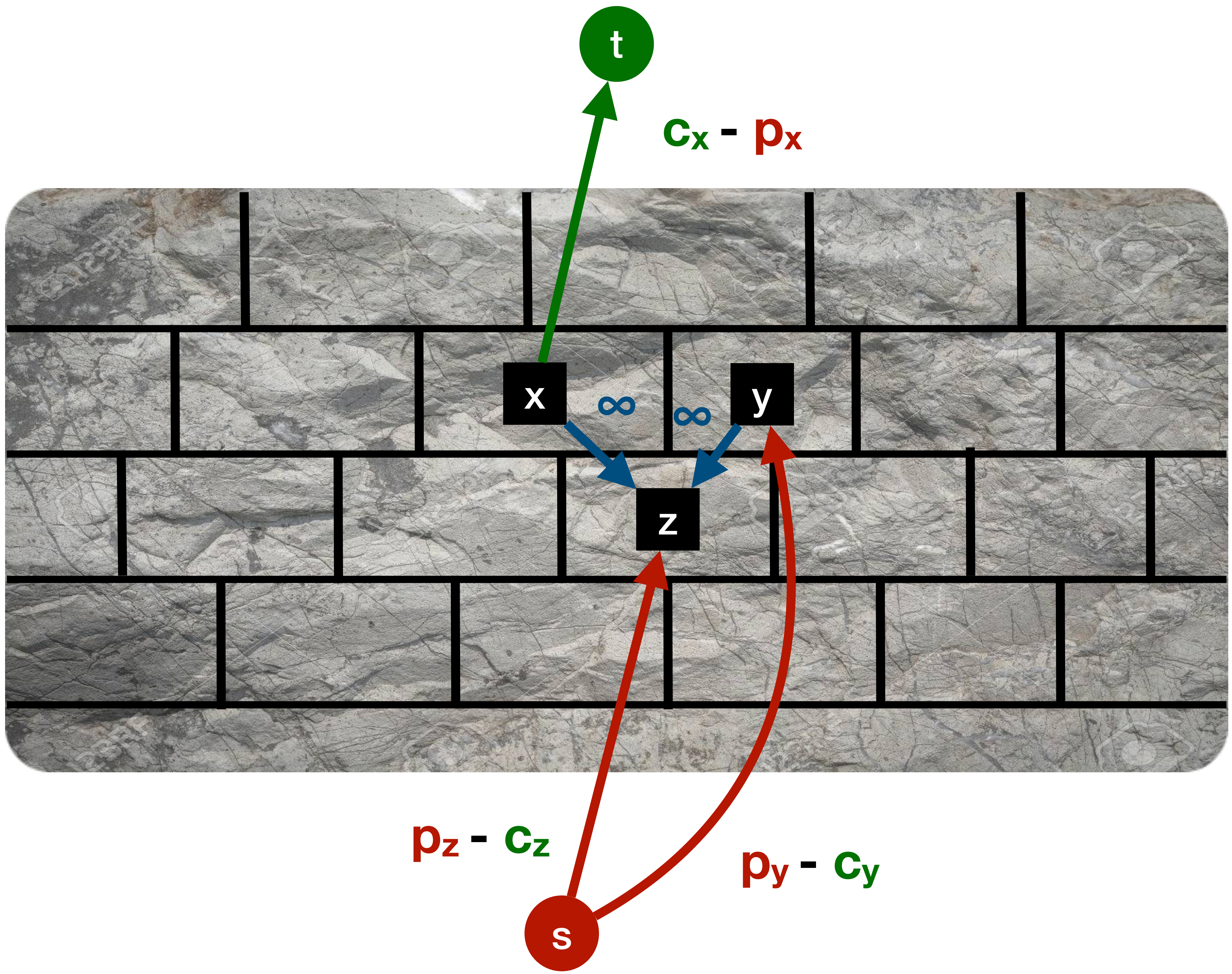

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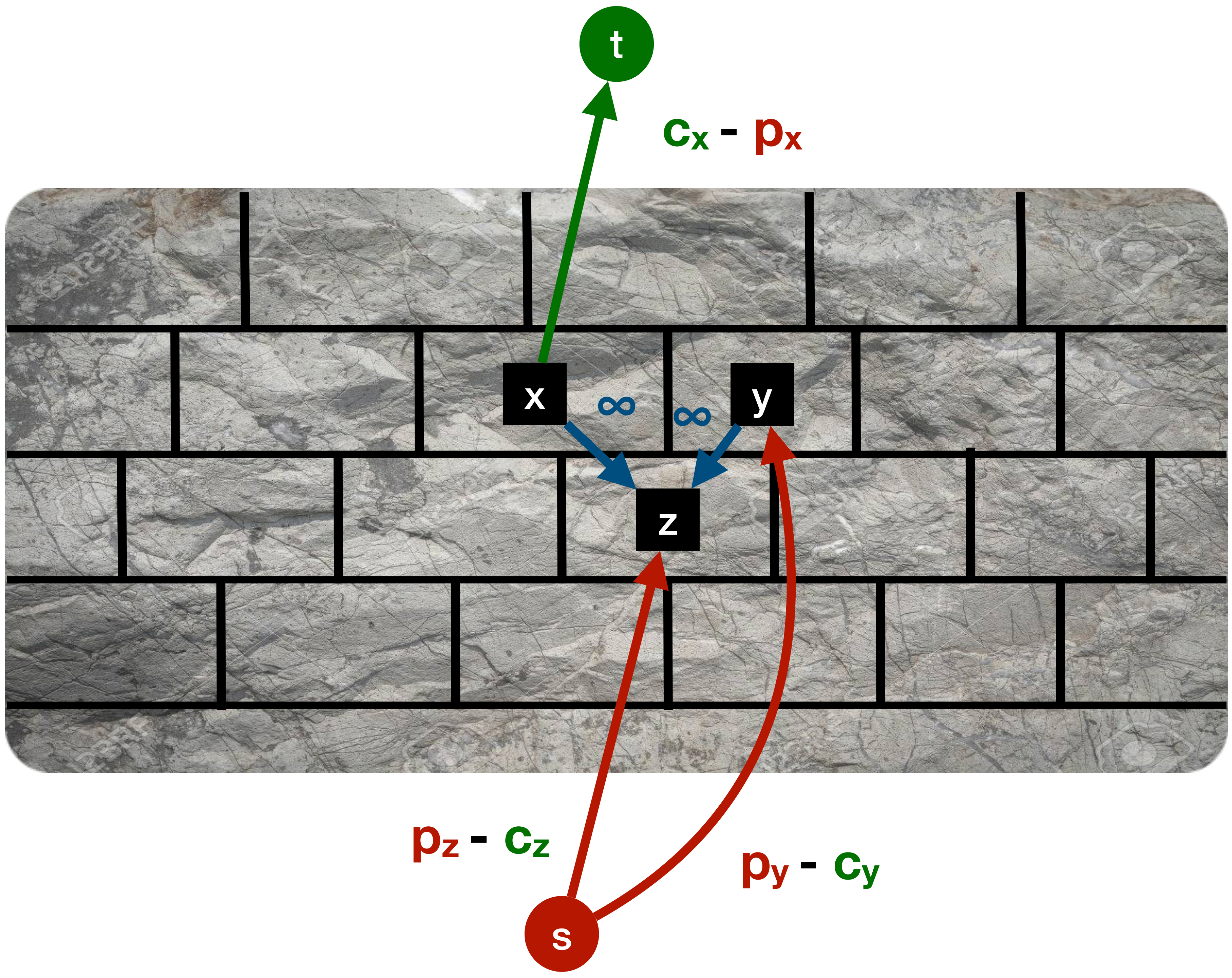

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Optimality?

Optimality of our mining set.

$$c(S, T) = \sum_{z \in T \,:\, p_z - c_z > 0} (p_z - c_z) + \sum_{z \in S \,:\, p_z - c_z < 0} (c_z - p_z)$$

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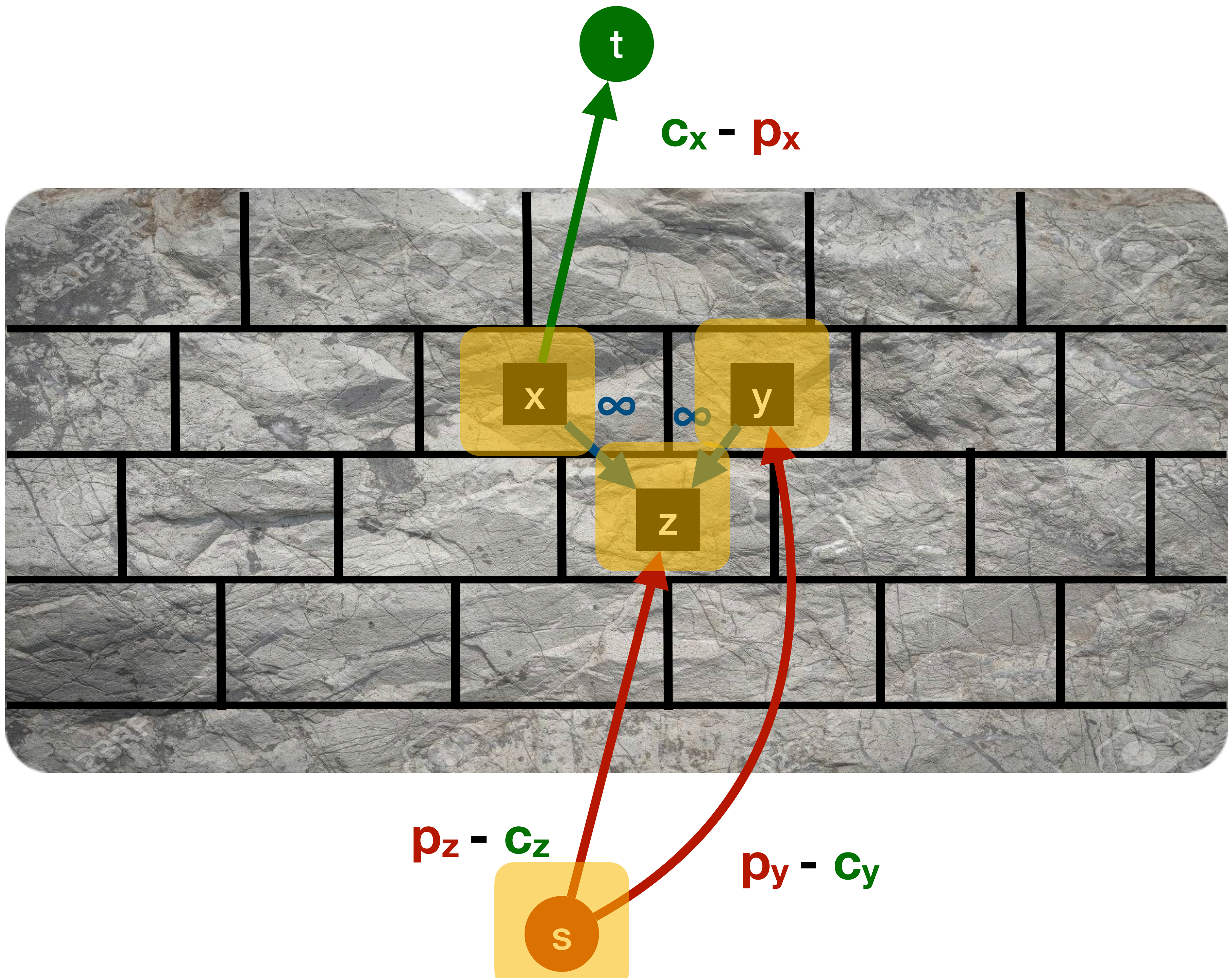

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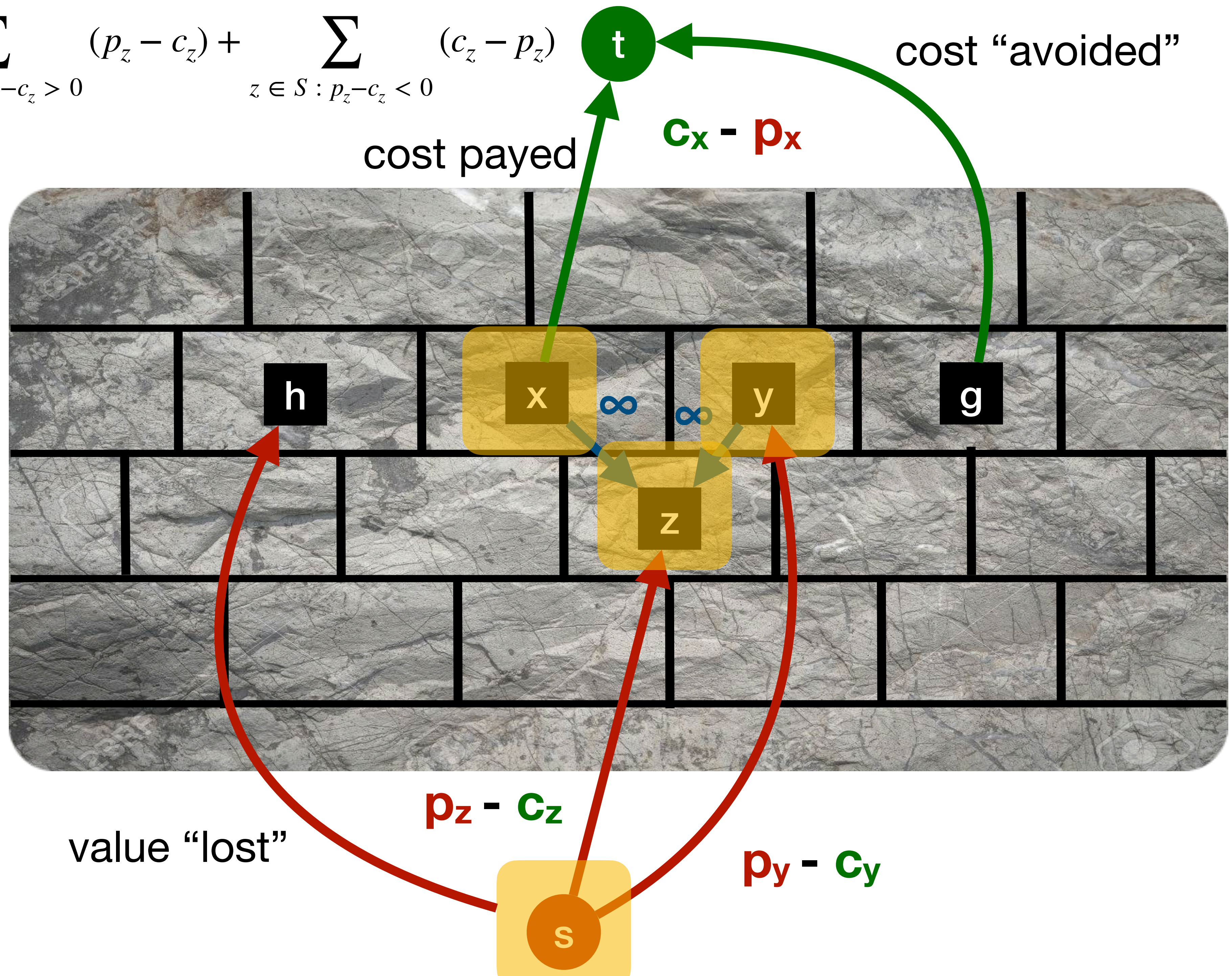

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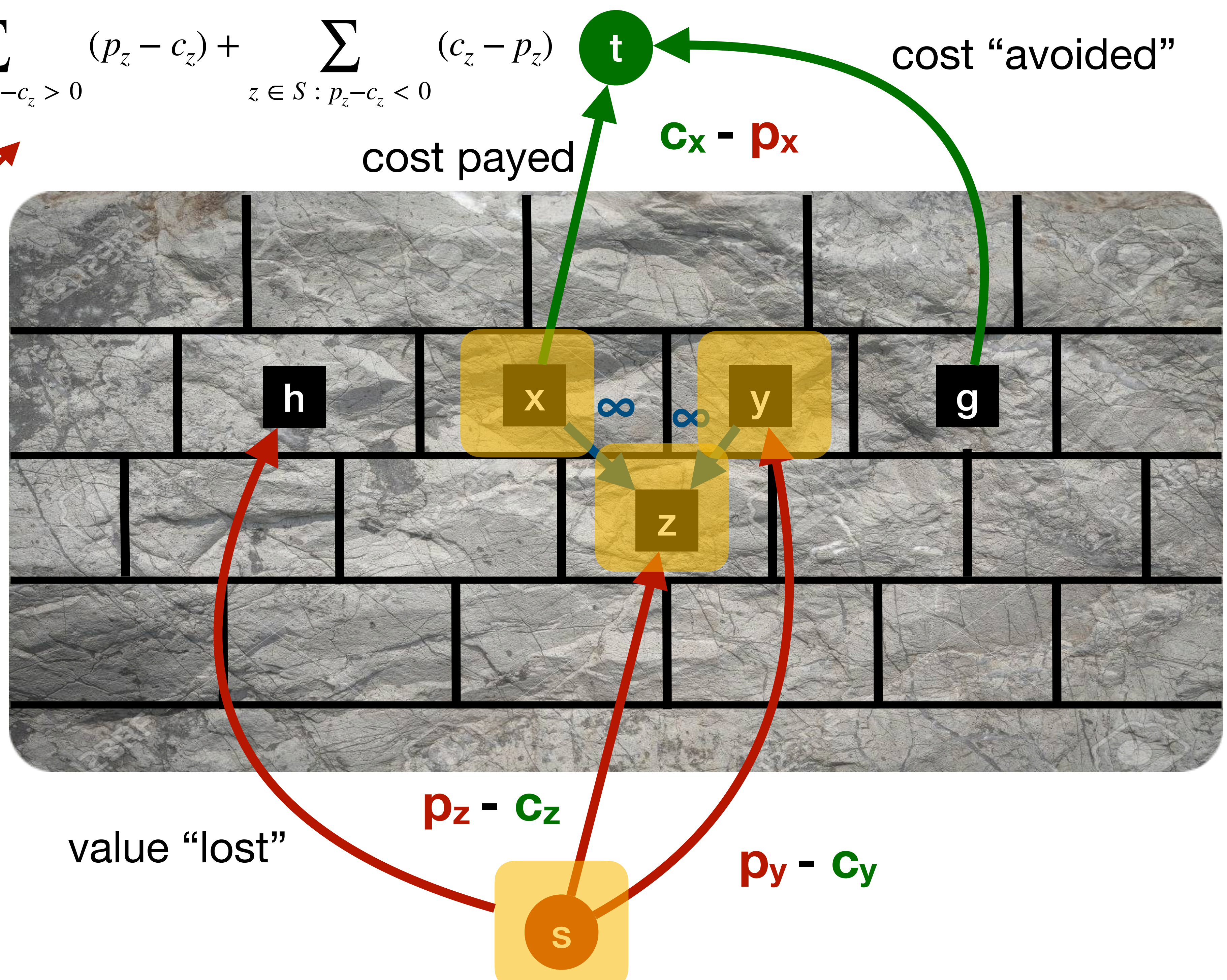
Sum of capacities of red edges crossing the cut.



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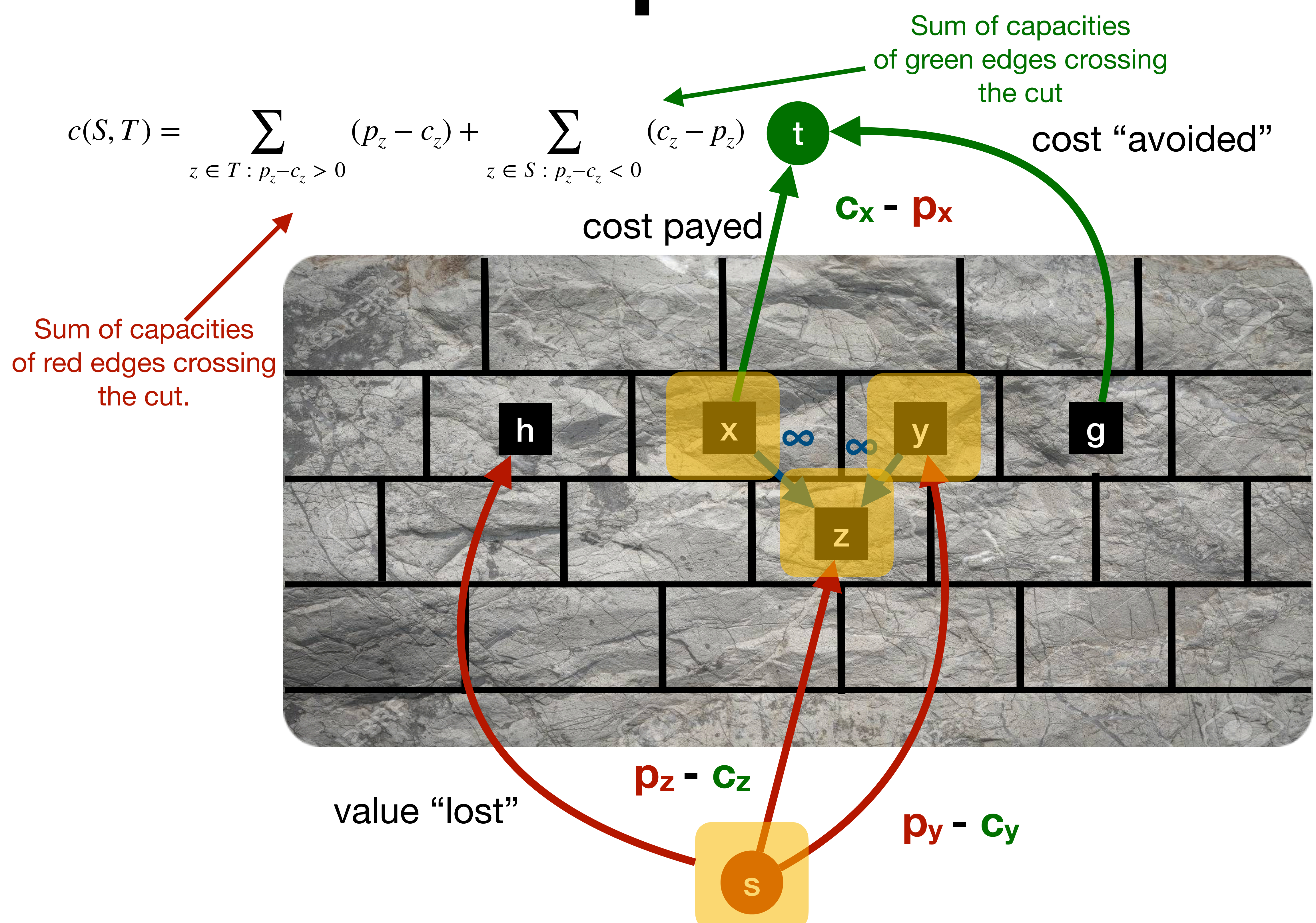

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constant

Mining profit

Open-pit mining - Summarising

Construct the flow network.

Run Ford-Fulkerson to find a maximum flow.

Find a minimum cut using the final residual graph.

Mine the blocks in the S part of the cut.