Algorithms and Data Structures

Average Case Analysis

Recall: The Quicksort algorithm

Quicksort first divides the array into two parts, such that the first part is “smaller” than the second part.

This is done via the Partition procedure.

Then it calls itself recursively.

The two parts are joined, but this is trivial.

The Partition procedure

```
Procedure Partition(A[i,...,j])

    Choose a pivot element x of A

    k = i

    For h = i to j do

        If A[h] < x

            Swap A[k] with A[h]
            k = k + 1

       Swap A[k] with A[h]

Return k
```

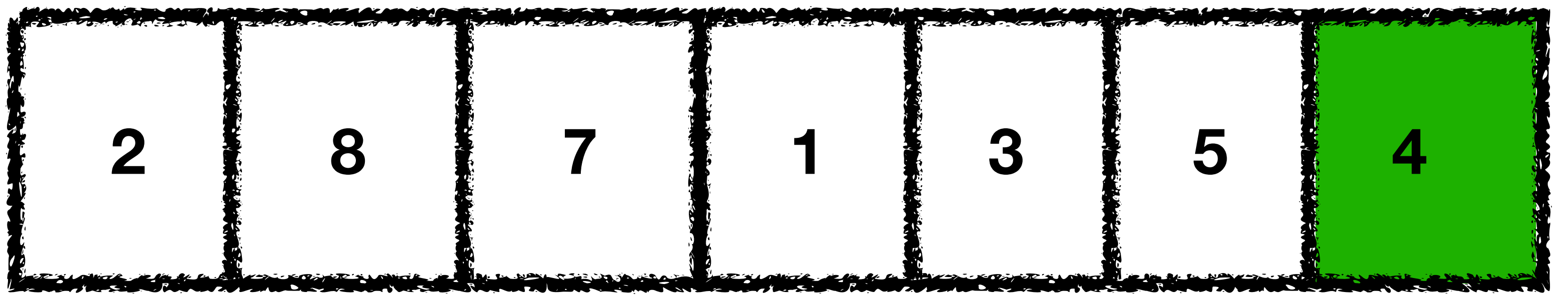
Correctness of Partition:
(CLRS p. 171-173)

Running time **O(n)**

The Quicksort algorithm

2	8	7	1	3	5	4

The Quicksort algorithm



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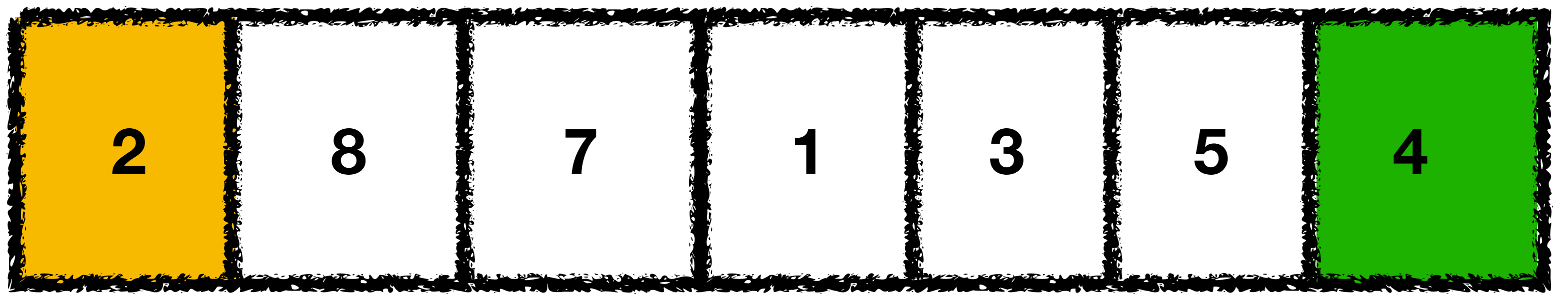

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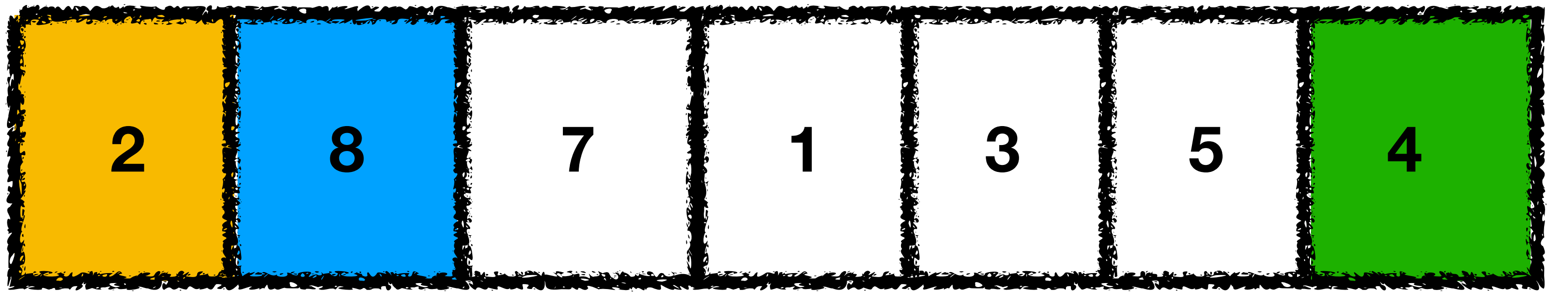

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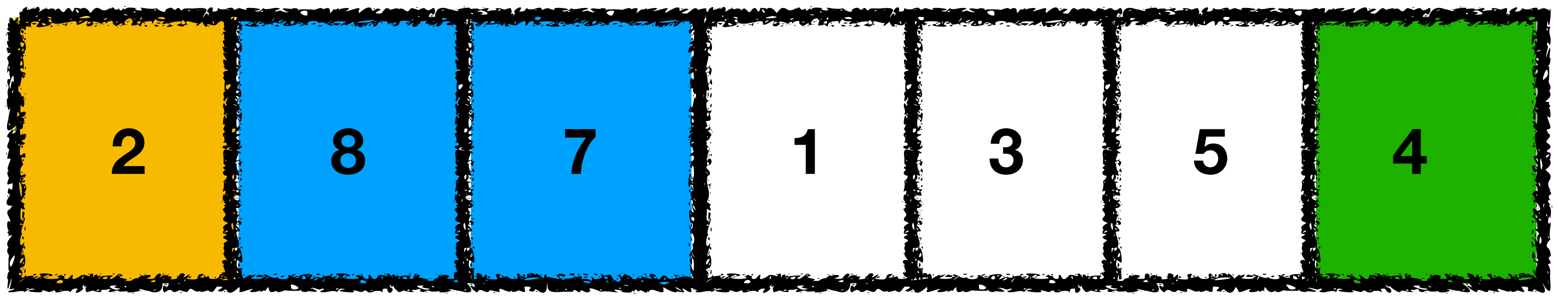

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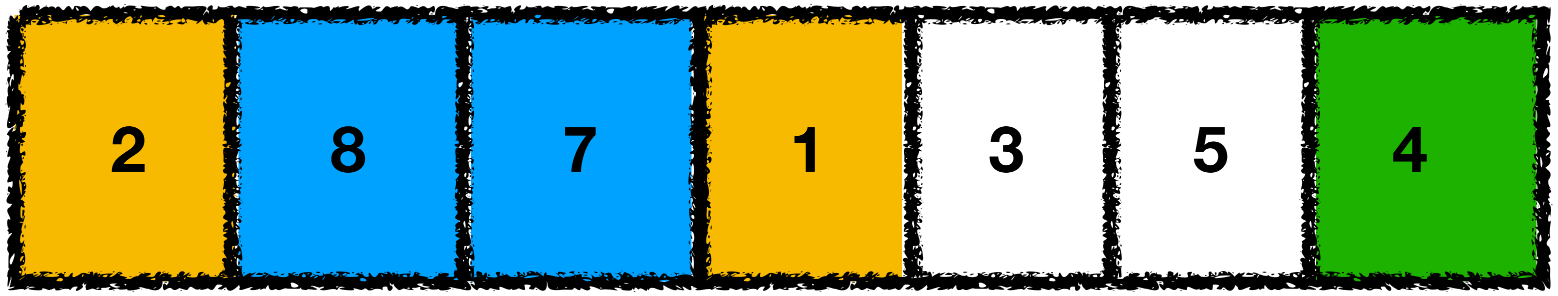

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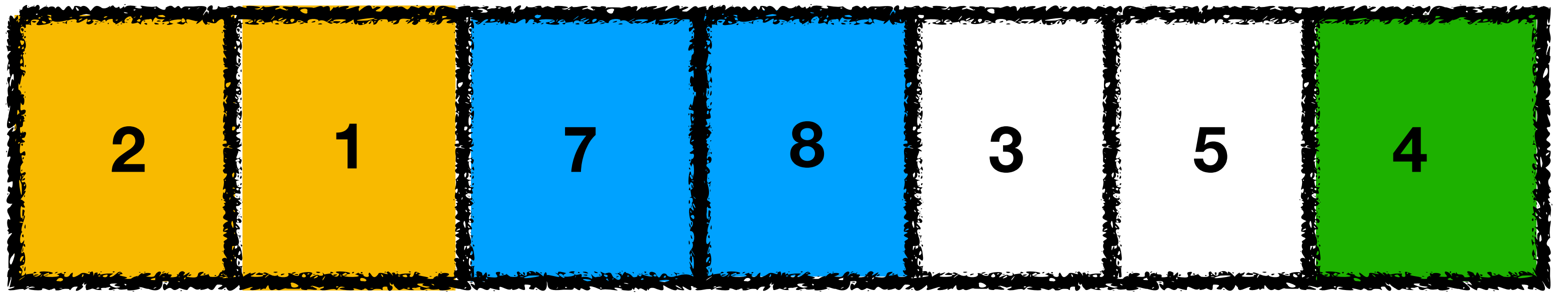

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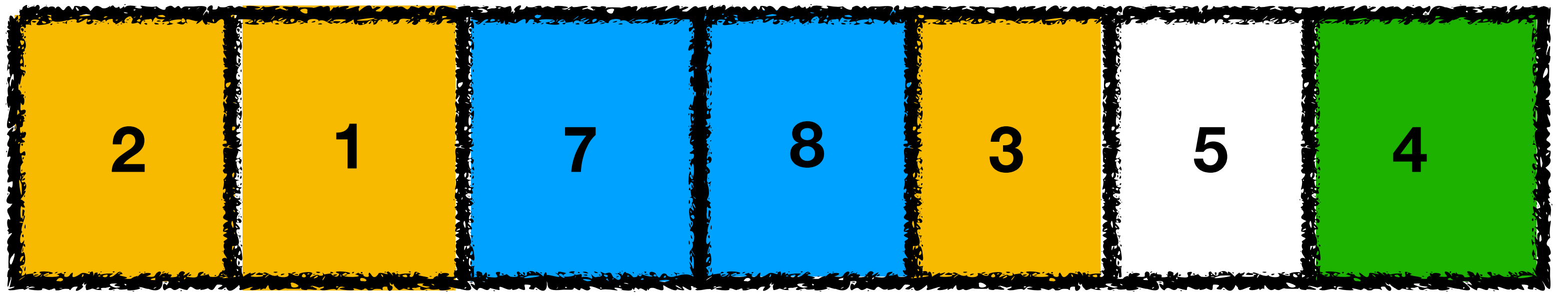

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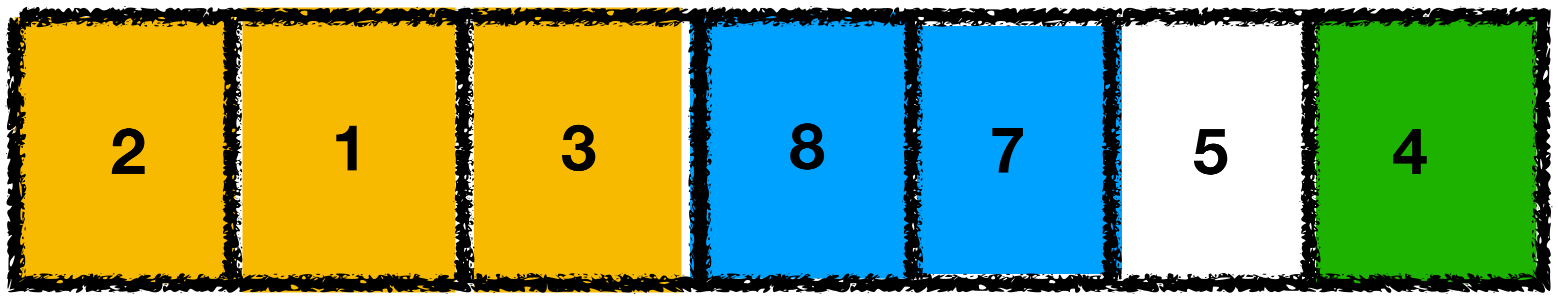

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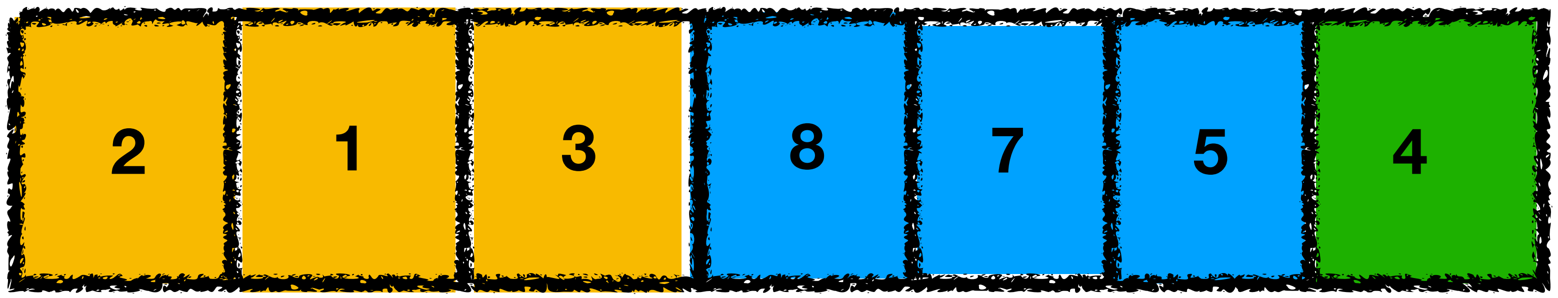

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y = **Partition**(**A**[$i,\ldots,j$])

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Running time of Quicksort

Quicksort: $T(n) \leq T(n_1) + T(n_2) + cn$

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This is the best-case running time.

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When $n_1 = n - 1$ $n_2 = 0$, we get $T(n) \leq T(n-1) + cn$ and the running time is $O(n^2)$.

This is the worst-case running time.

Running time of Quicksort

Quicksort: $T(n) \leq T(n_1) + T(n_2) + cn$

What about the average-case running time?

Worst vs Best vs Average Case

Convention: When we say "the running time of Algorithm A", we mean the worst-case running time, over all possible inputs to the algorithm.

We can also measure the best-case running time, over all possible inputs to the problem.

In between: average-case running time.

Running time of the algorithm on inputs which are chosen at random from some distribution.

The appropriate distribution depends on the application (usually the uniform distribution - all inputs equally likely).

Running time of Quicksort

Quicksort: $T(n) \leq T(n_1) + T(n_2) + cn$

What about the average-case running time?

Assume that the input sequence of n numbers is drawn uniformly at random from a distribution over all $n!$ possible inputs.

Unbalanced Partitions

Quicksort: $T(n) \leq T(n_1) + T(n_2) + cn$

Assume that we use a pivot element that results in a 9-to-1 split, i.e., $n_1 = 9n/10$ and $n_2 = n/10$.

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Q: Can you work out what the recurrence relation evaluates to? Use the unrolling technique.

Unbalanced Partitions

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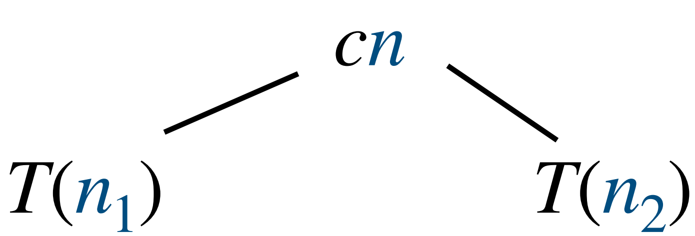
Main message: Bad partitions are rather unlikely to happen. Most partitions are good partitions.

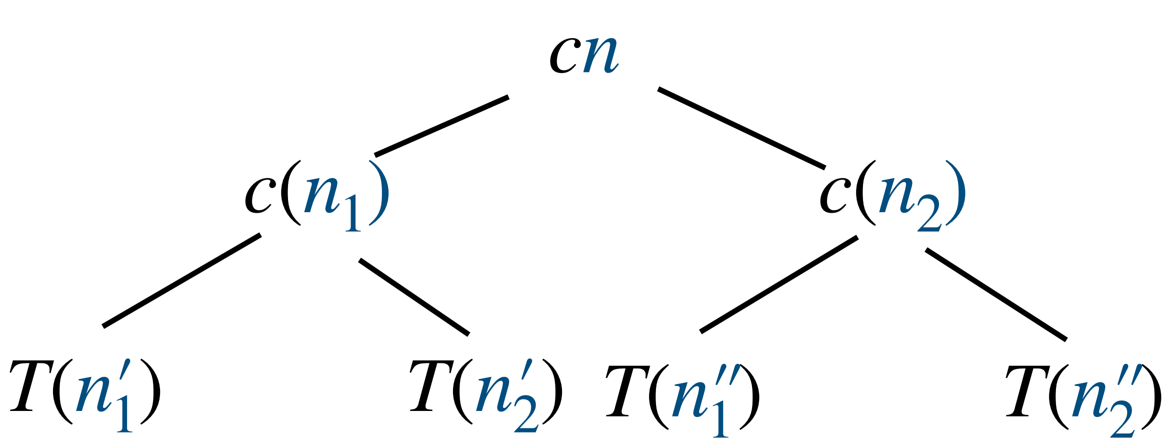
For the sake of intuition

Consider the recursion tree of **Quicksort**.

First iteration



Second iteration



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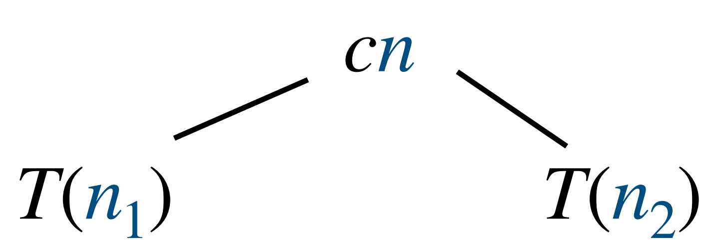

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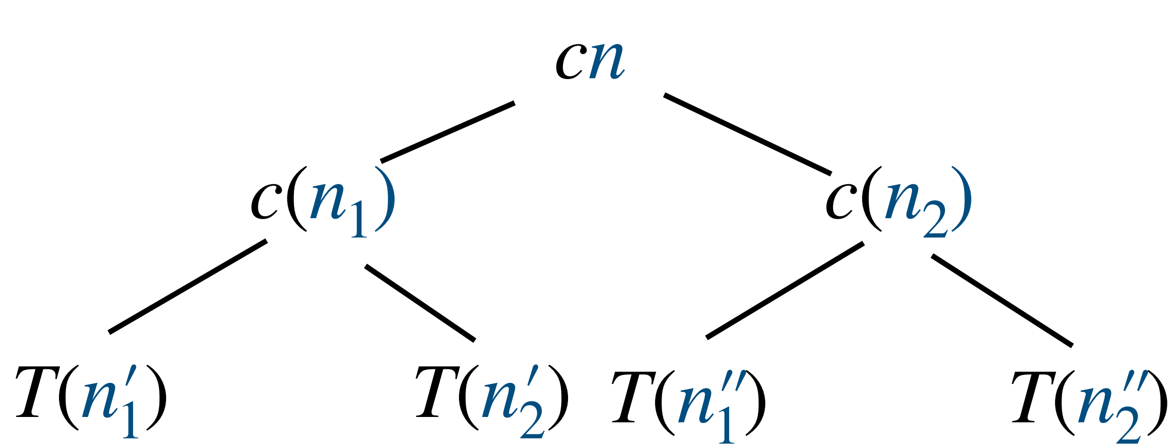

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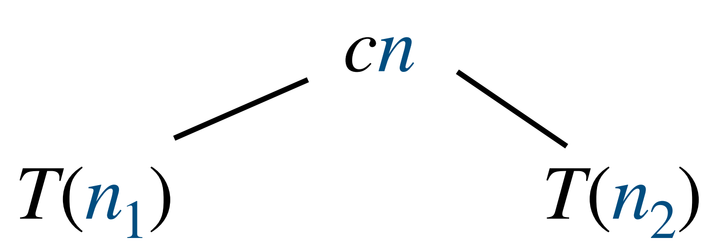

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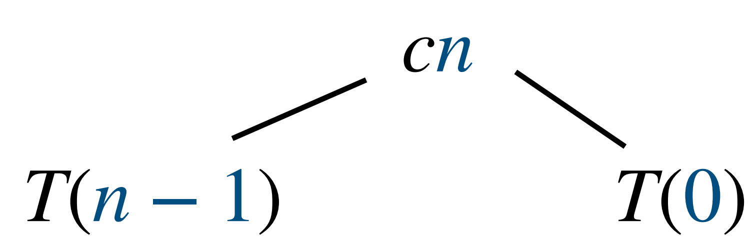

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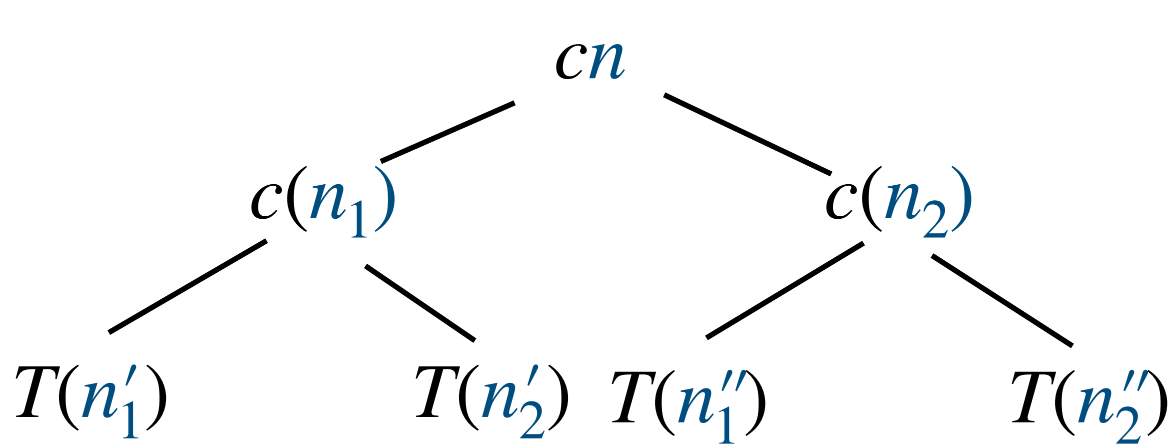

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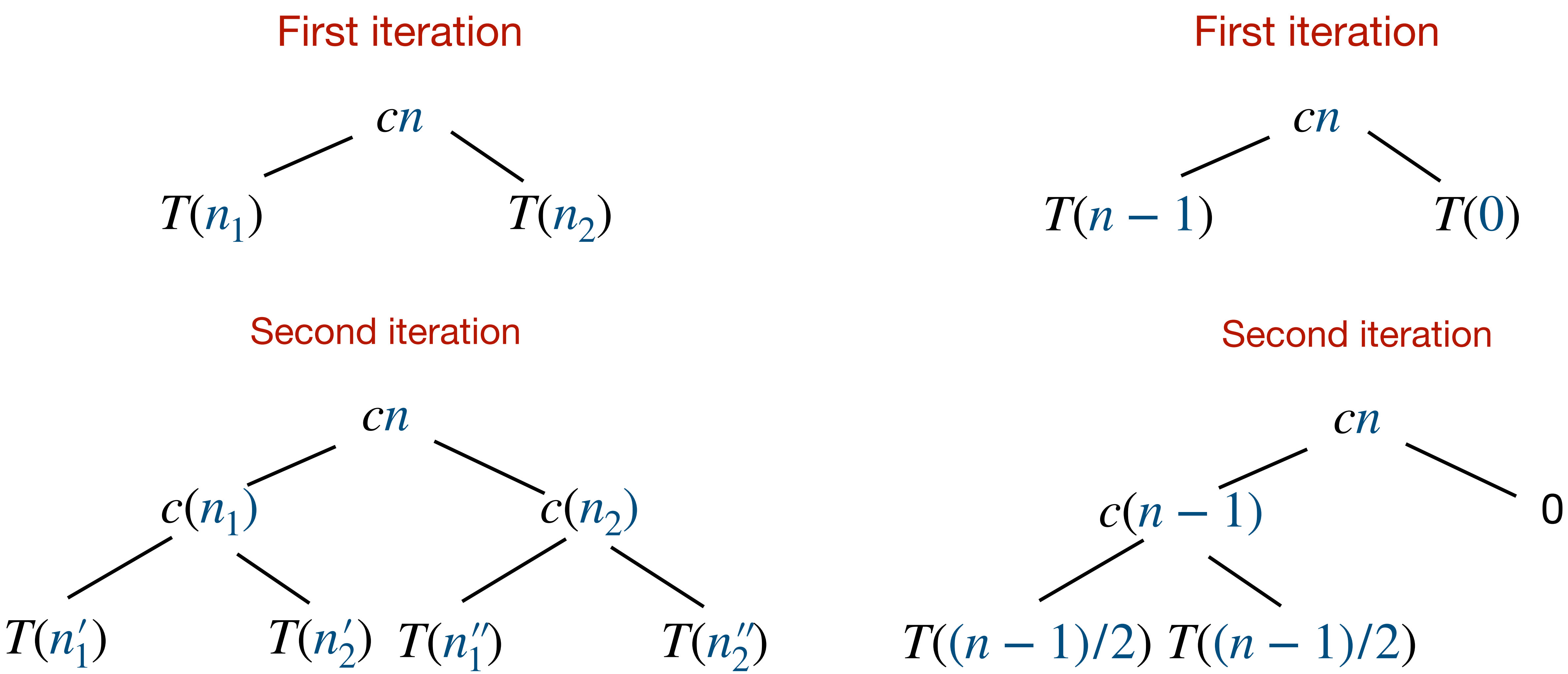

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$$T(n) \leq T(n-1) + cn \leq 2T(n-1/2) + c(2n-1)$$

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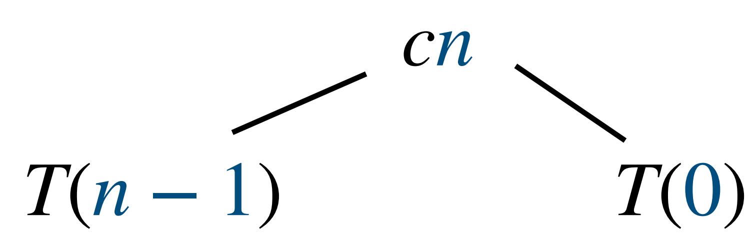

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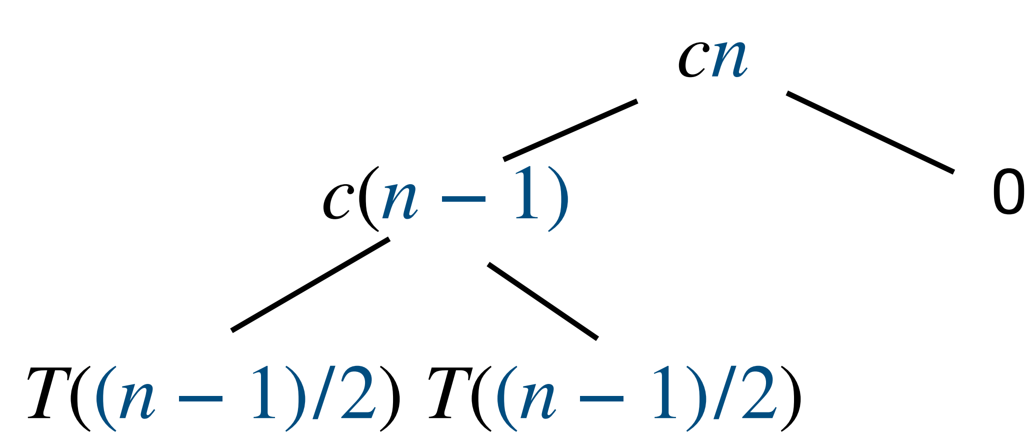

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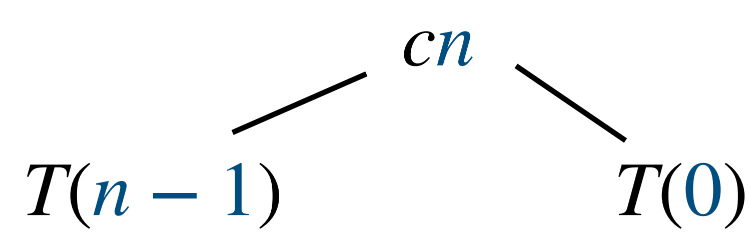

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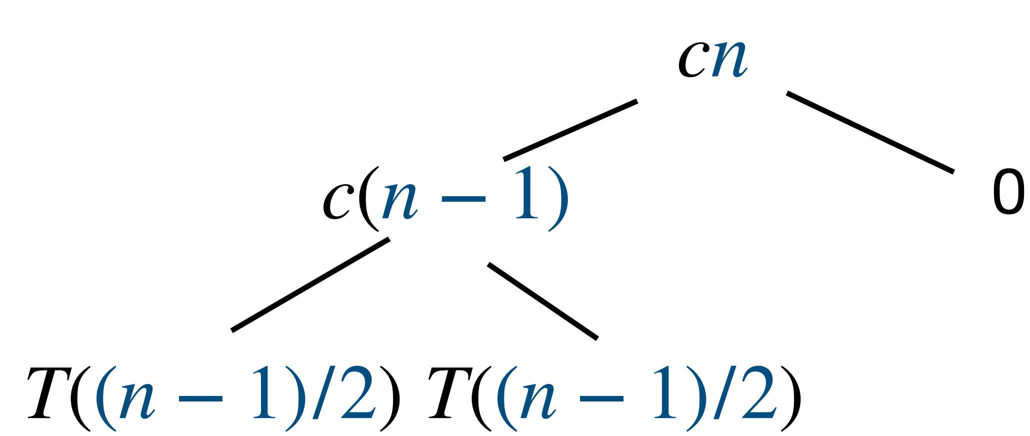

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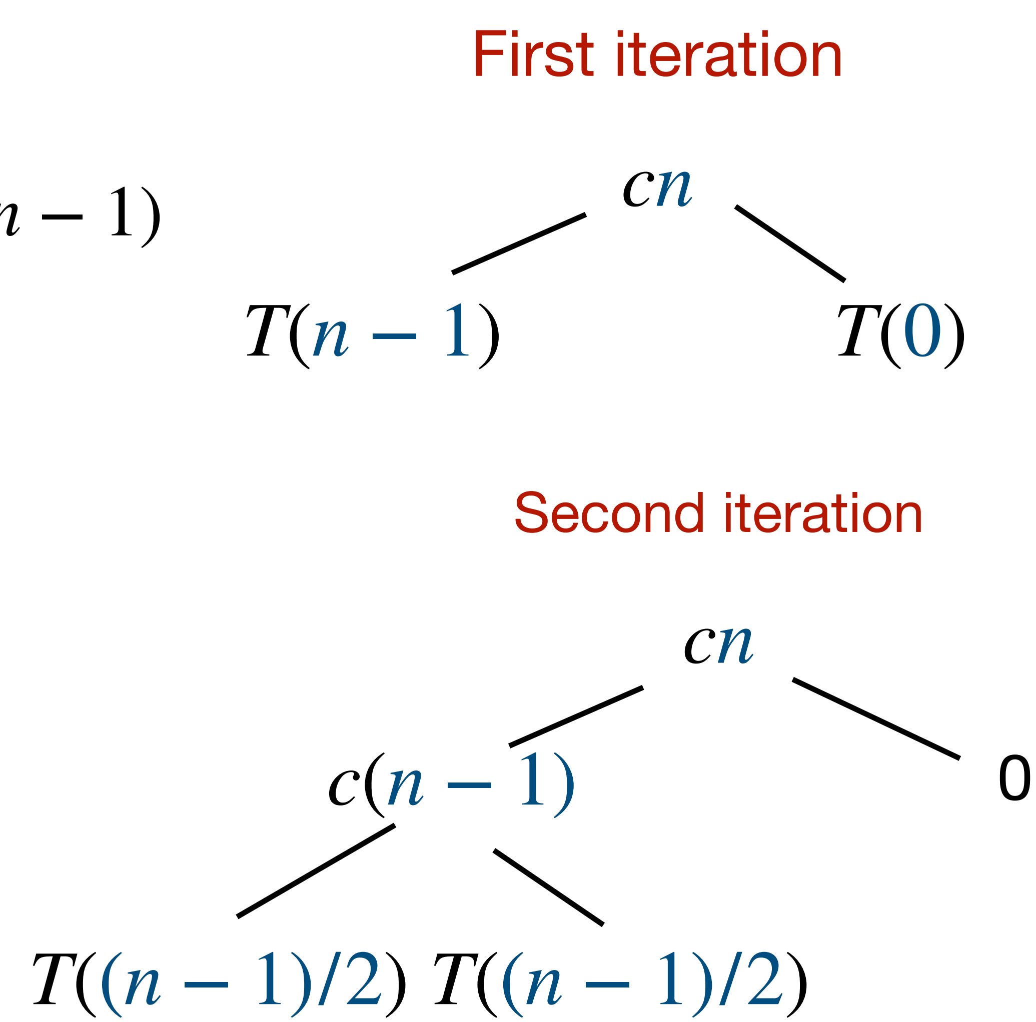

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We only pay extra in constants.

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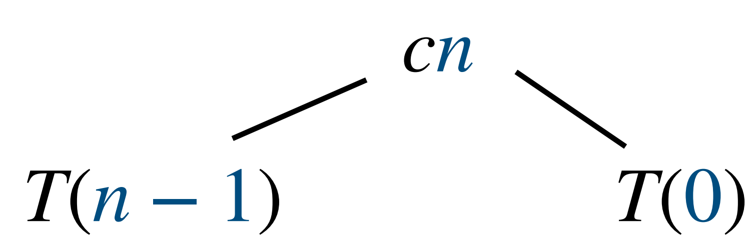

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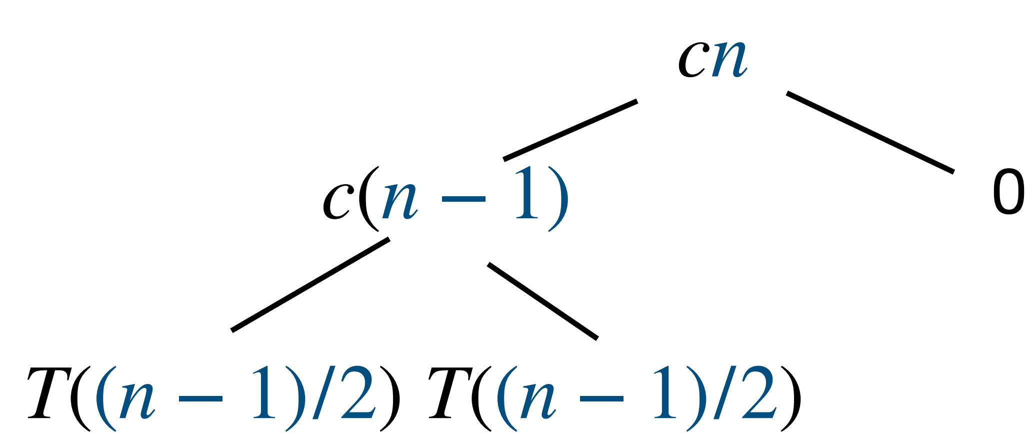

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Choose a **pivot element** **x** of **A**

$k = i$

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How many operations in total? $O(n + X)$

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We need to compute its expectation $\mathbb{E}[X]$.

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Let $Z_{ij} = \{z_i, z_{i+1}, \ldots, z_j\}$ contain the elements of a subsequence of the sorted array.

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Lemma: During the execution of the algorithm, an element z_i is compared with an element z_j, where $i < j$ iff one of them is chosen as the pivot before any other element in the set Z_{ij}. Moreover, no two elements are ever compared more than once.

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Proof:

$\Leftarrow$ If none of z_i and z_j is chosen as the pivot before any other element $z \in Z_{ij}$, then they are not compared with each other.

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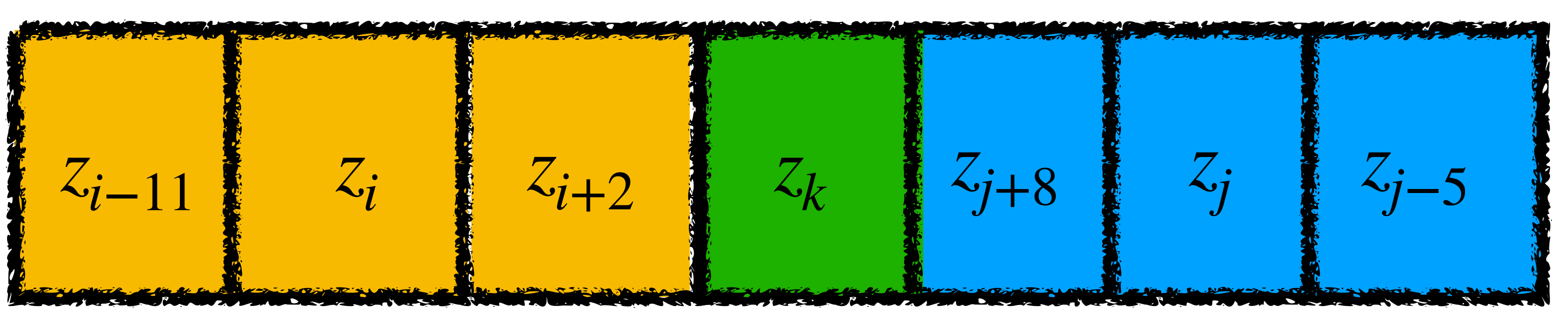

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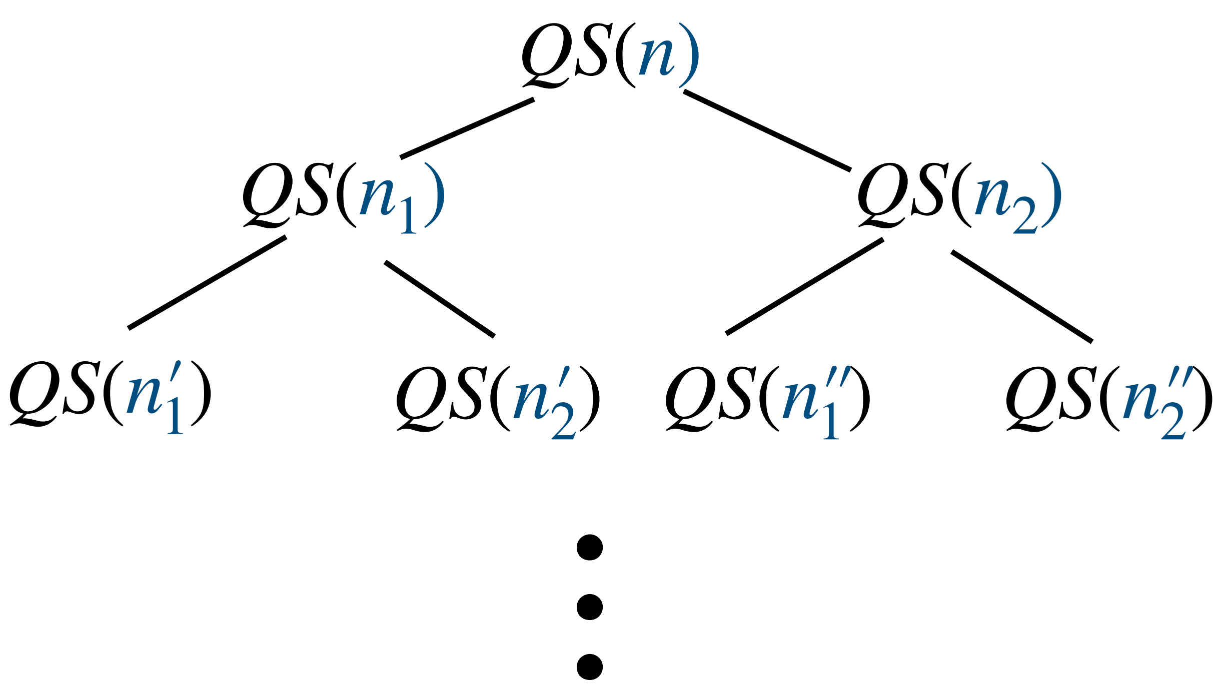

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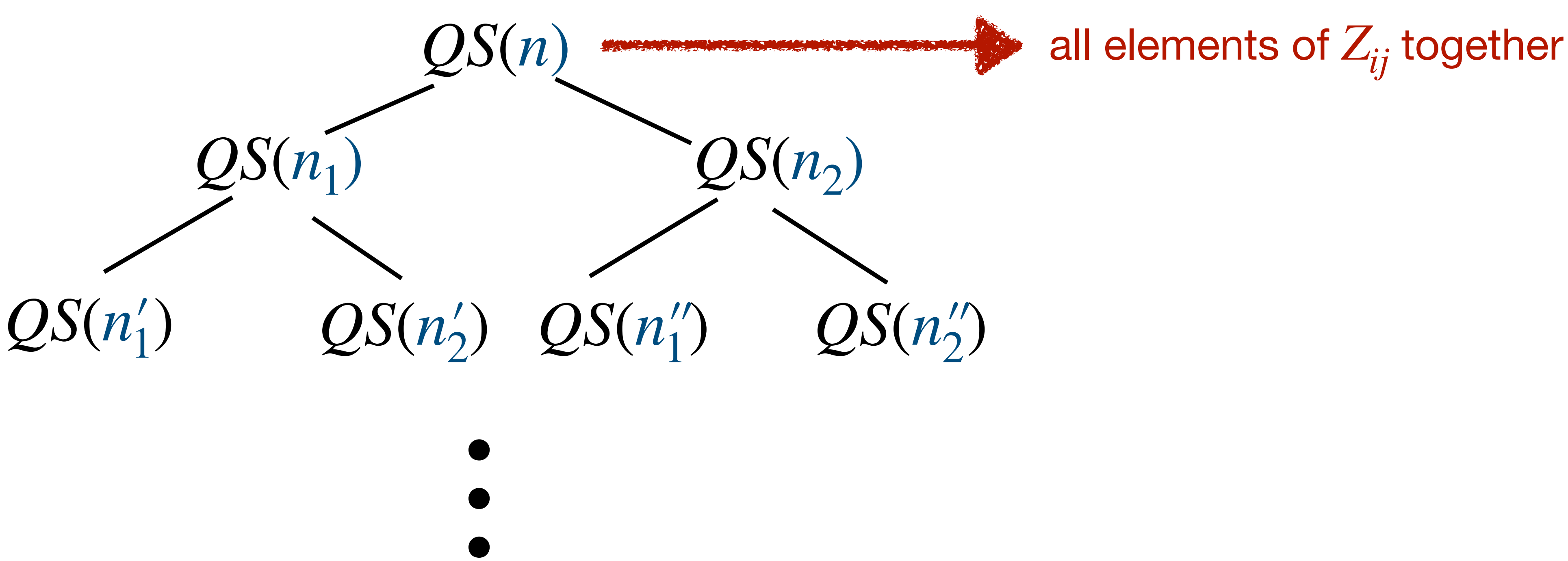

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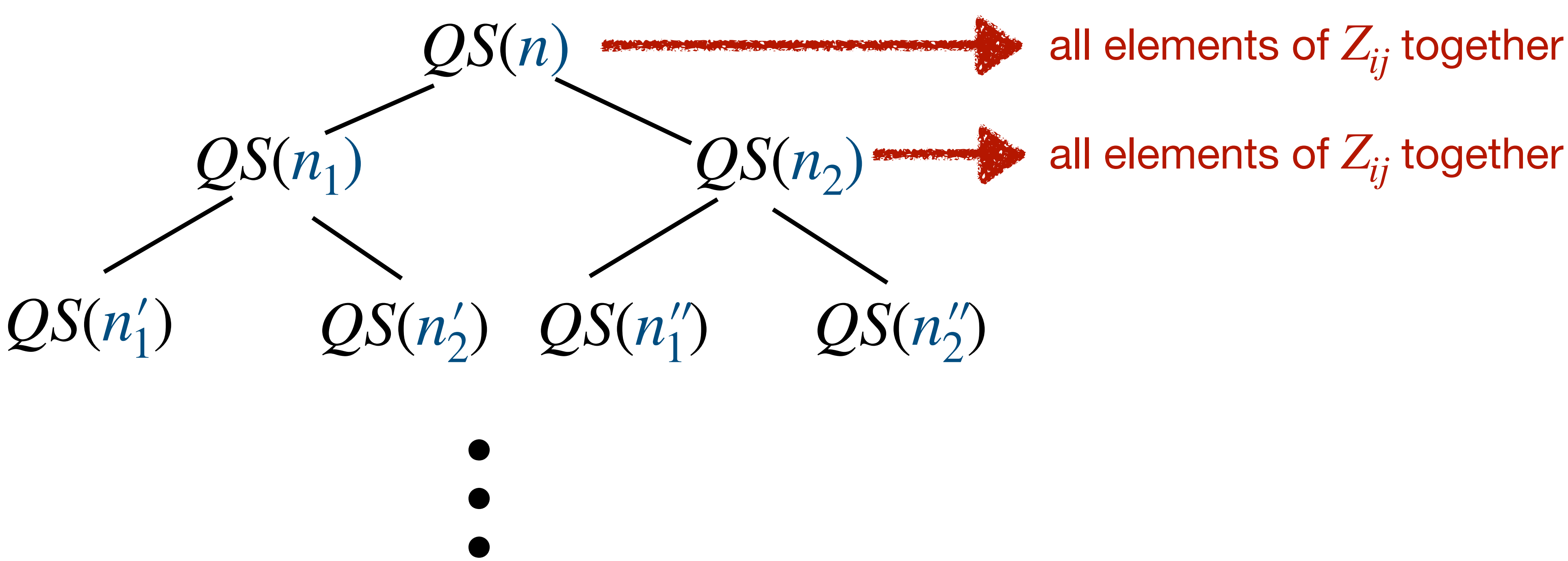

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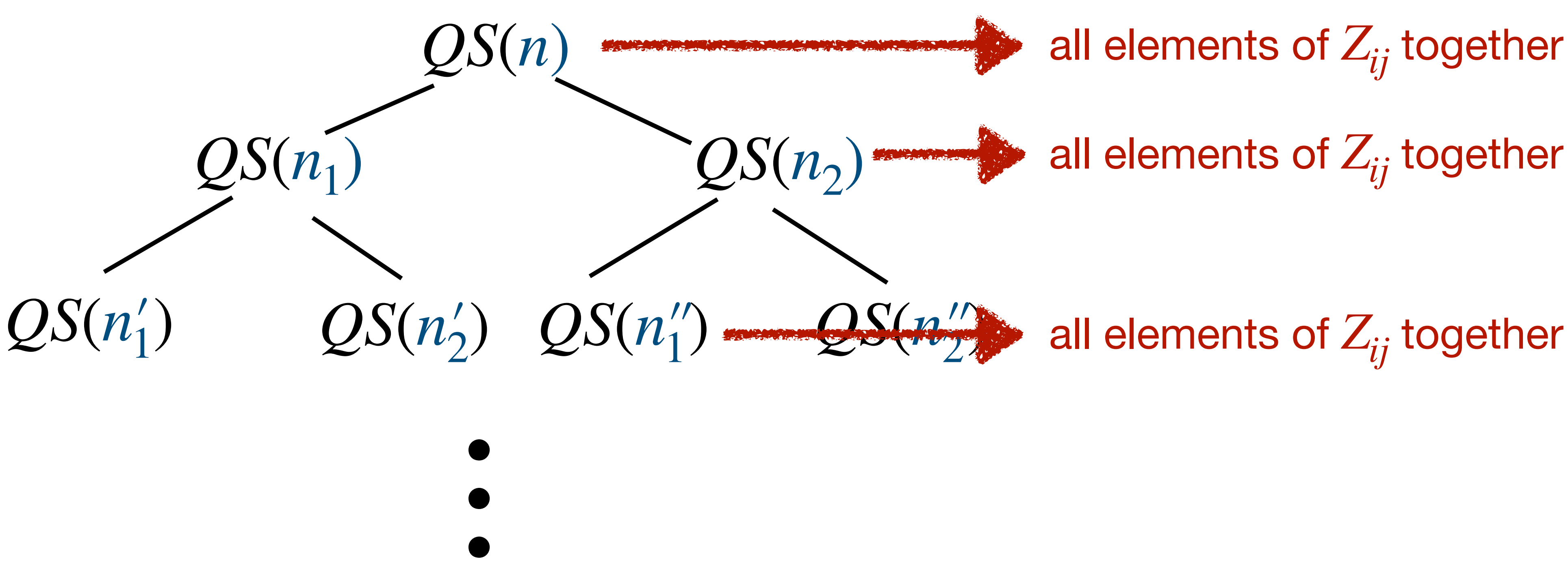

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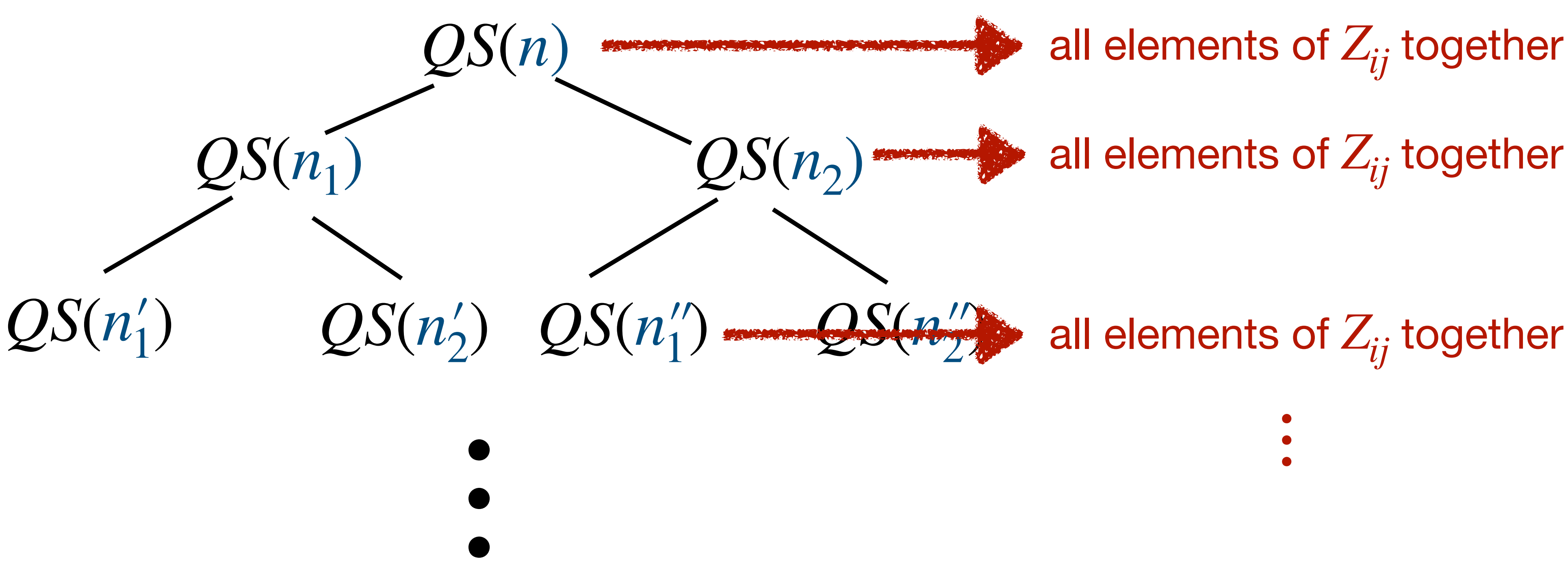

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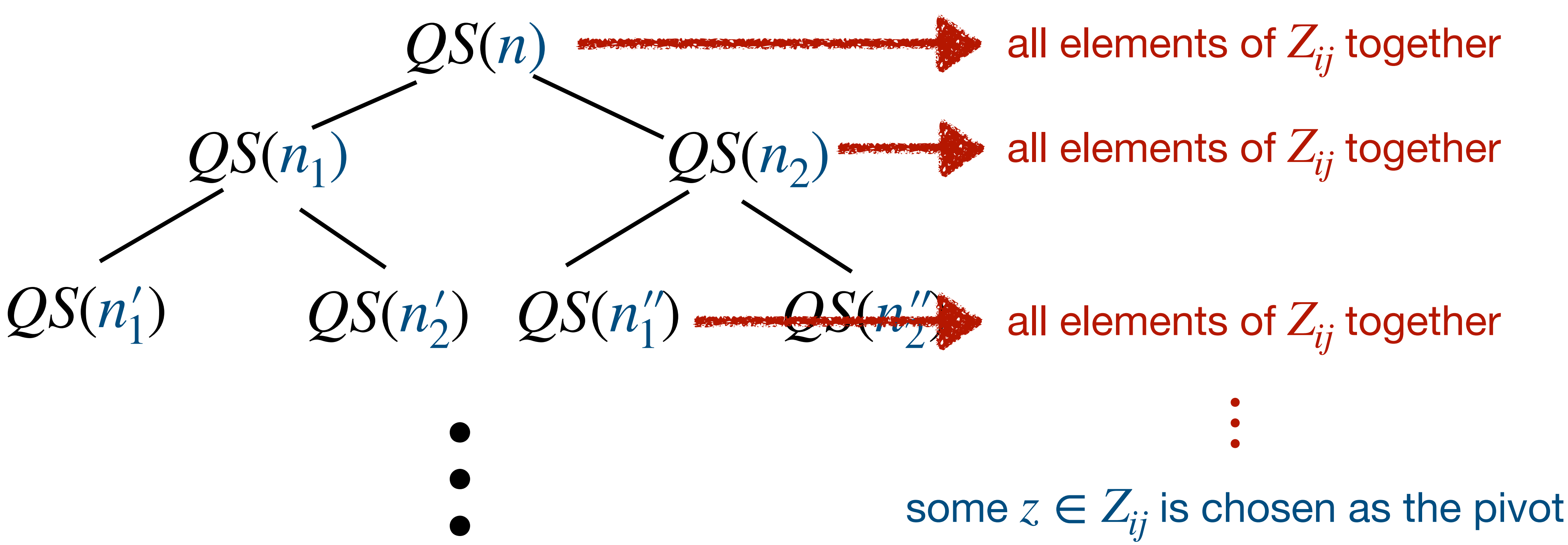

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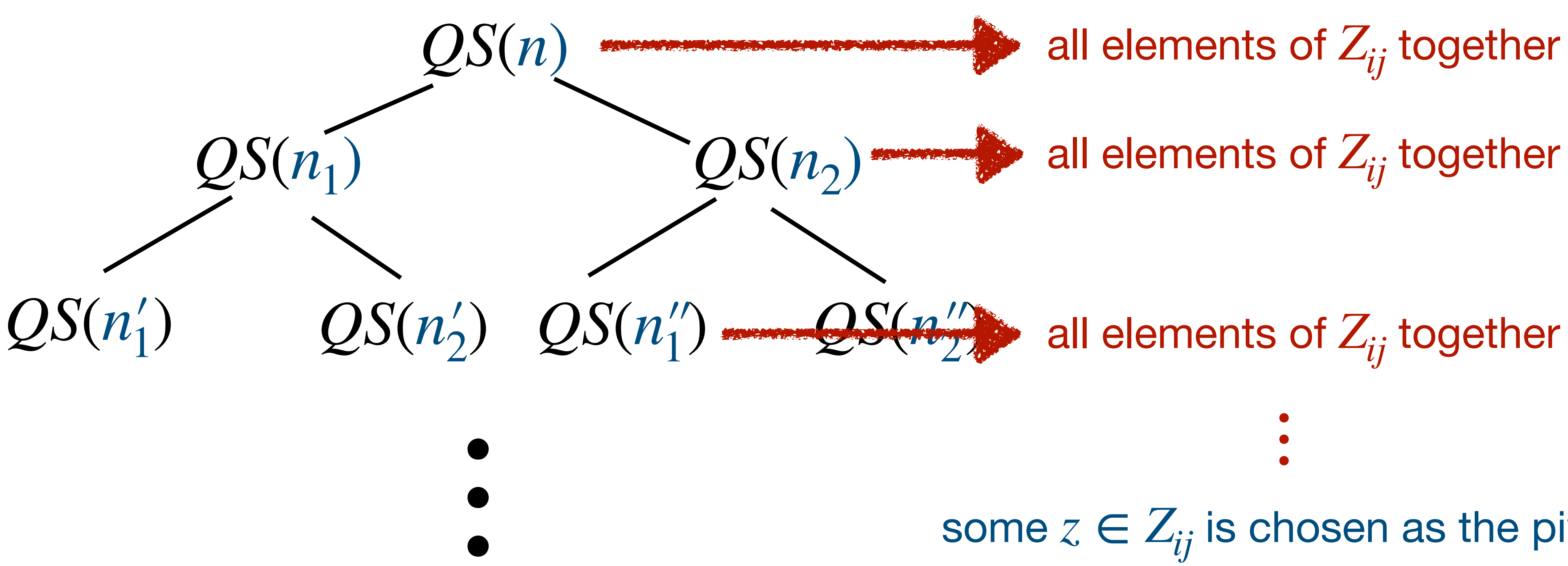

some $z \in Z_{ij}$ is chosen as the pivot

at this point every $z \in Z_{ij}$ is equally likely to be chosen

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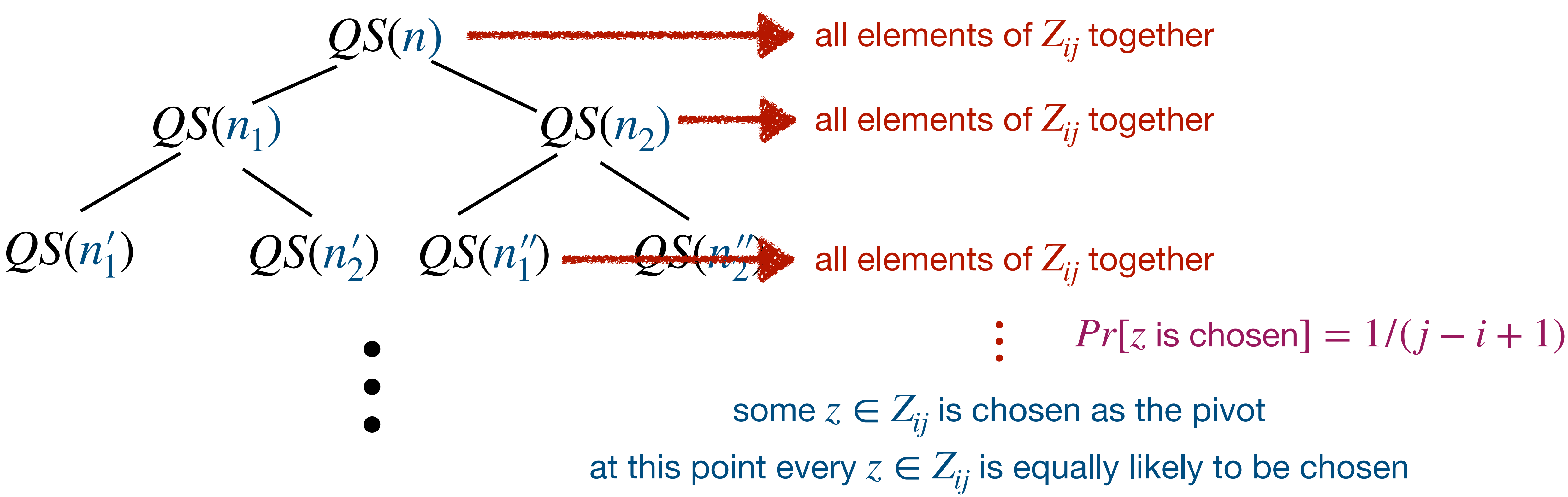

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$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \quad \text{and} \quad \mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right]$$

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$$= O(n \log n)$$

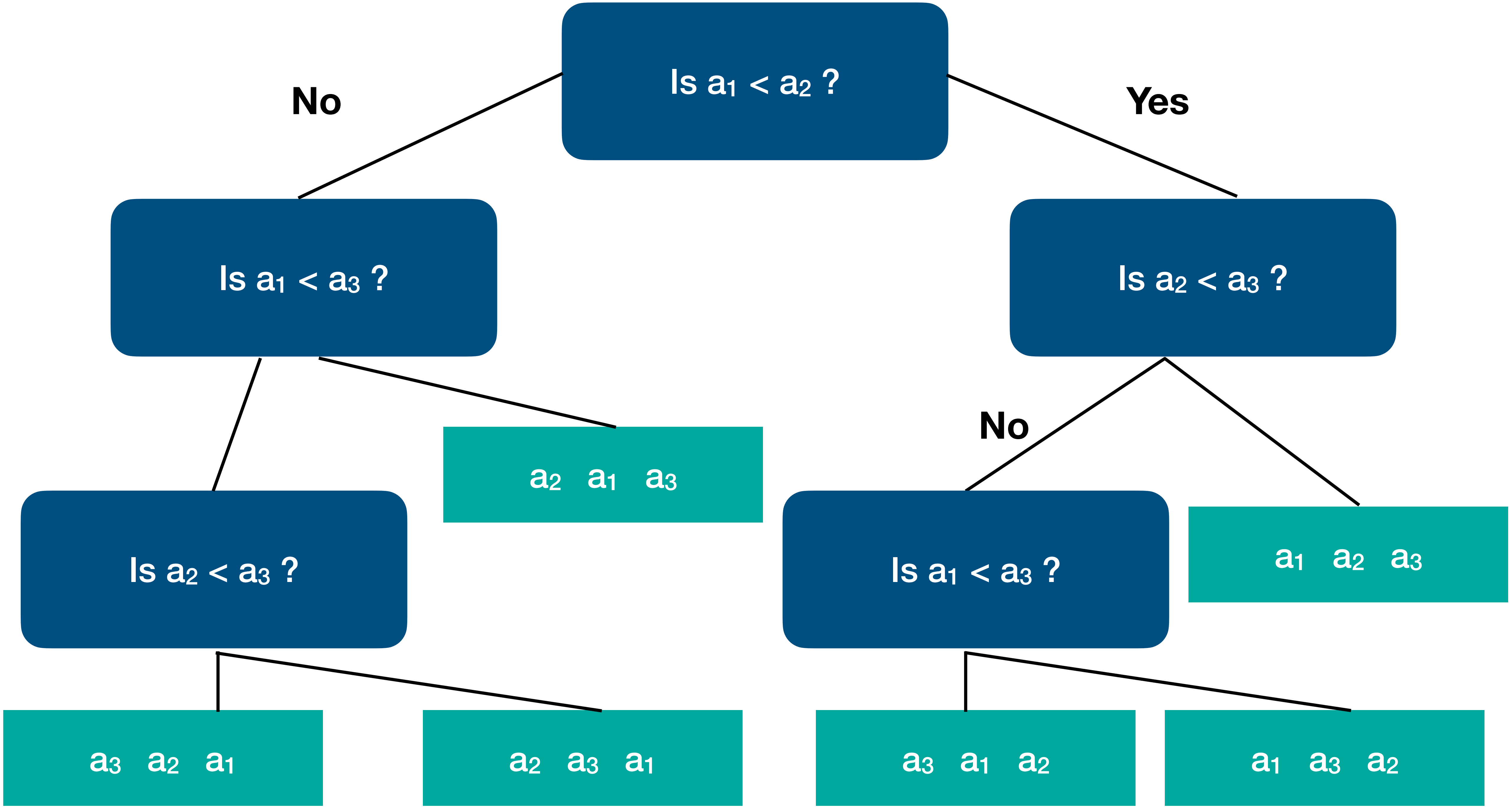
The Quicksort algorithm

```
Procedure Partition(A[i,...,j])

    Choose a pivot element x of A

    k = i

    For h = i to j do

        If A[h] < x

            Swap A[k] with A[h]
            k = k + 1

        Swap A[k] with A[h]

Return k
```

```
Algorithm Quicksort(A[i,...,j])

    y = Partition(A[i,...,j])
        Quicksort(A[i,...,y-1])
        Quicksort(A[y+1,...,j])
```

Q: Can you think of a version of the algorithm that will have worst-case running time $O(n \log n)$?

Lower bound for sorting



Lower bound for sorting

We need as many comparisons as the *depth* of the tree (length of the longest path from the root to the leaves).

The decision tree has $n!$ leaves

A leaf is a permutation of $\{a_1, a_2, \ldots, a_n\}$

Every possible permutation can appear as a leaf, since every possible permutation is a valid output.

Lower bound for sorting

Fact: Every binary tree of depth d has at most 2^d leaves.

Therefore the minimum number of comparisons is $\log_2(n!)$

We claim that $\log_2(n!) = \Omega(n \log n)$

$$
\begin{aligned}
\log_2(n!) &= \log_2(1 \cdot 2 \cdot , \ldots, \cdot n) \\
&= \log_2(1) + \log_2(2) + \ldots + \log_2(n) \\
&\geq \log_2(n/2) + \ldots + \log_2(n) \quad \textit{(half)} \\
&\geq \log_2(n/2) + \ldots + \log_2(n/2) = (n/2)\log_2(n/2)
\end{aligned}
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Worst-case lower bound for sorting

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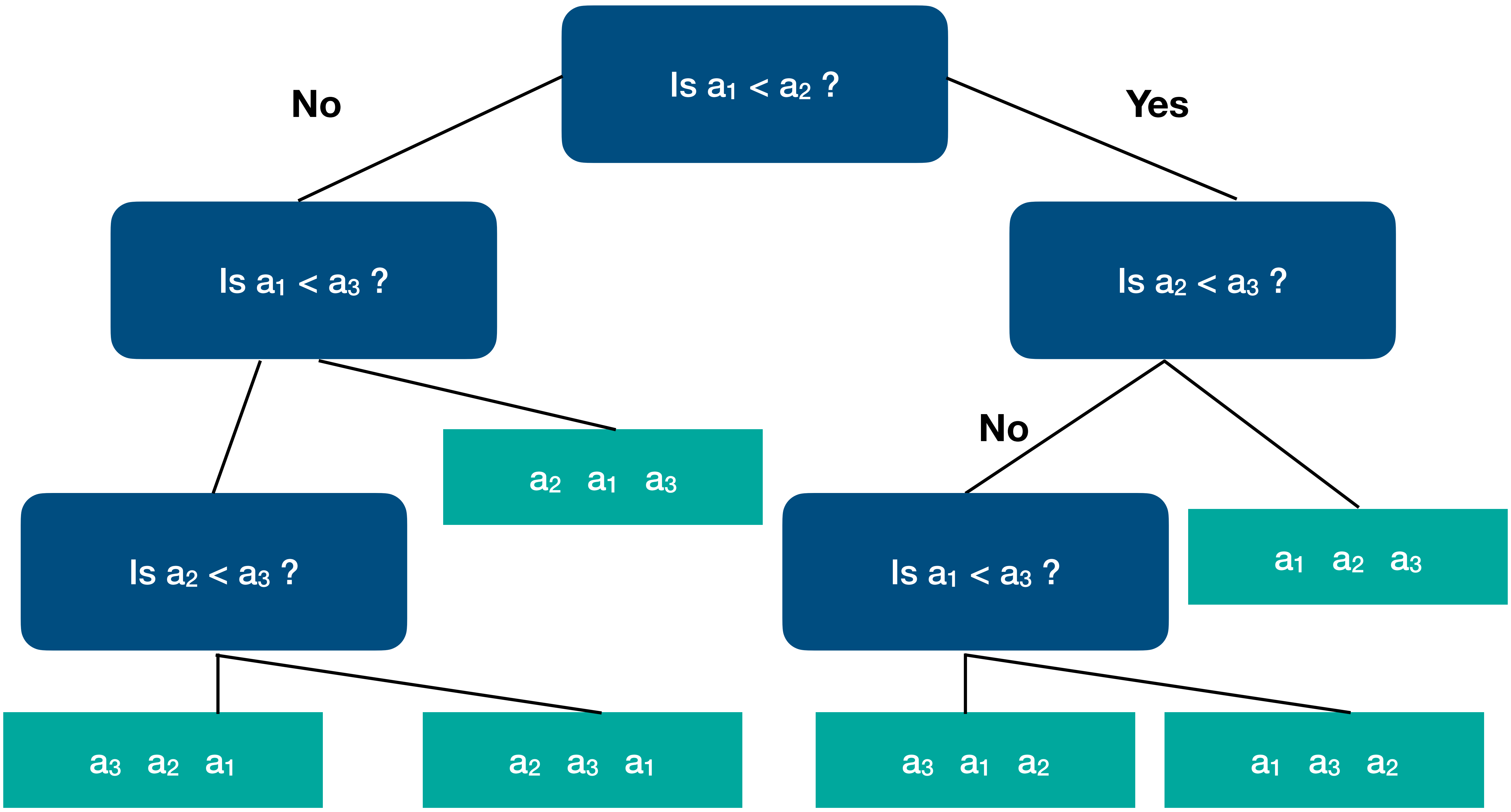
We need as many comparisons as the *average depth* of the tree (average length of the longest path from the root to a the leaves).

Average depth



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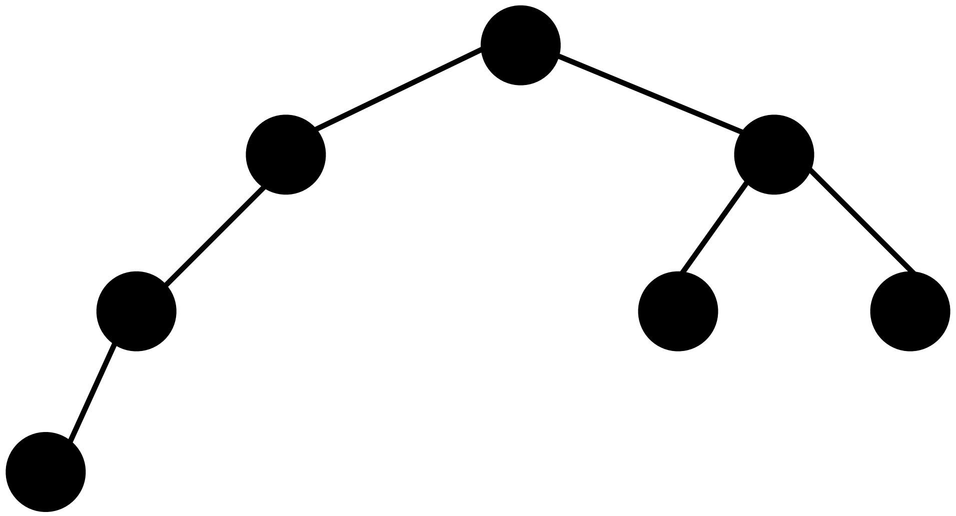

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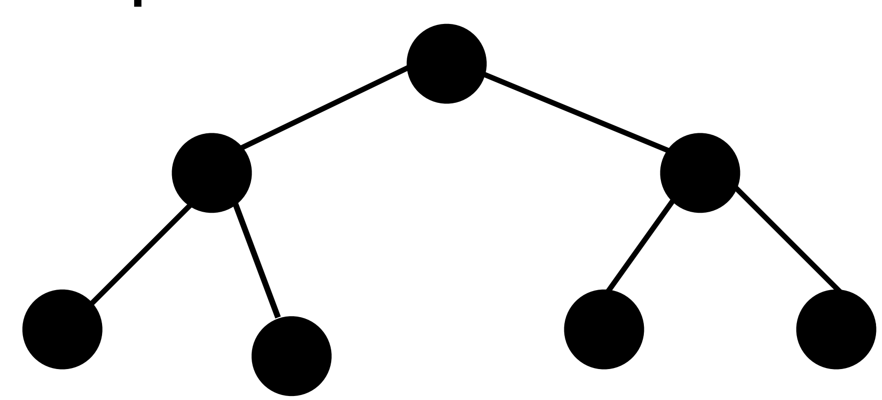

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The depth of a balanced tree is $\Theta(\log_2 n!)$ and the analysis goes through as before.