# Algorithms and Data Structures

## The Greedy Approach and Minimum Spanning Trees

# The Greedy approach

- The goal is to come up with a global solution.
- The solution will be built up in small consecutive steps.
- For each step, the solution will be the best possible myopically, according to some criterion.

# Graph Theory Basics

# Graph Definitions

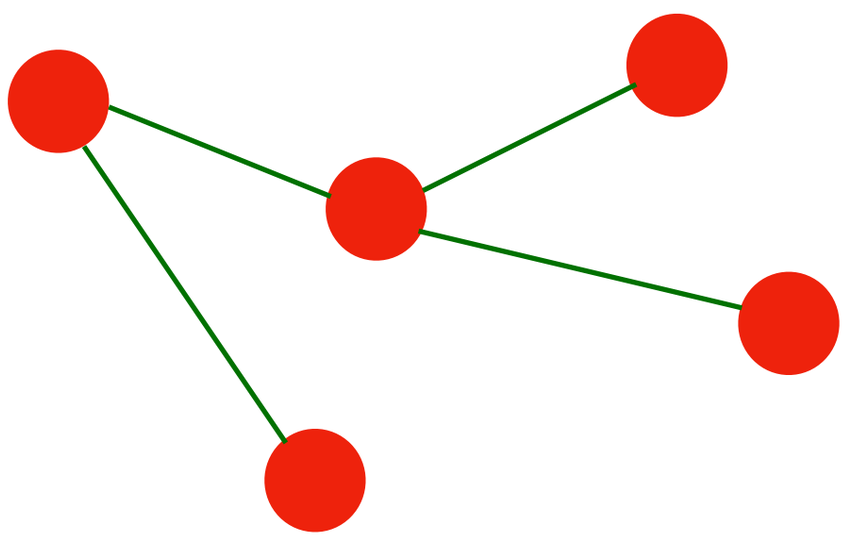
Graph $G = (V,E)$

Set of nodes (or vertices) $V$, with $|V| = n$

Set of edges $E$, with $|E| = m$

Undirected: edge $e = \{v,w\}$

Directed: edge $e = (v,w)$



# Graph Definitions

**Neighbours of $v$ :** Set of nodes connected by an edge with $v$

**Degree of a node:** number of neighbours

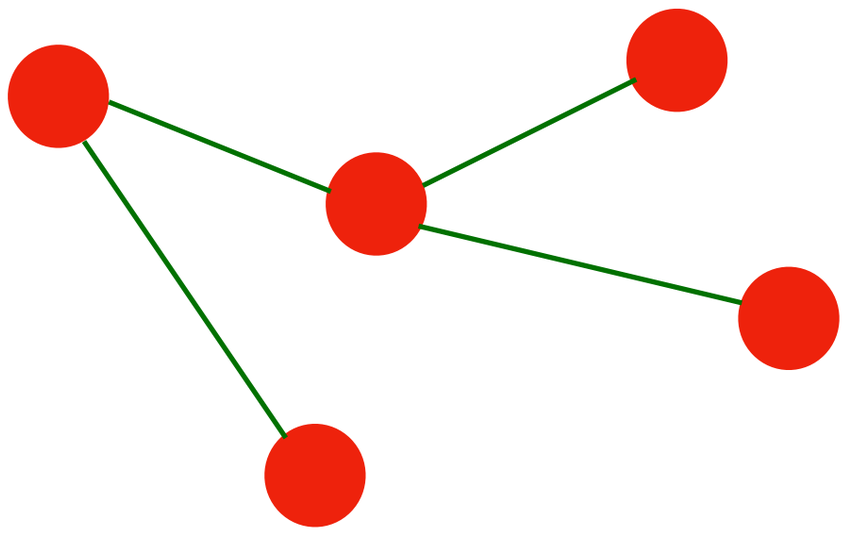
Directed graphs: *in-degree* and *out-degree*

**Path:** A sequence of (non-repeating) nodes with consecutive nodes being connected by an edge.

Length: # nodes - 1

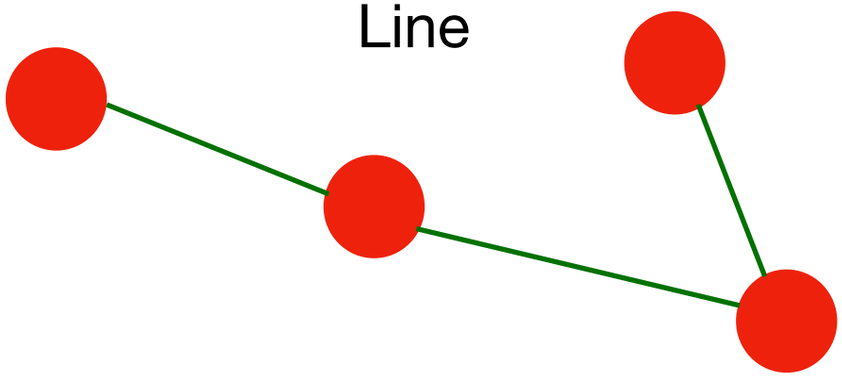
**Distance between $u$ and $v$ :** length of the shortest path $u$ and $v$,

**Graph diameter:** The longest distance in the graph

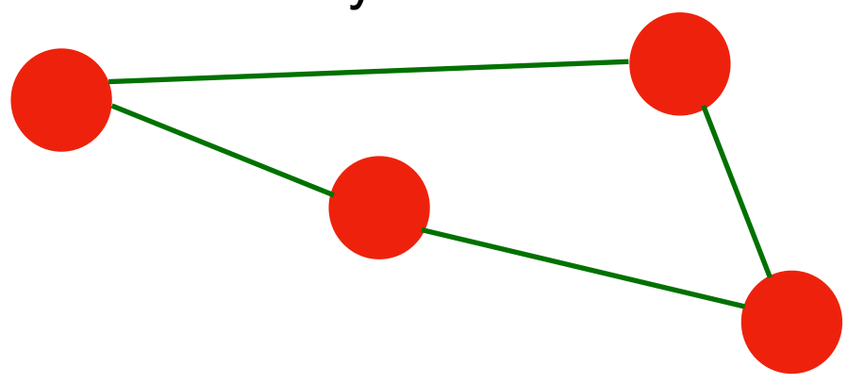


# Lines, cycles, trees and cliques

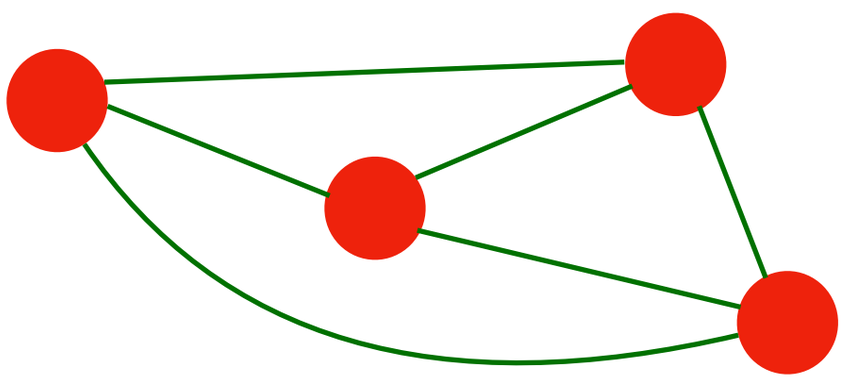
Line



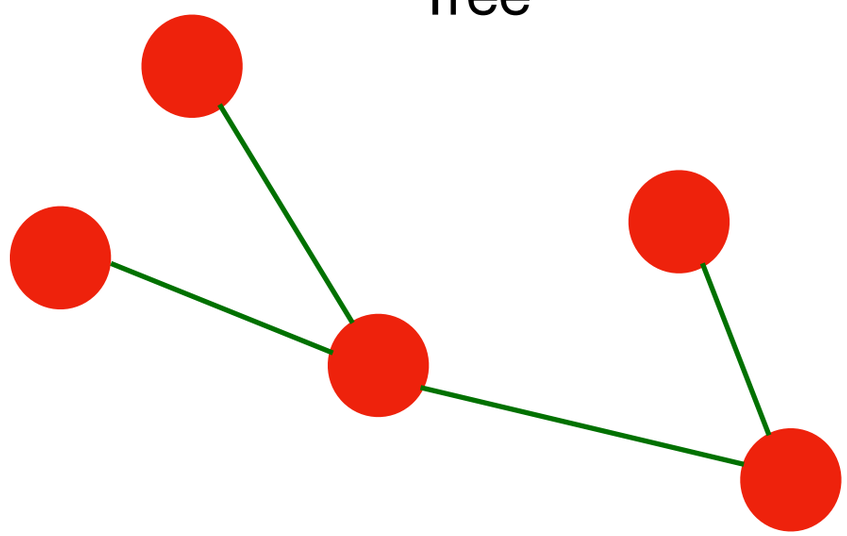
Cycle



Clique

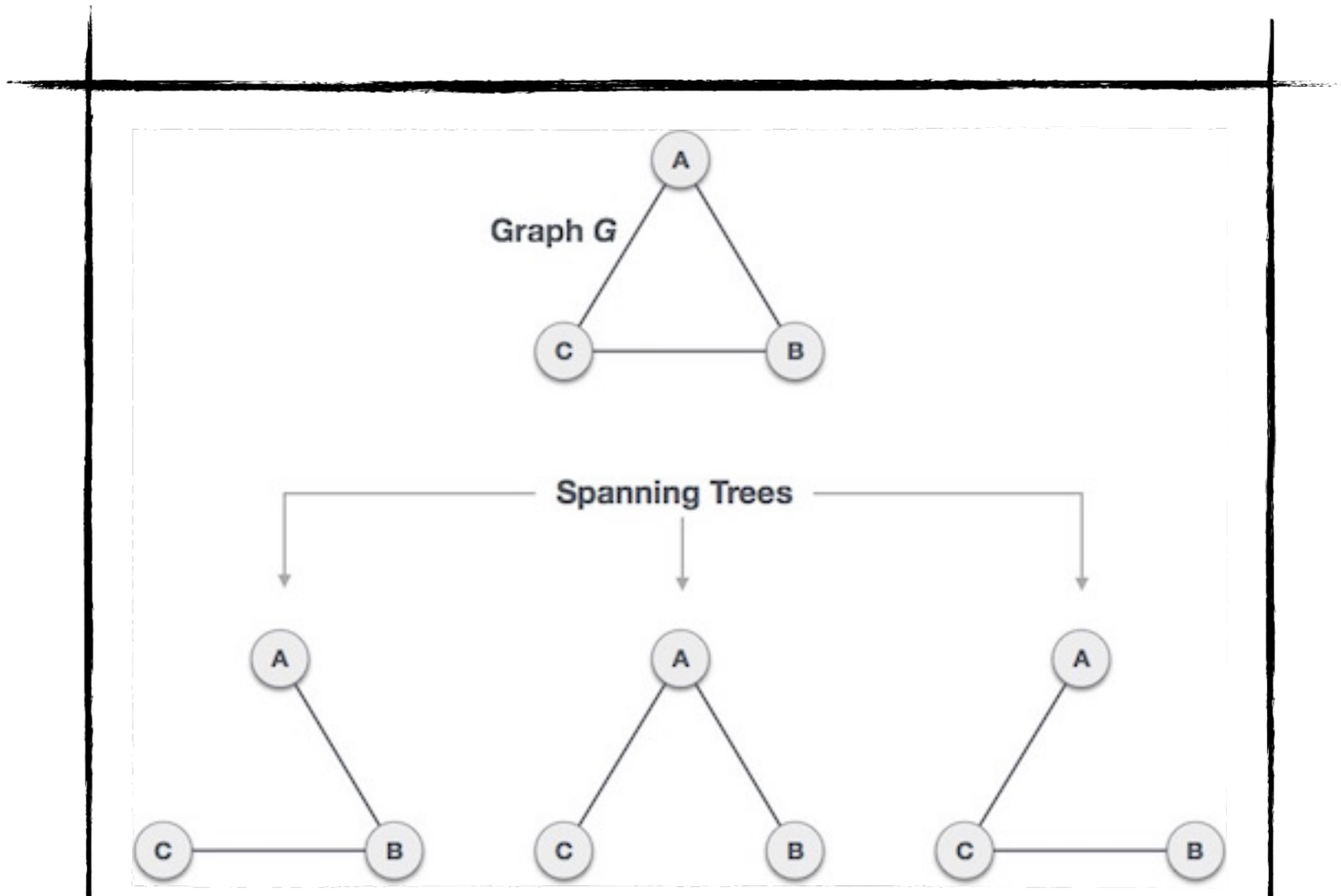


Tree



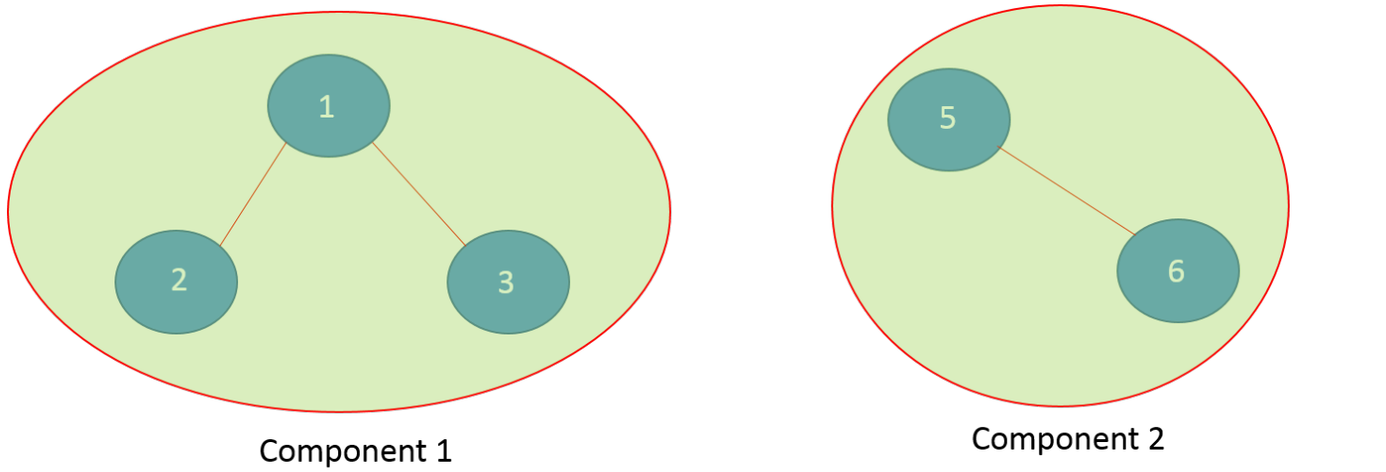
# Definitions

A **spanning tree** of a graph $G$ is a tree containing all the nodes of $G$.



# Definitions

A **connected component** of a graph $G$ is subgraph such that any two vertices are connected via some path.



# Graph Representations

How do we represent a graph $G = (V,E)$?

Adjacency Matrix

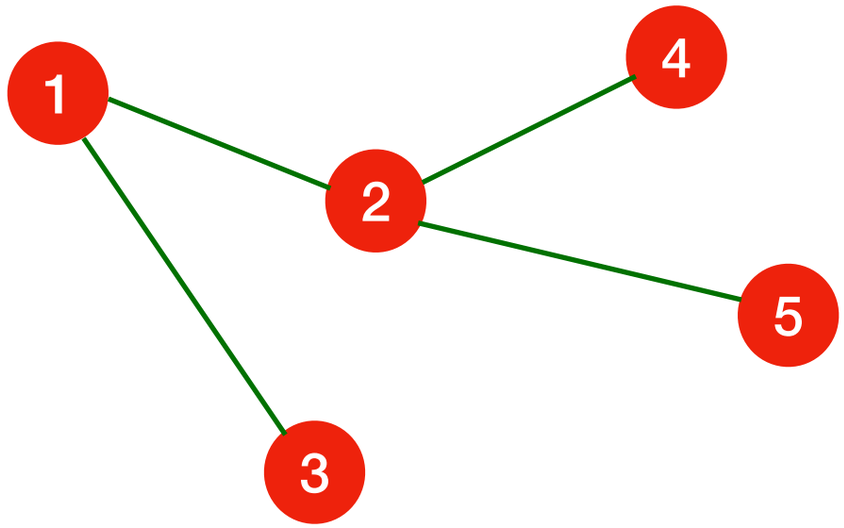
Adjacency List

# Adjacency Matrix **A**

The $i^{\text{th}}$ node corresponds to the $i^{\text{th}}$ row and the $i^{\text{th}}$ column.

If there is an edge between $i$ and $j$ in the graph, then we have **A**[$i$,$j$] = 1, otherwise **A**[$i$,$j$] = 0.

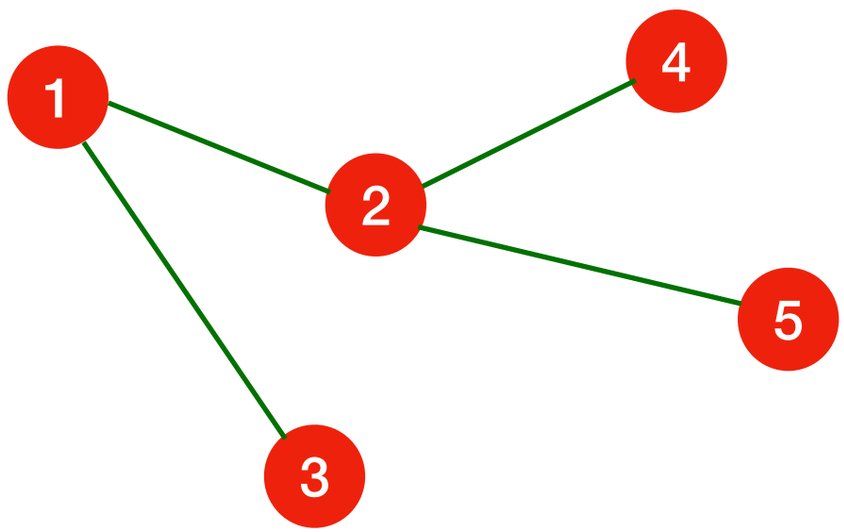
For undirected graphs, necessarily **A**[$i$,$j$] = **A**[$j$,$i$]. For directed graphs, it could be that **A**[$i$,$j$] ≠ **A**[$j$,$i$].

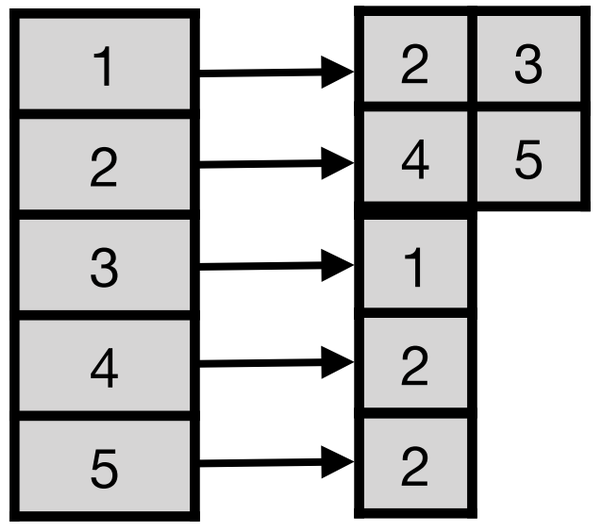



| | | | | |
|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 |

# Adjacency List L

- Nodes are arranged as a list, each node points to the neighbours.
- For undirected graphs, the node points only in one direction.
- For directed graphs, the node points in two directions, for in-degree and for out-degree



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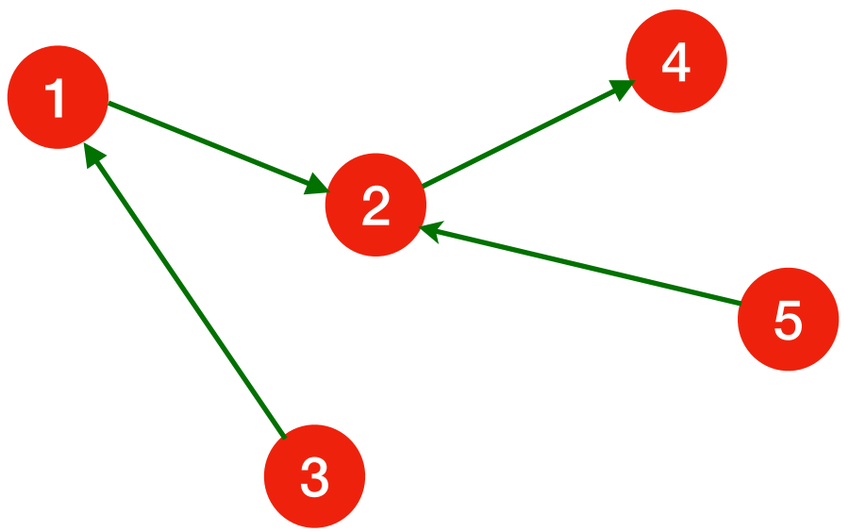


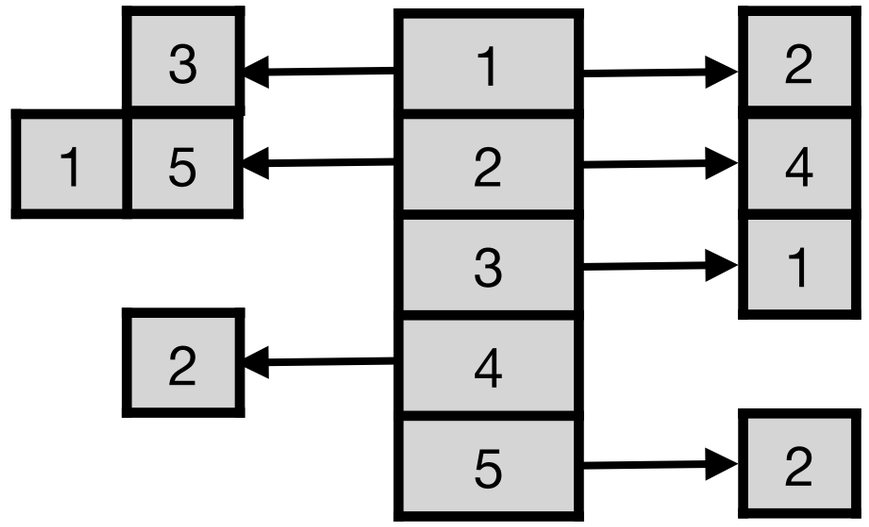

# Adjacency Matrix vs Adjacency List

**Adjacency Matrix**

Memory: $O(n^2)$

Checking *adjacency* of $u$ and $v$
Time: O(1)

Finding *all adjacent nodes* of $u$
Time: O(n)

**Adjacency List**

Memory: $O(m+n)$

Checking *adjacency* of $u$ and $v$
Time: $O(\min(\deg(u),\deg(v))$

Finding *all adjacent nodes* of $u$
Time: $O(\deg(u))$

# Graph Traversal (Search)

We would like to go over all the possible nodes of an (undirected) graph.

There are different ways of doing that.

Two systematic ways:

- Depth-First Search
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KT Chapter 3.2.

CLRS Chapter 20.2, 20.3

# Minimum Spanning Tree via Greedy

# Application

We have a set of locations.

We want to build a communication network, joining all of them.

We want to do it as cheaply as possible.

Every direct connection between two locations has a cost.

We want to have everything connected at the minimum cost.

# Minimum Spanning Tree

Consider a *connected* graph $G=(V, E)$, such that for every edge $e=\{v,w\}$ of $E$, there is an associated positive cost $c_e$.

Goal: Find a subset $T$ of $E$ so that the graph $G'=(V, T)$ is connected and the total cost $\sum_{e \in T} c_e$ is minimised.

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($V$, $T$-{$e$}) is a valid solution, and it is cheaper. **Contradiction!**

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# Minimum Spanning Tree

$G'=(V, T)$ is a spanning tree and the problem is called the Minimum Spanning Tree problem.

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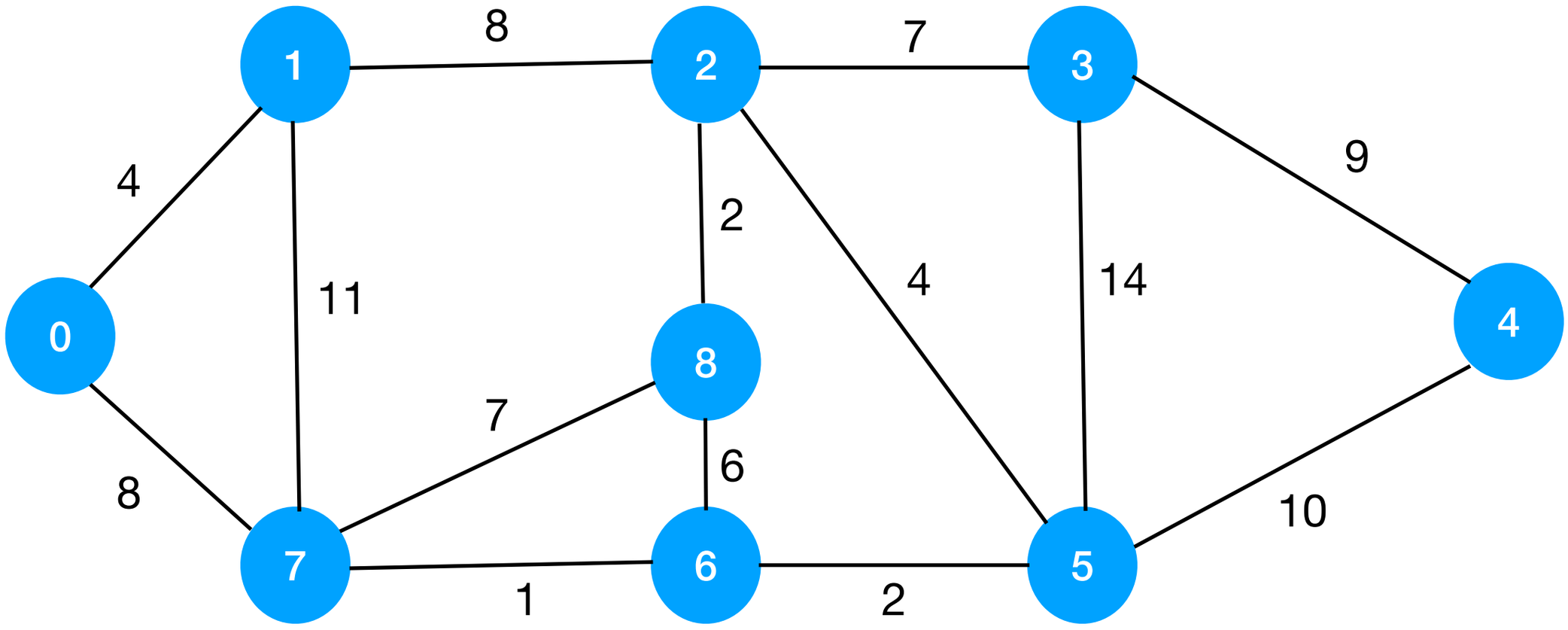

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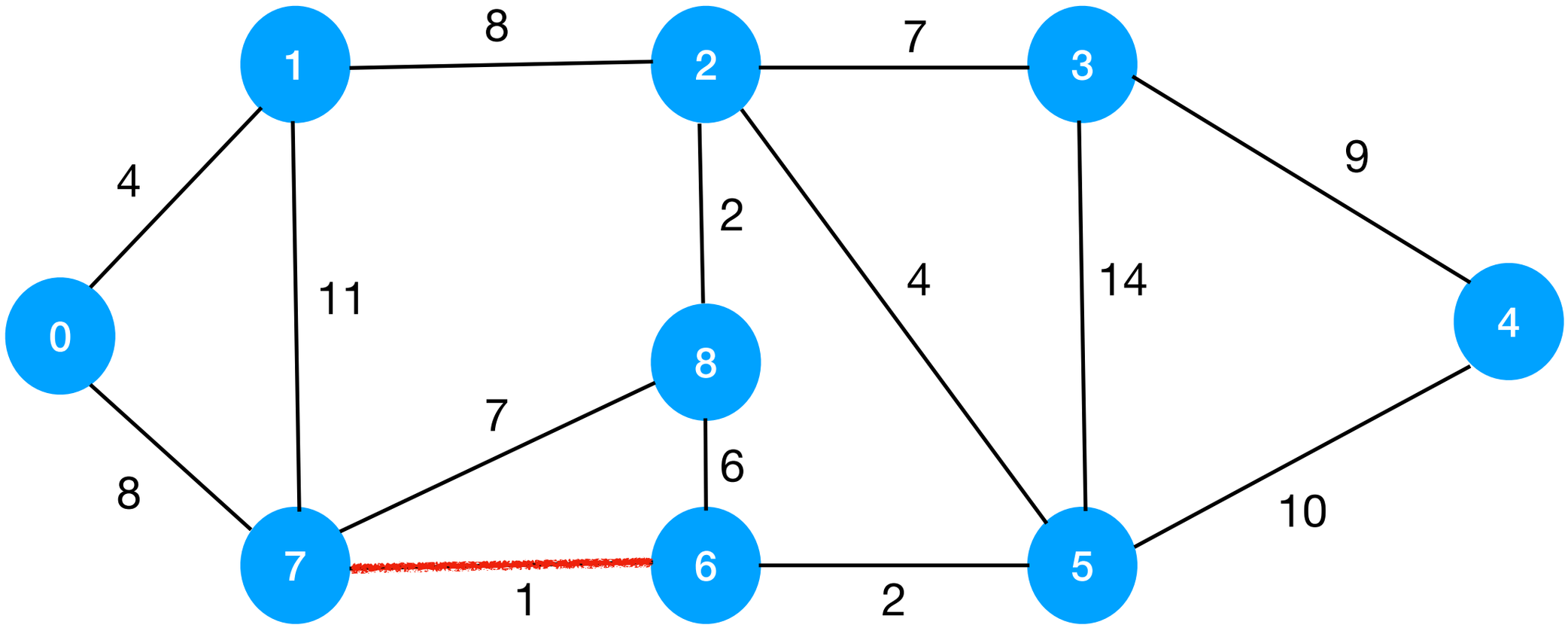

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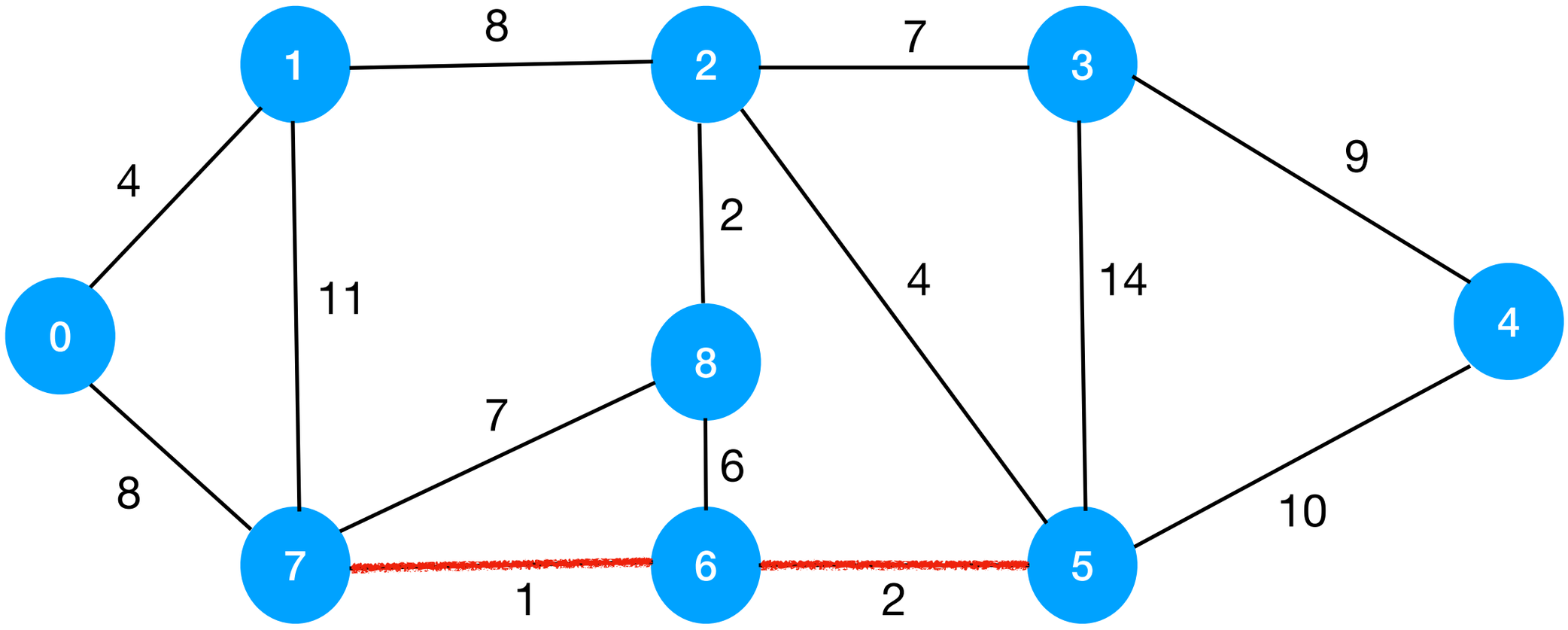

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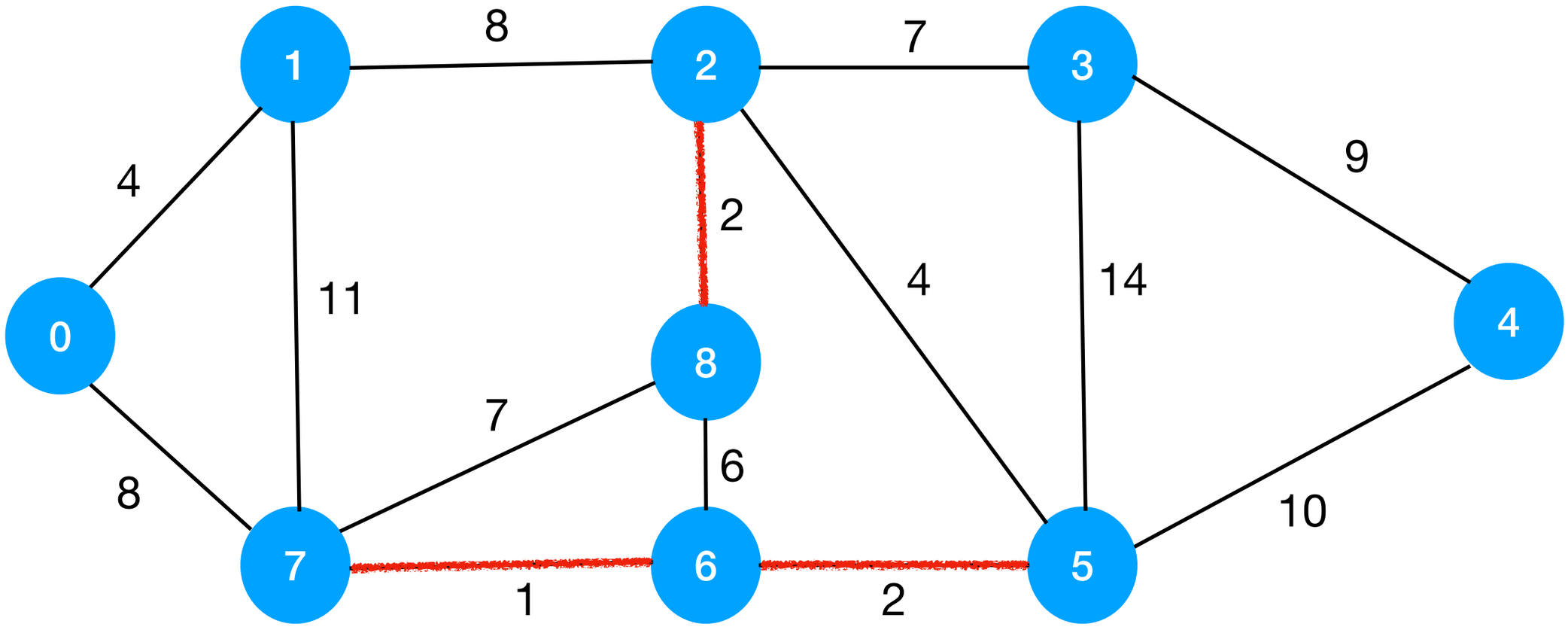

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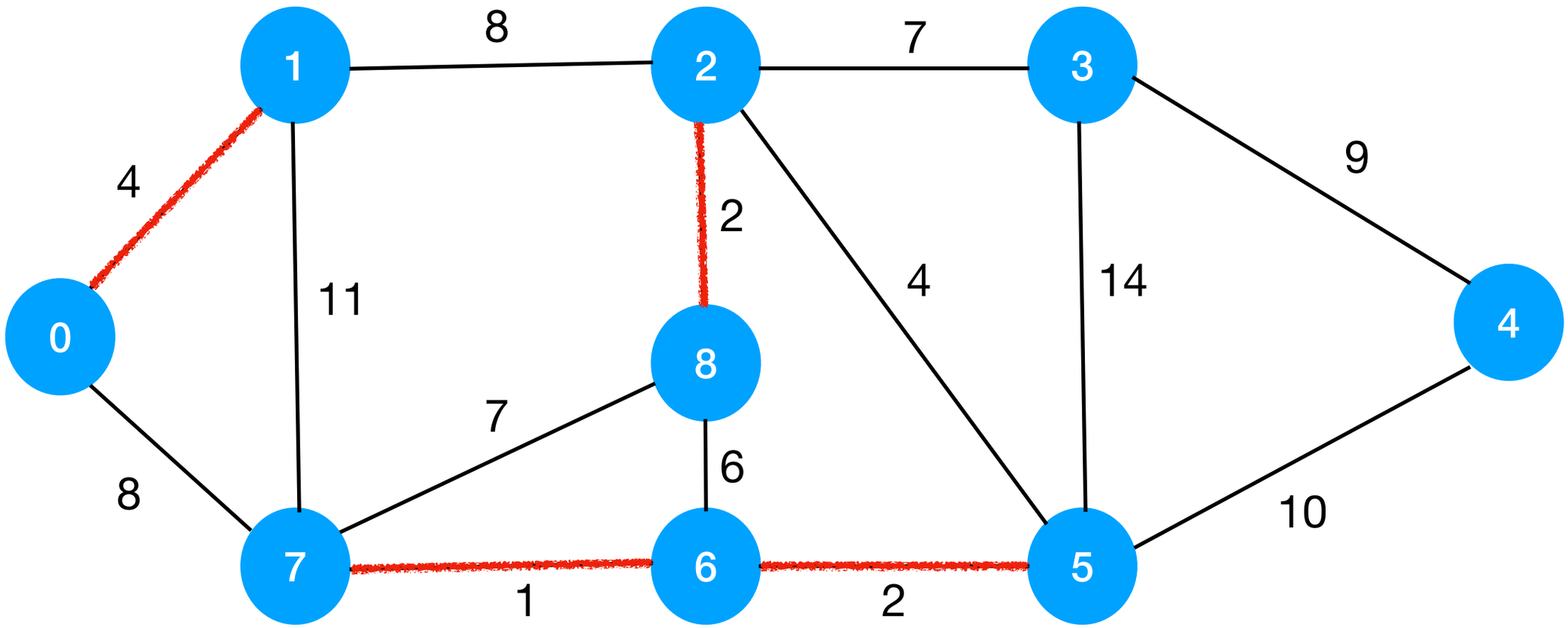

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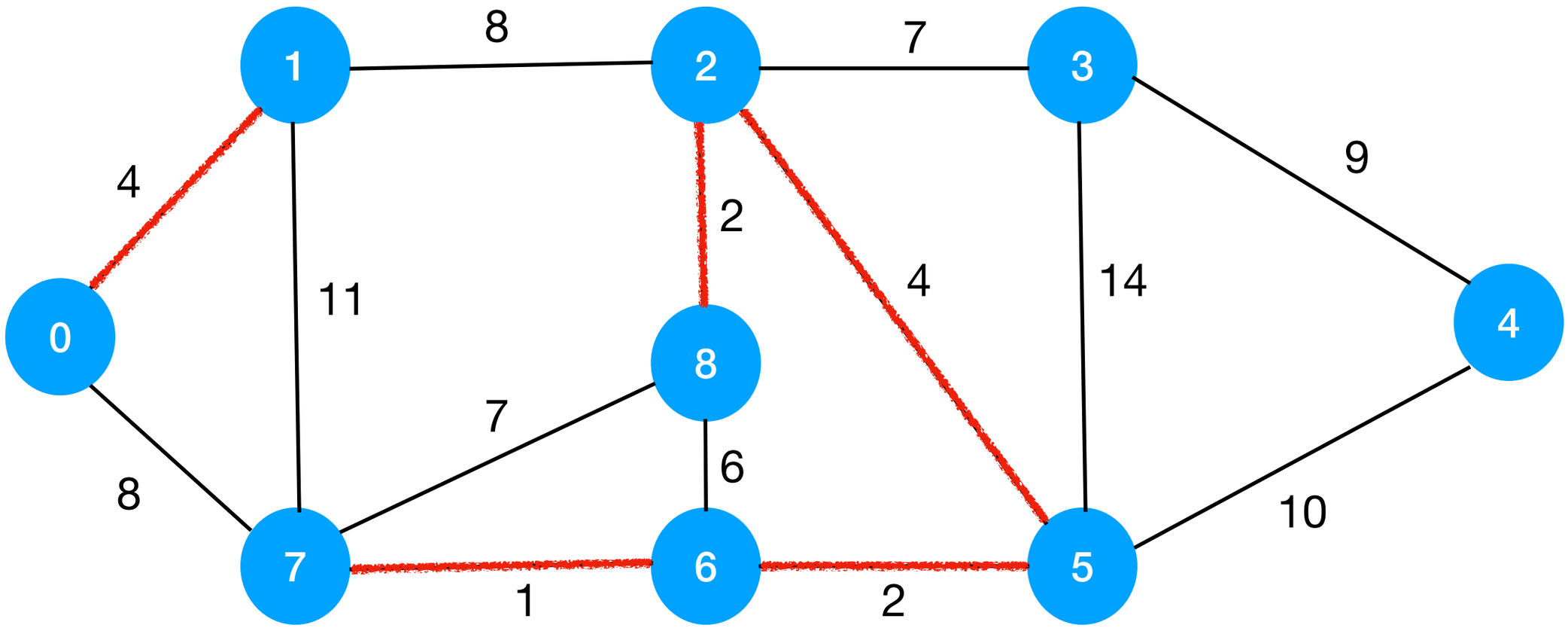

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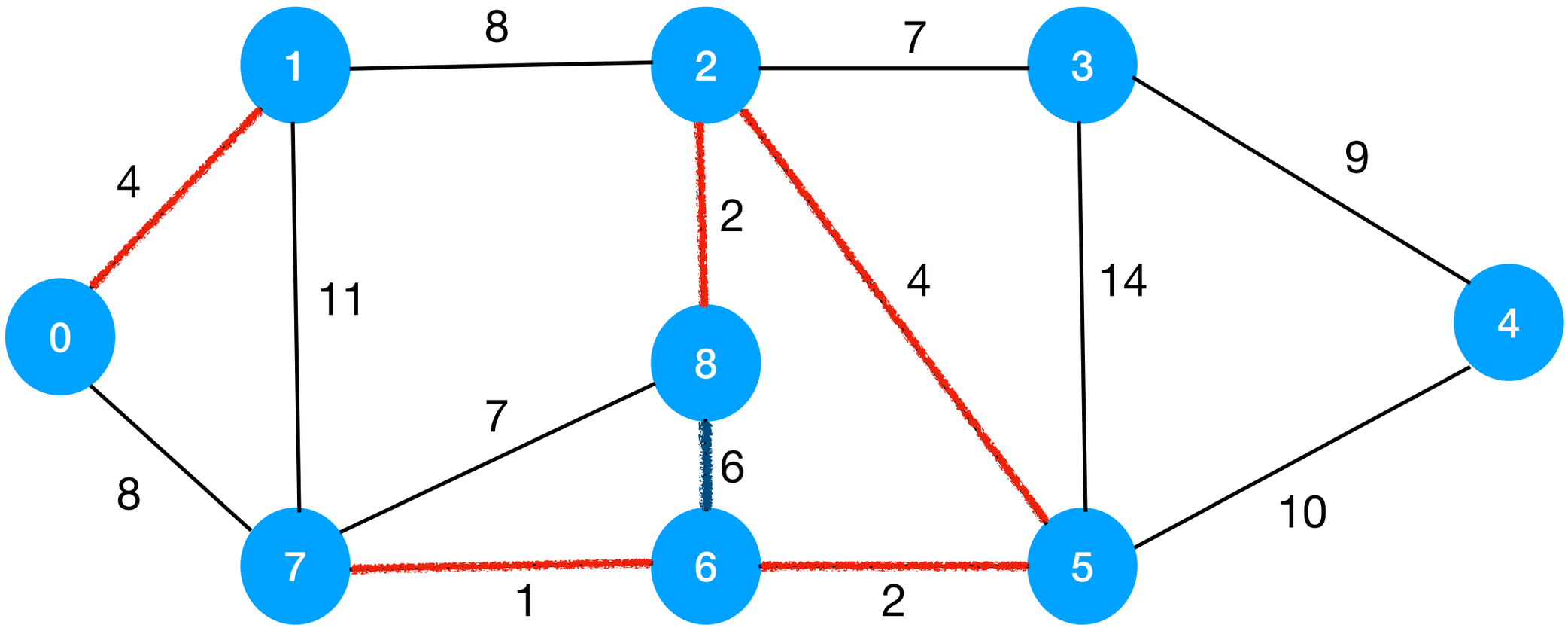

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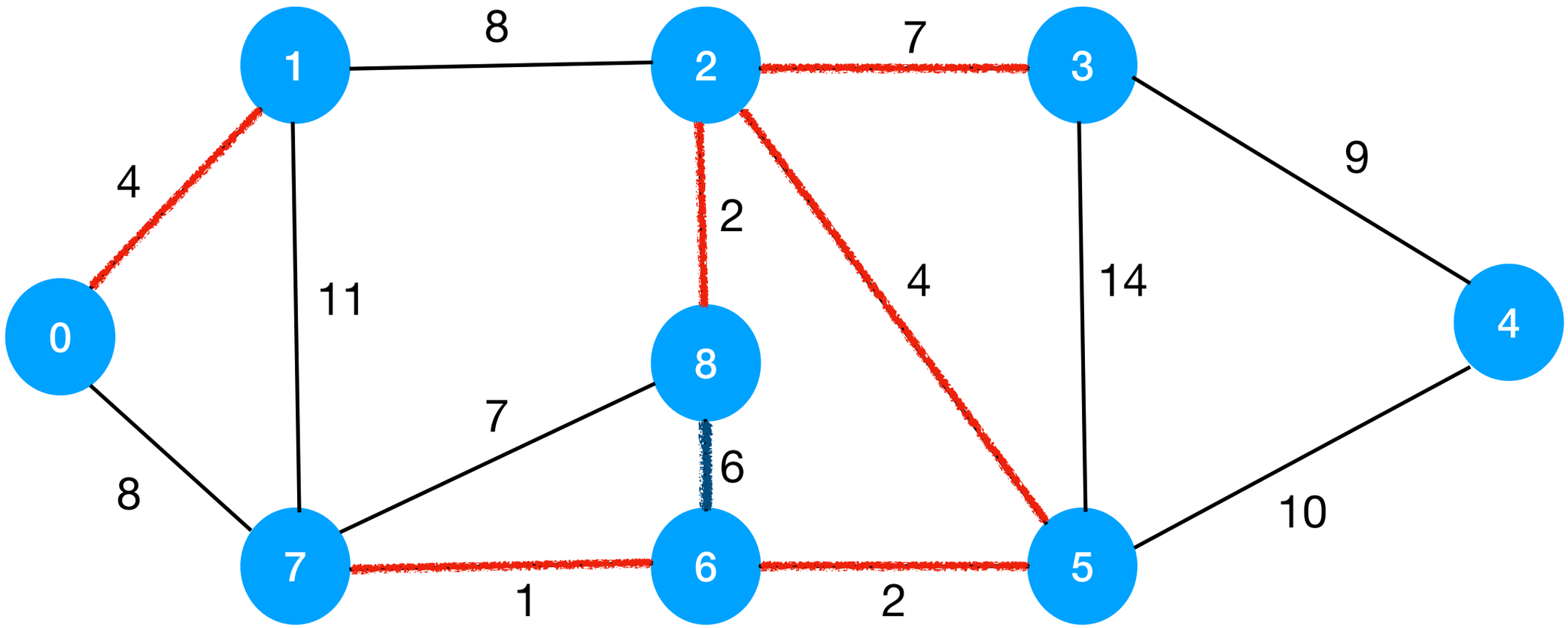

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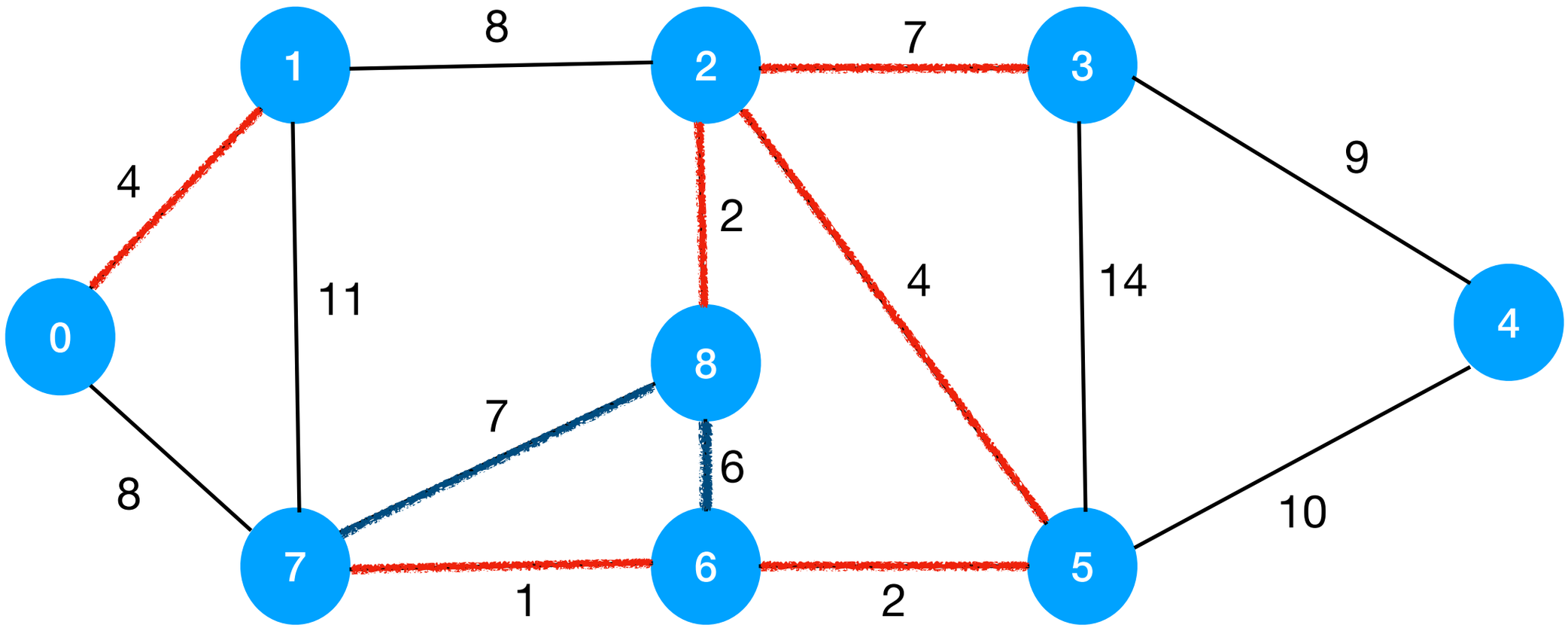

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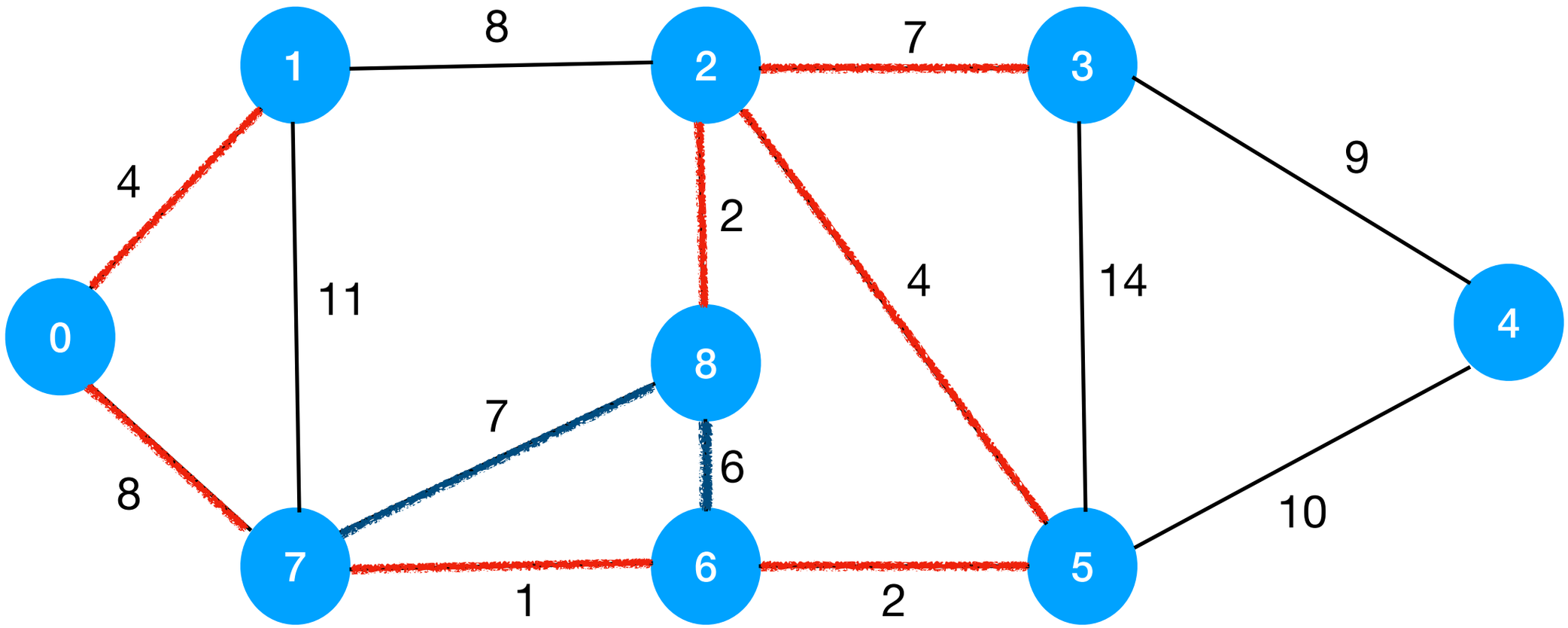

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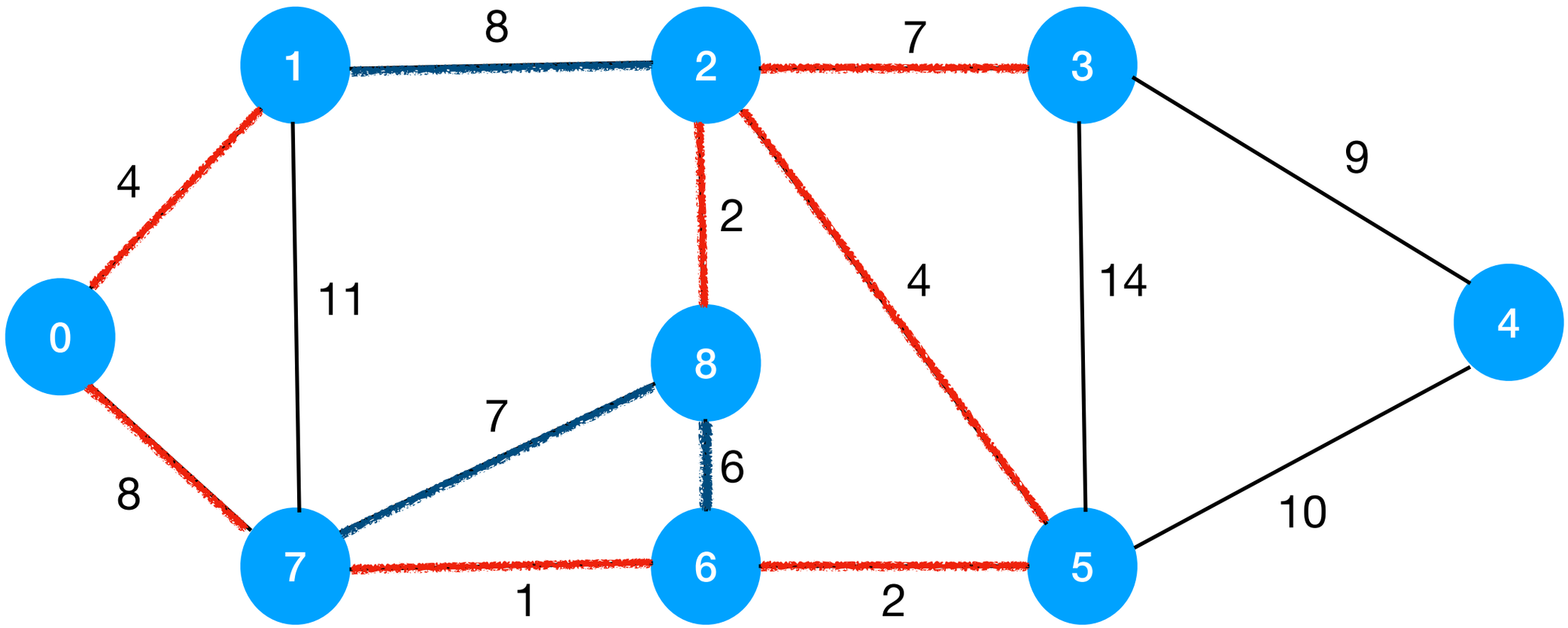

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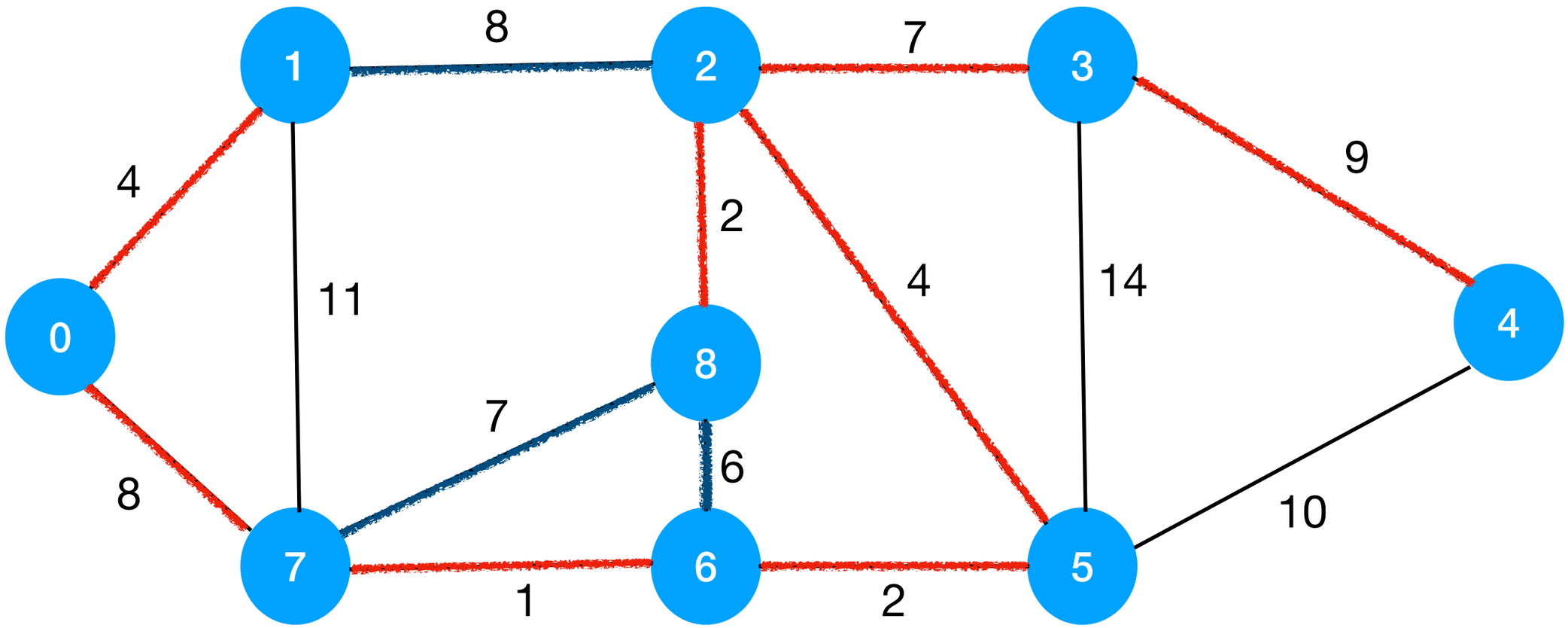

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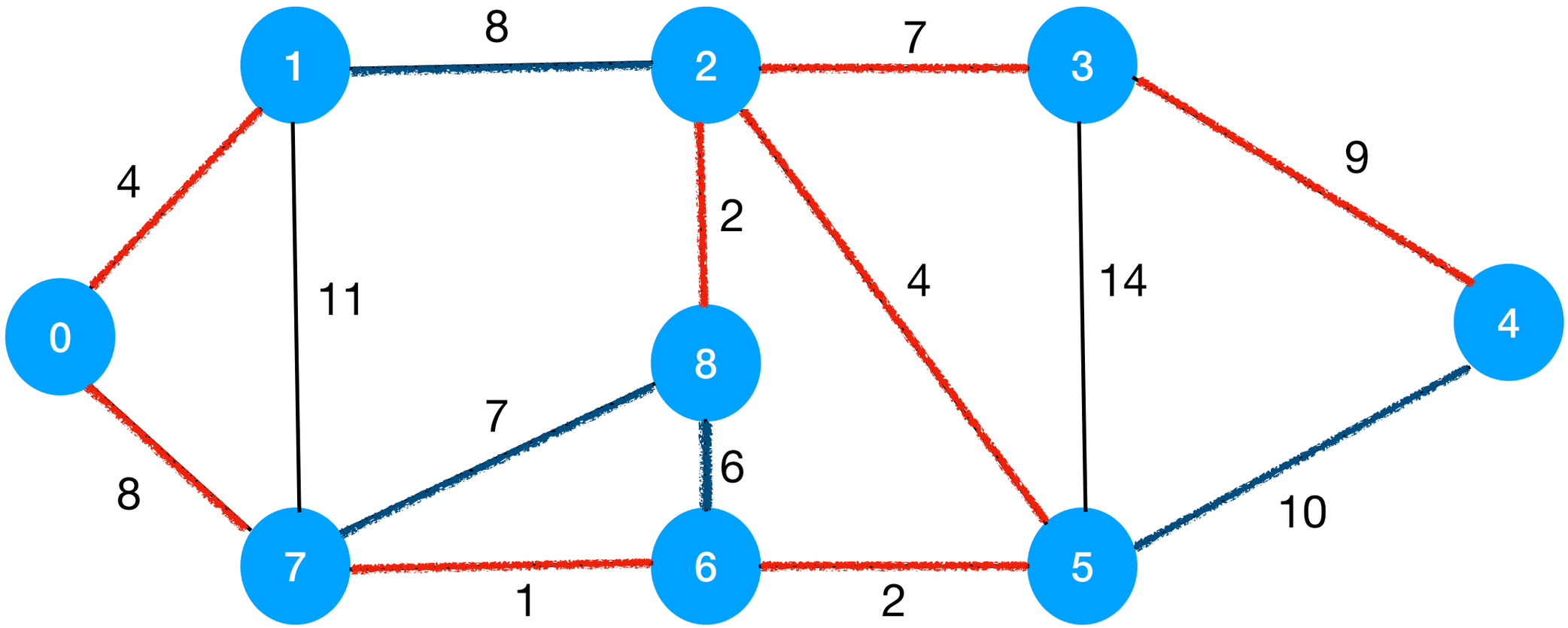

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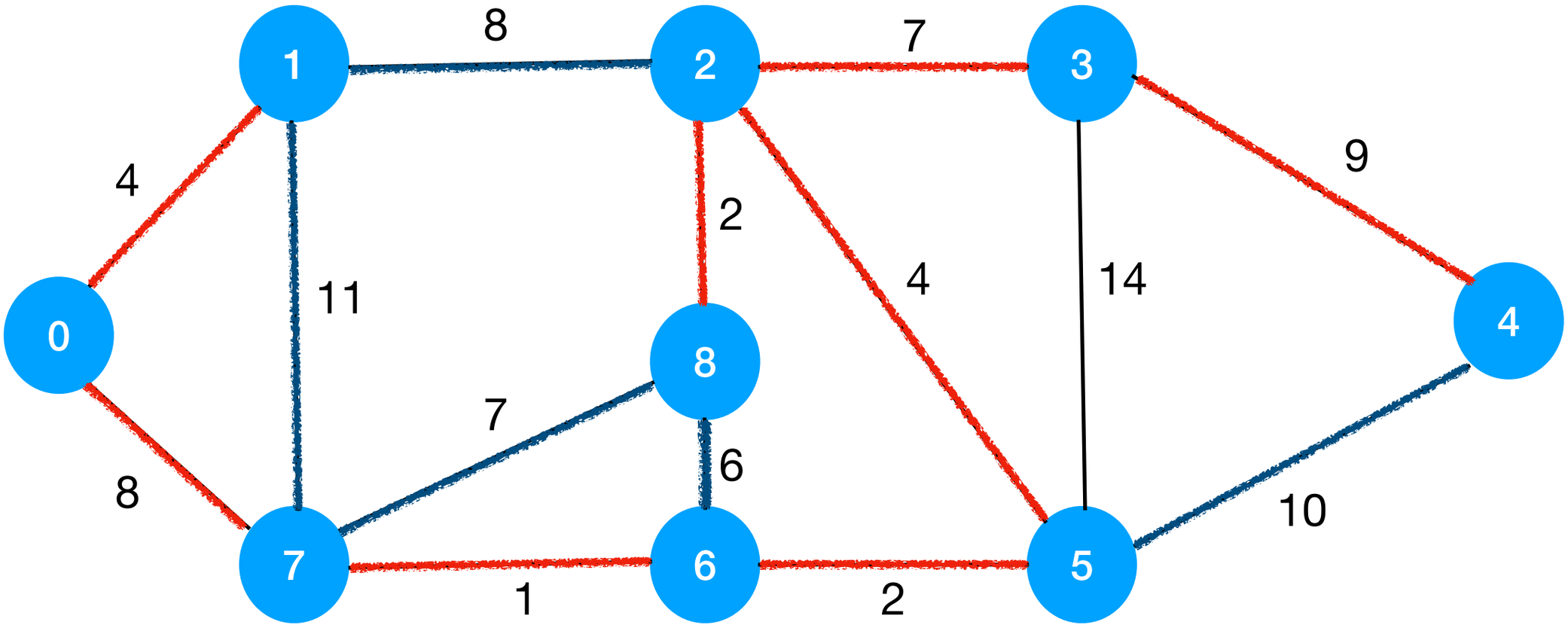

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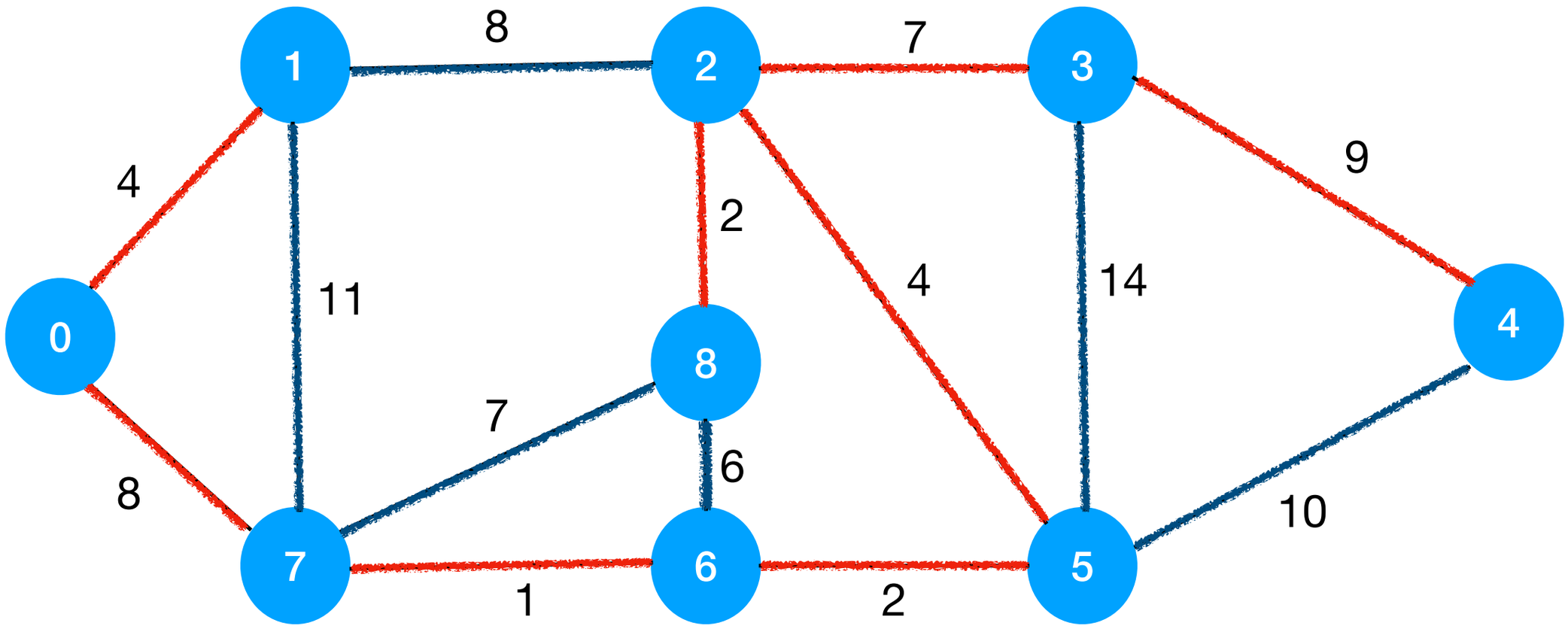

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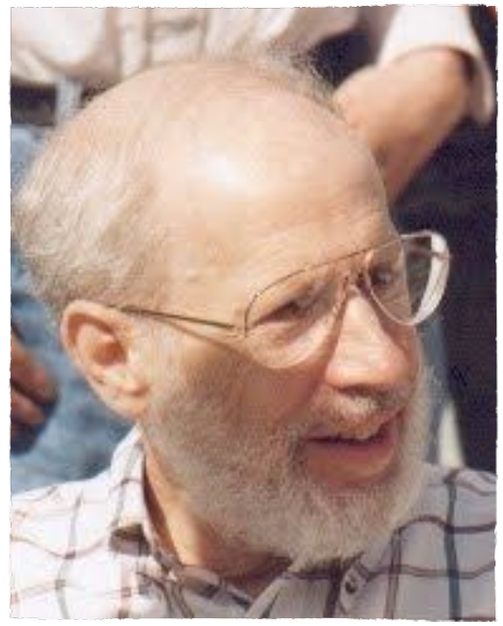
# Kruskal's Algorithm



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# Greedy Approach #2

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Start with an empty set of edges $T$.

Start with a node $s$.

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Add an edge $e=\{s,w\}$ to $T$.

Which one?

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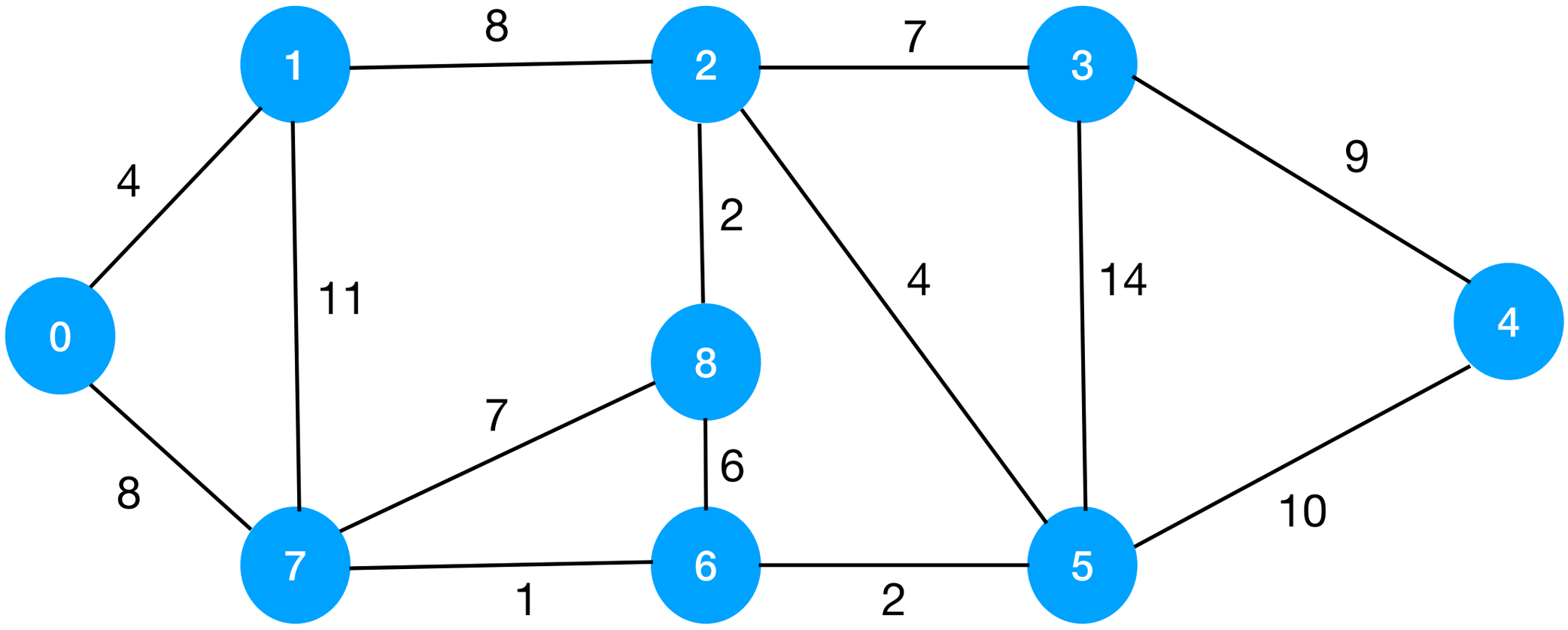
We continue like this.

We only consider edges to neighbours that are not in the spanning tree.

# Example



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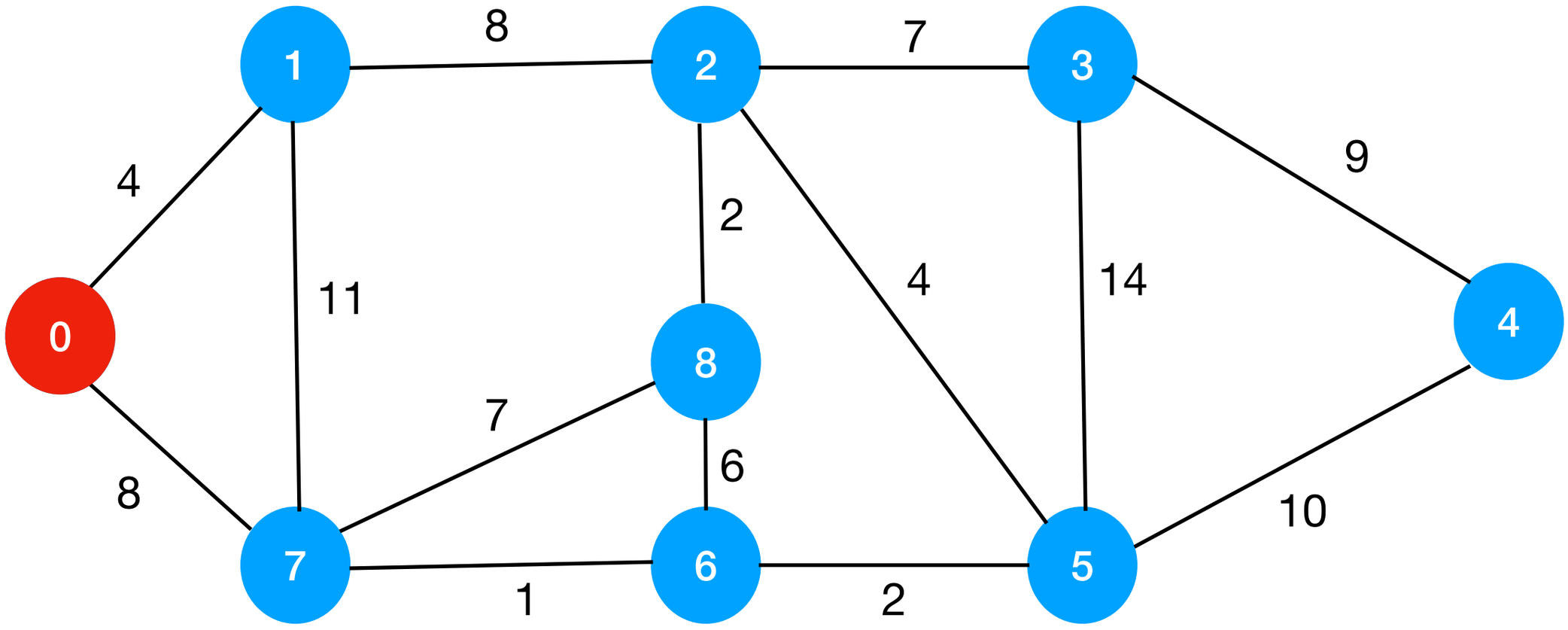

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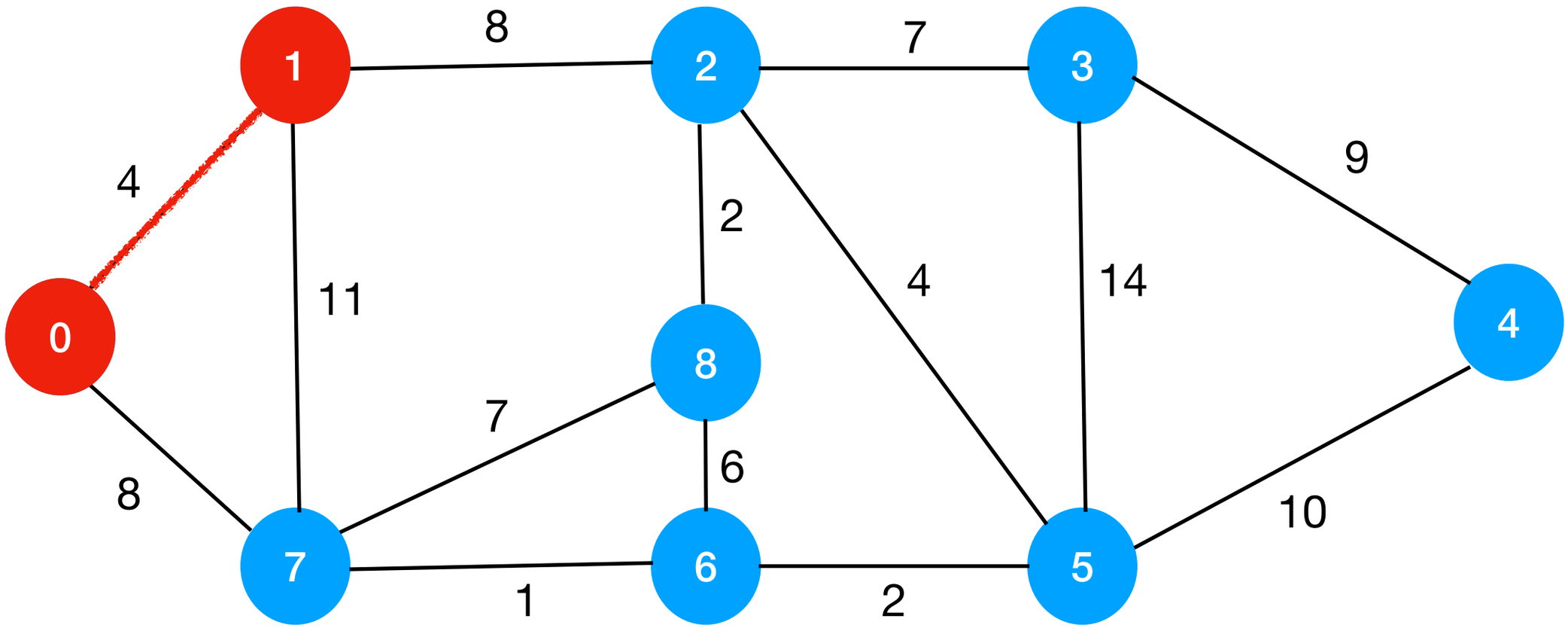

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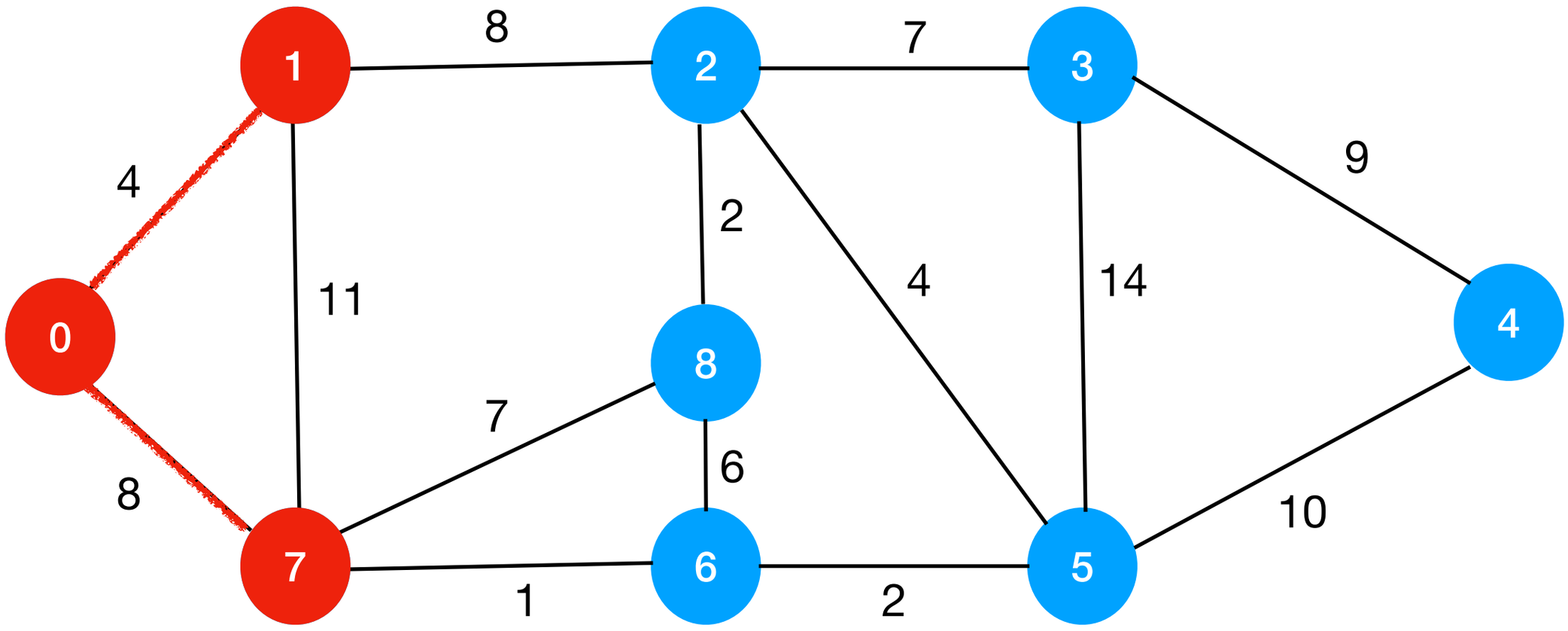

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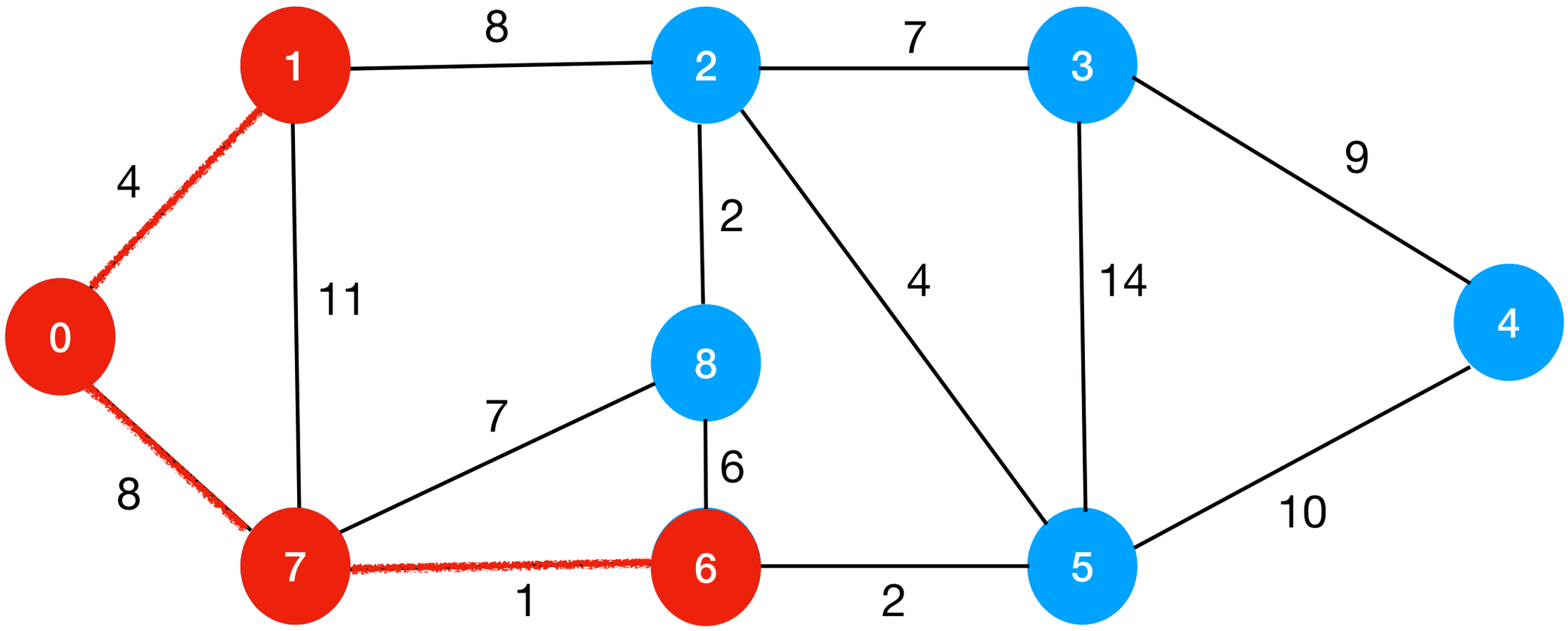

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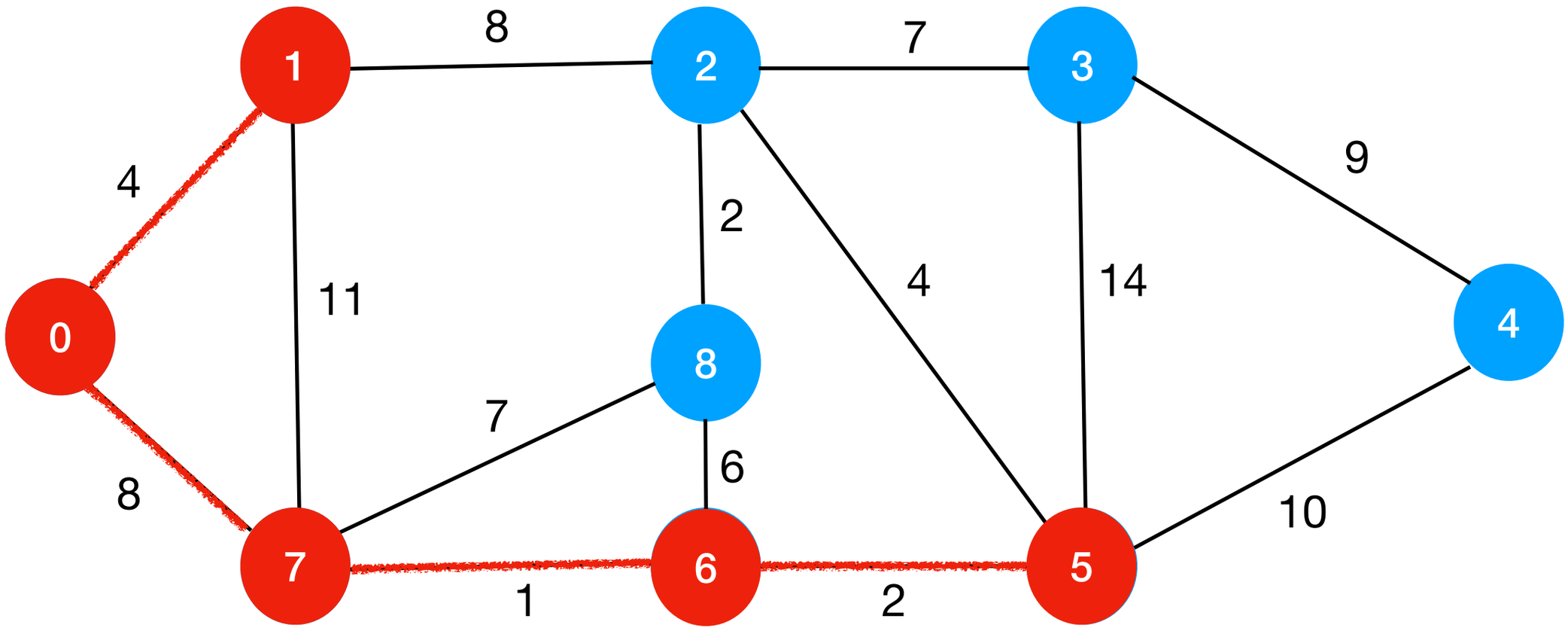

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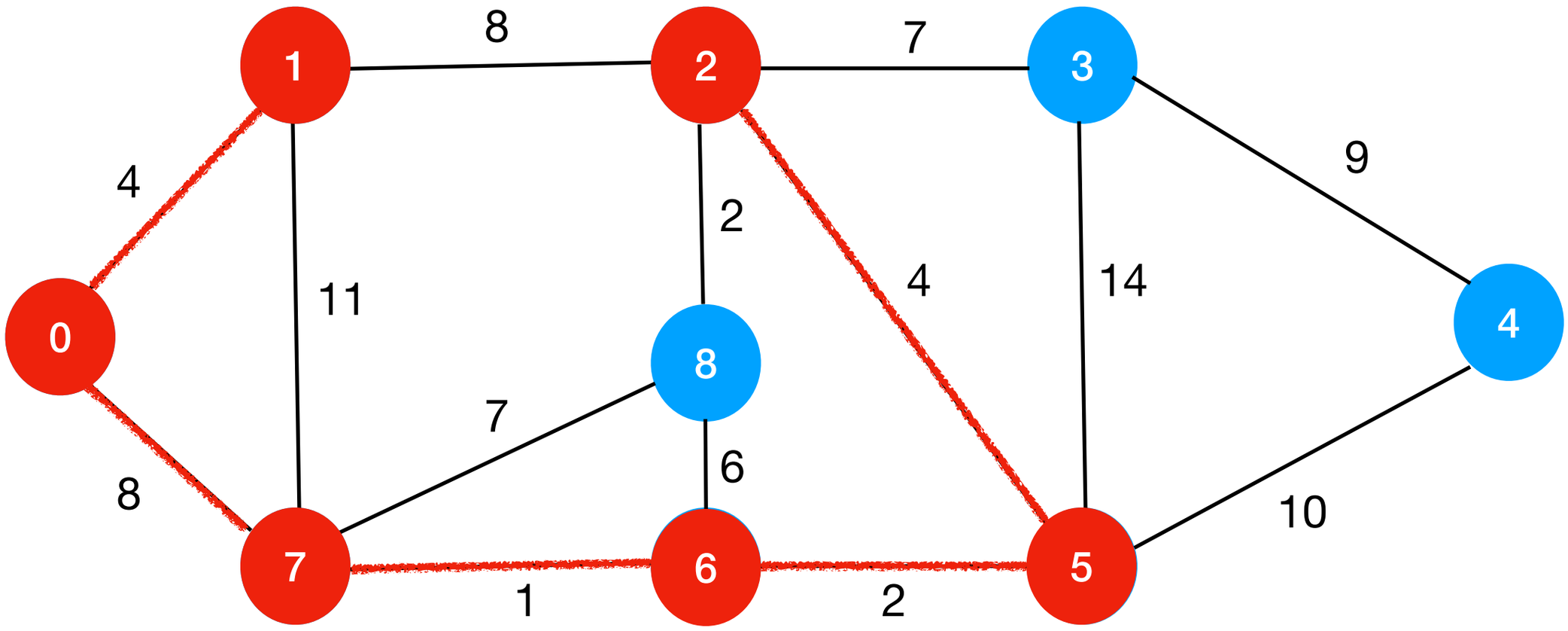

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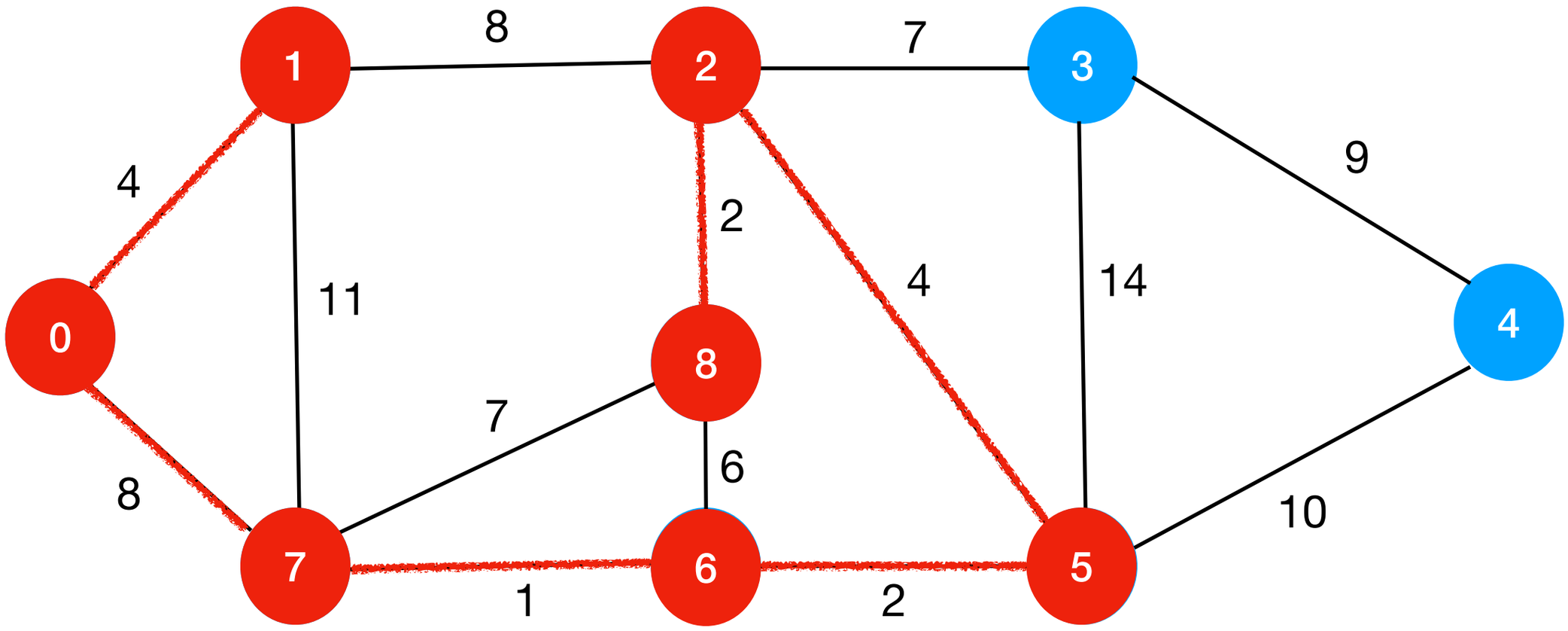

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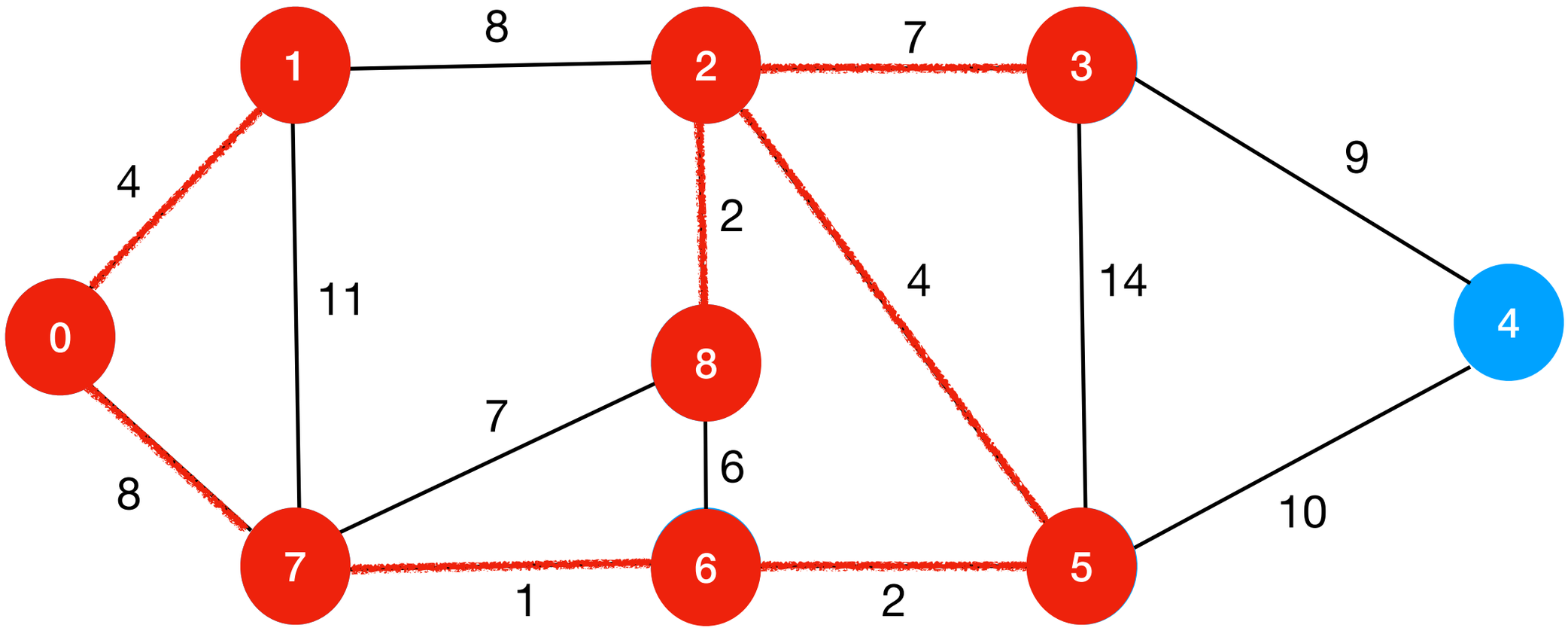

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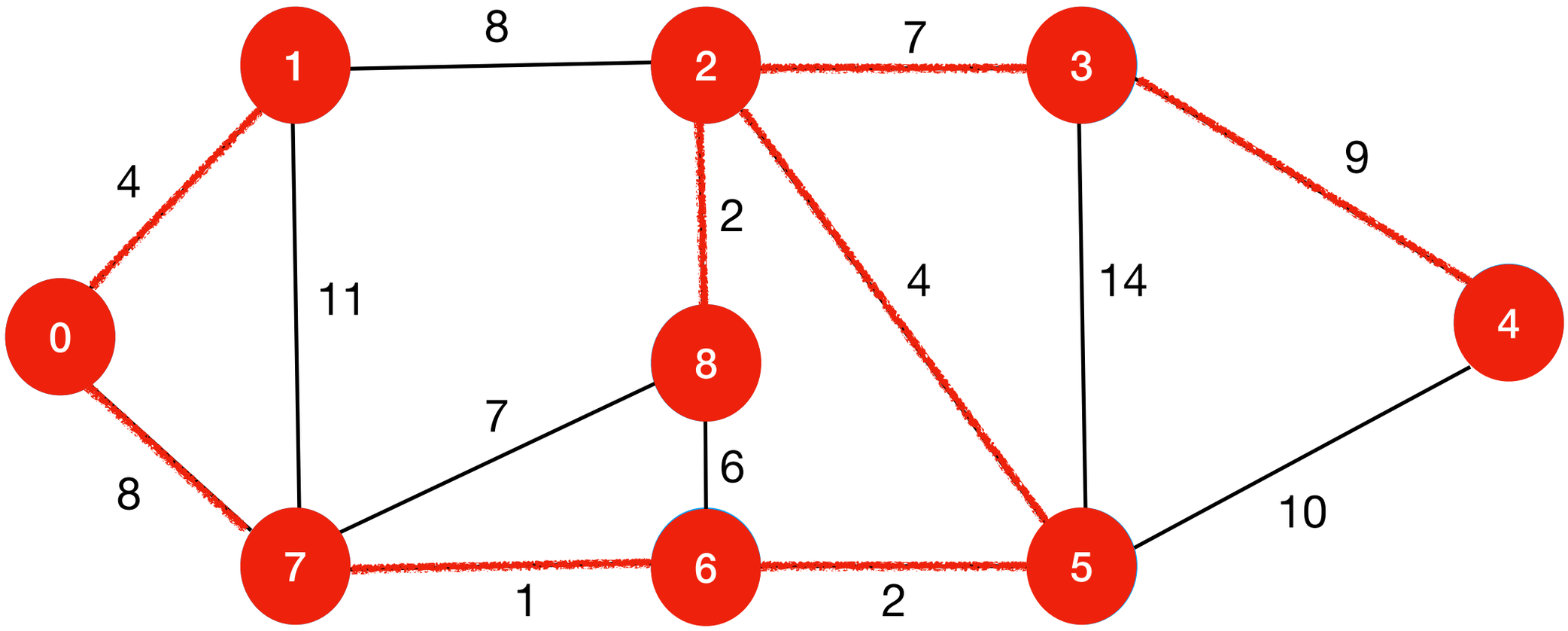

# Greedy Approach #2

Start with an empty set of edges $T$.

Start with a node $s$.

Add an edge $e=\{s,w\}$ to $T$.

Which one?

The one with the minimum cost $c_e$.

We continue like this.

We only consider edges to neighbours that are not in the spanning tree.

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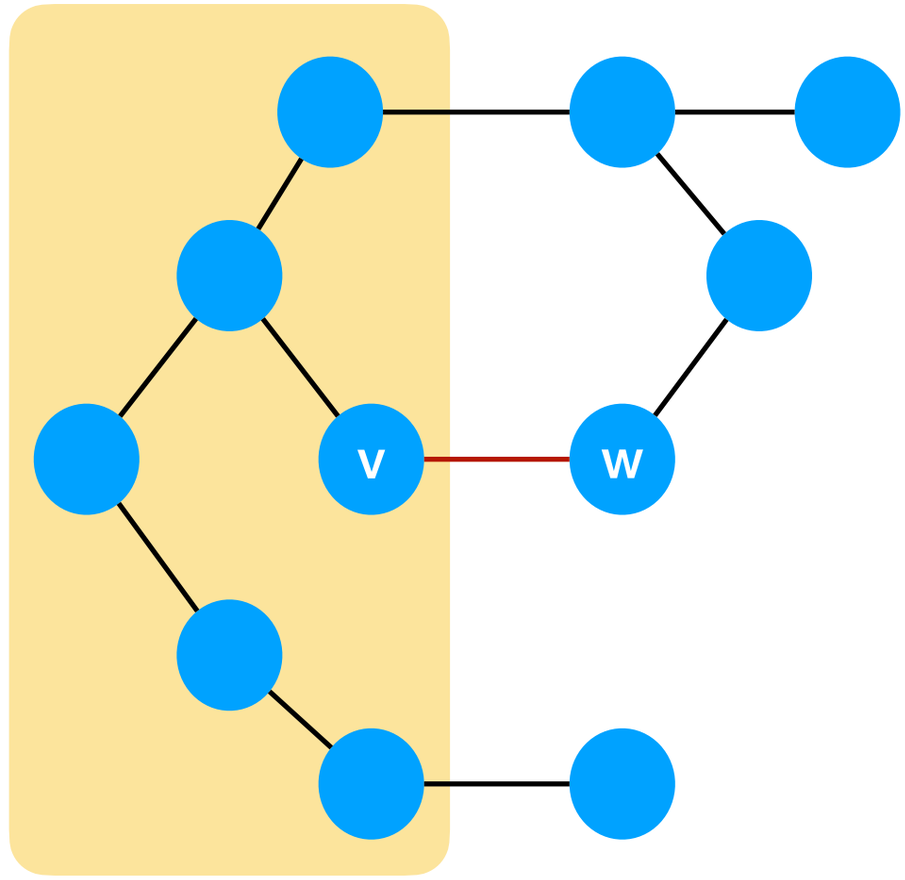

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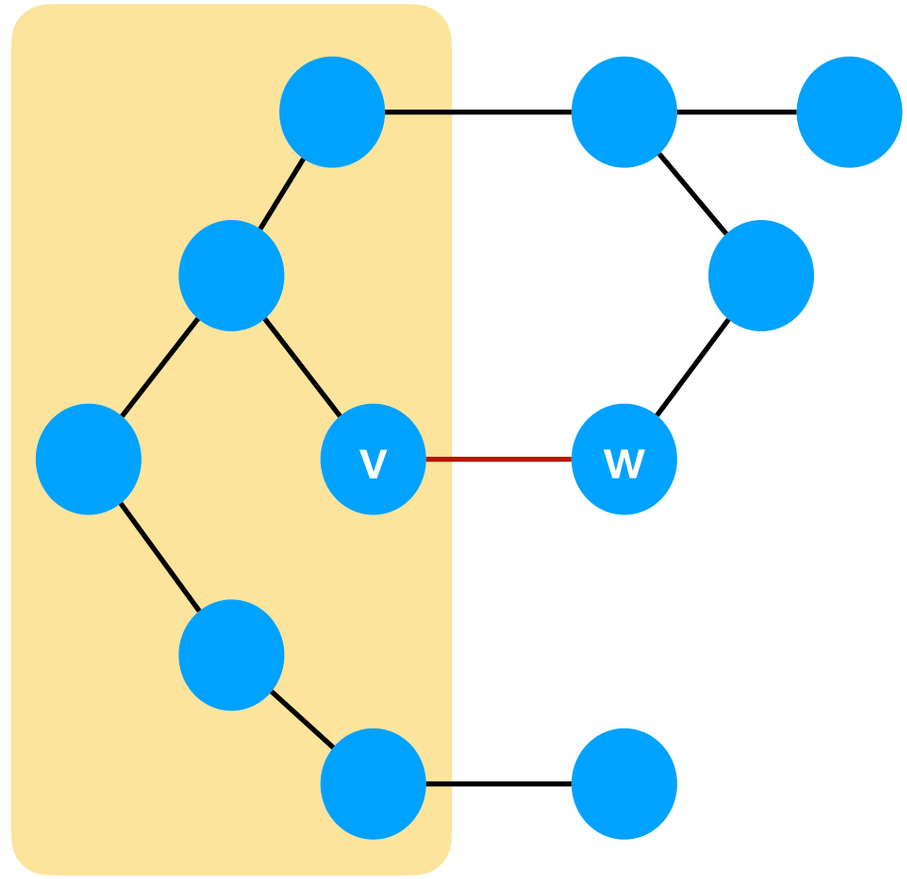

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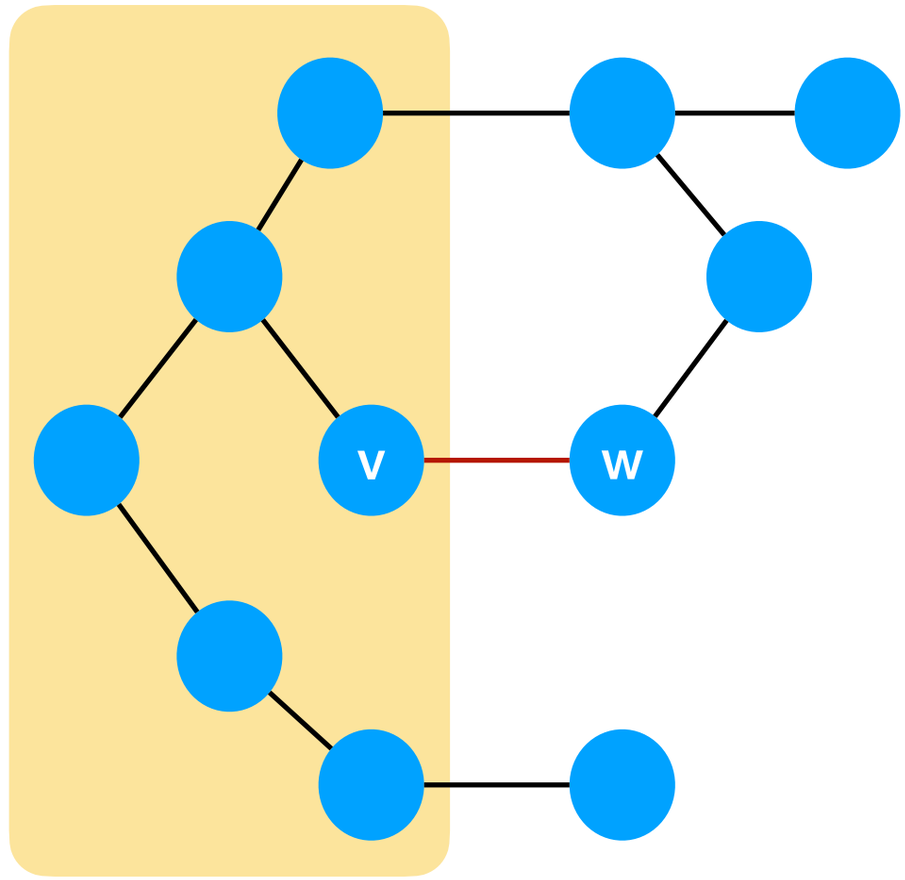

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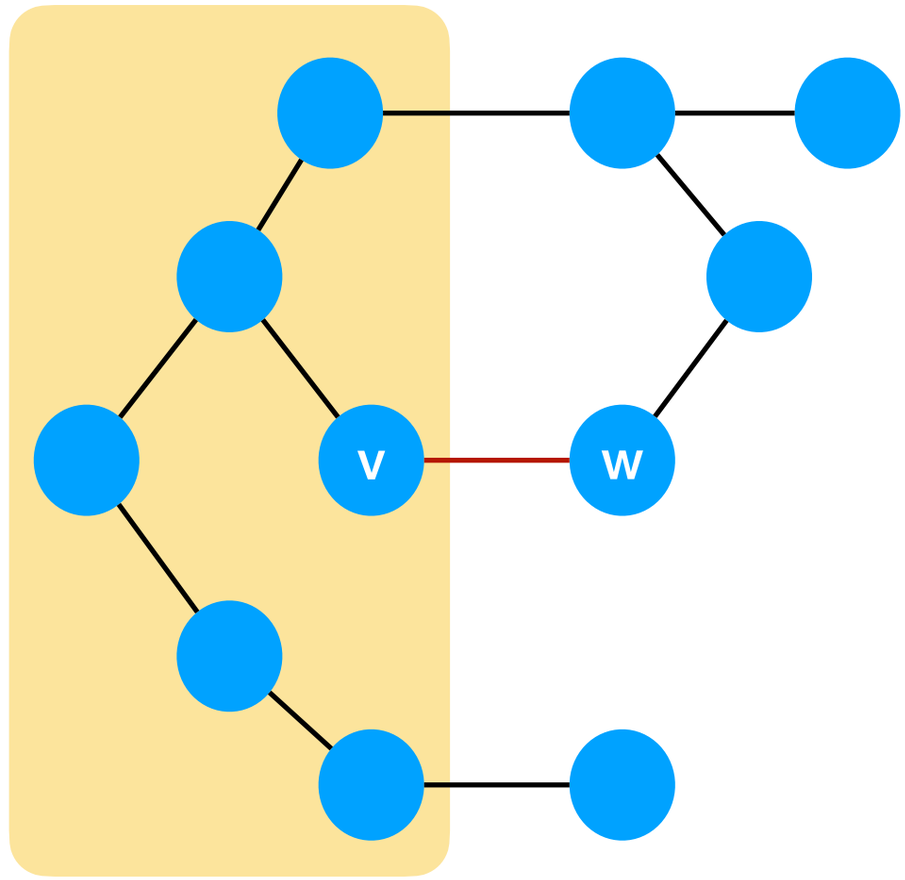

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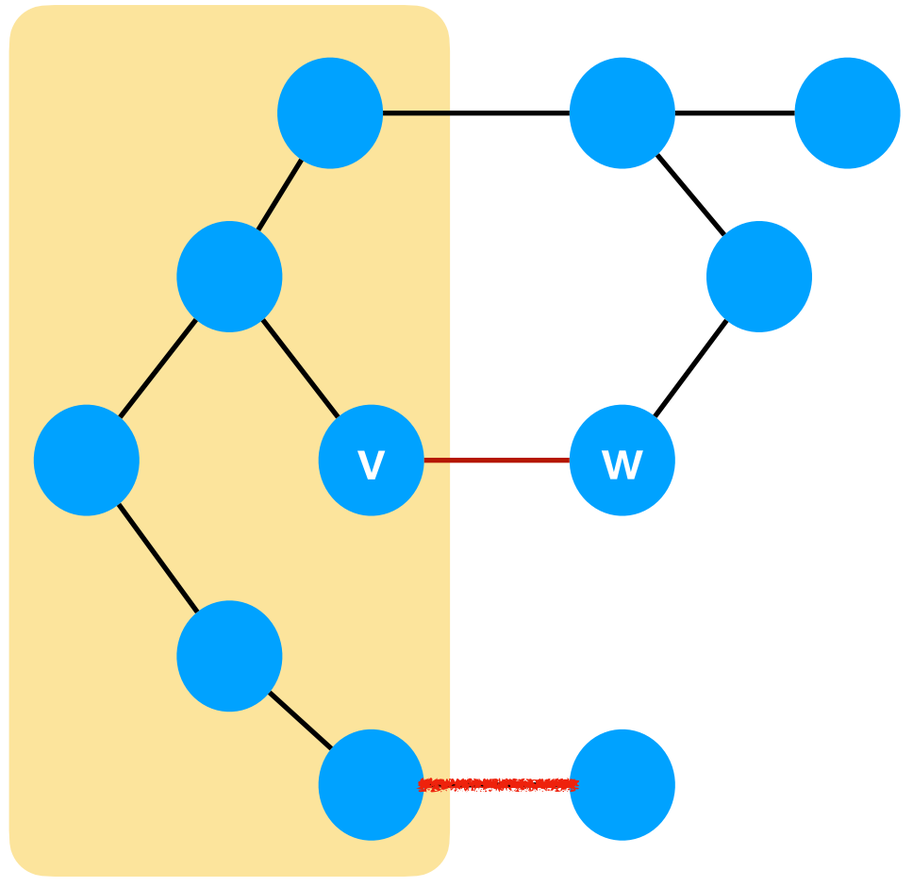

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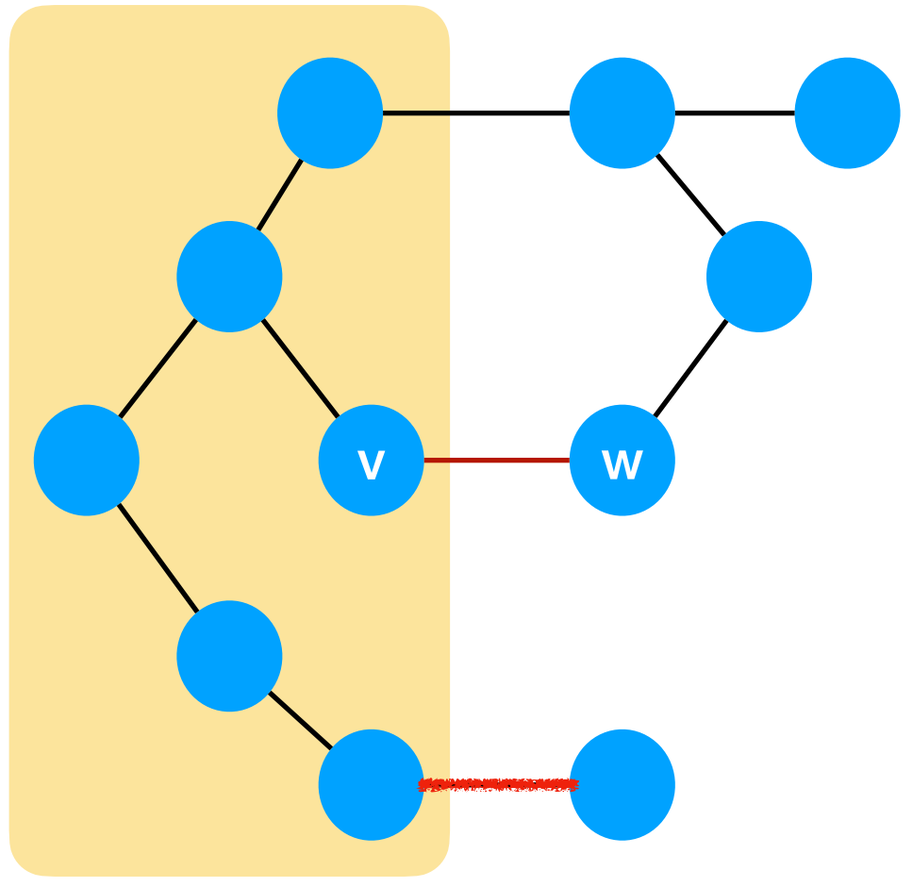

# The cut property

**No, $G_T$ - { $f$ } U { $e$ } might not be a spanning tree!**

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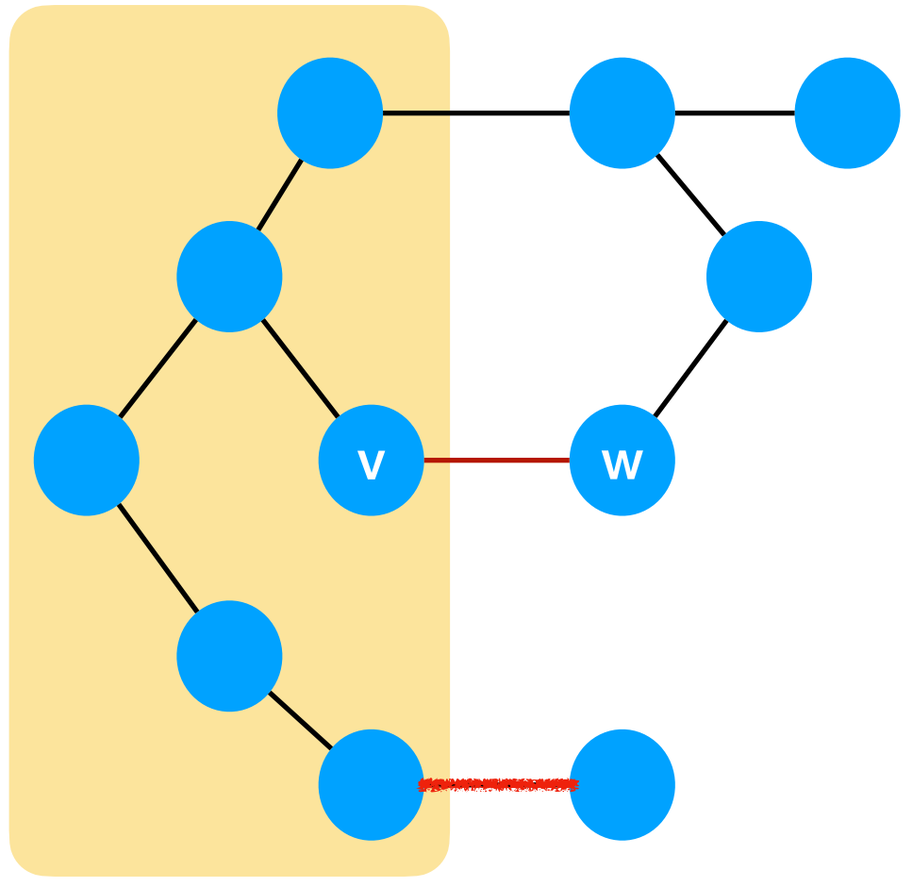

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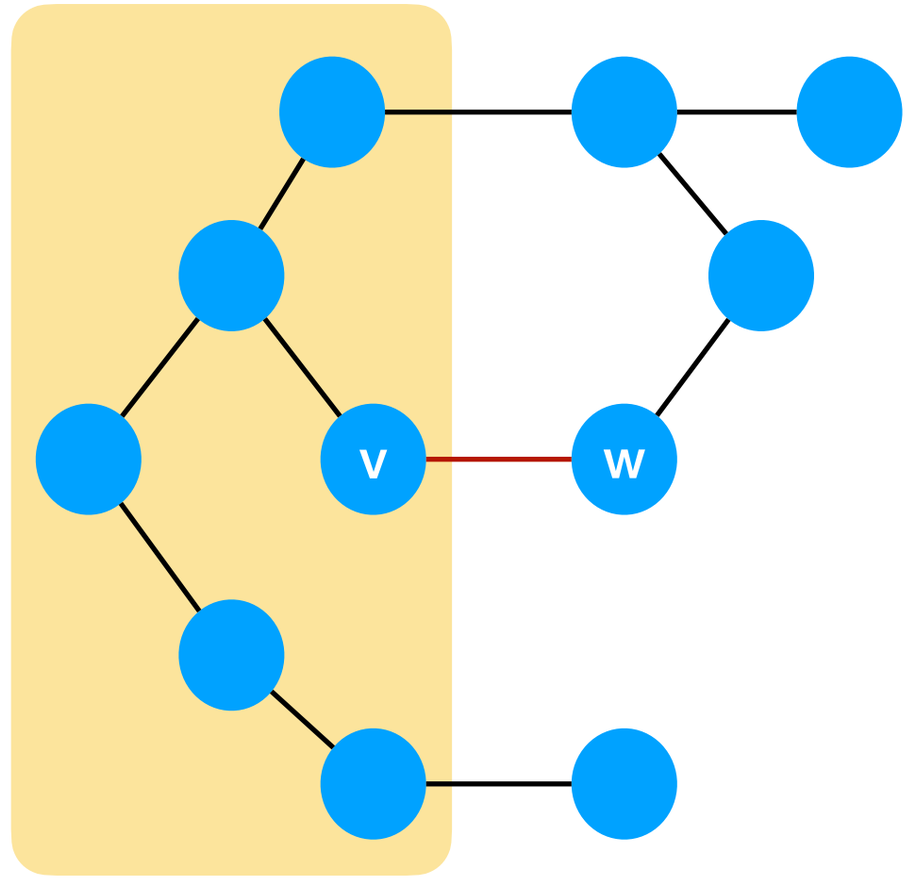

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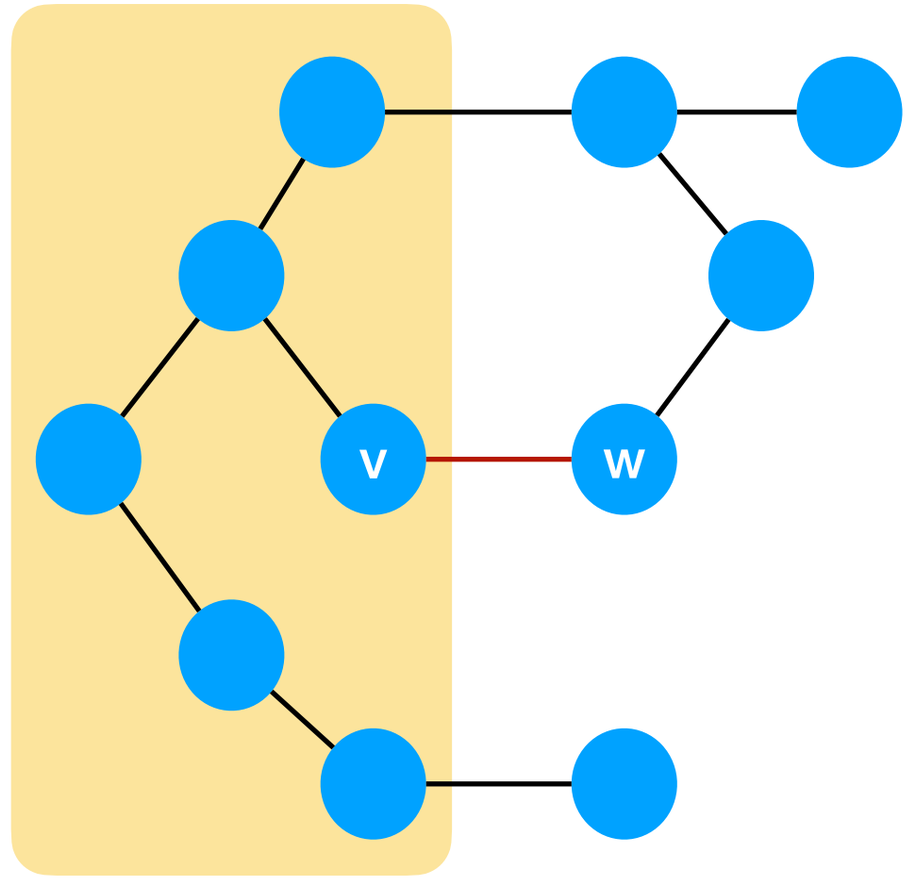

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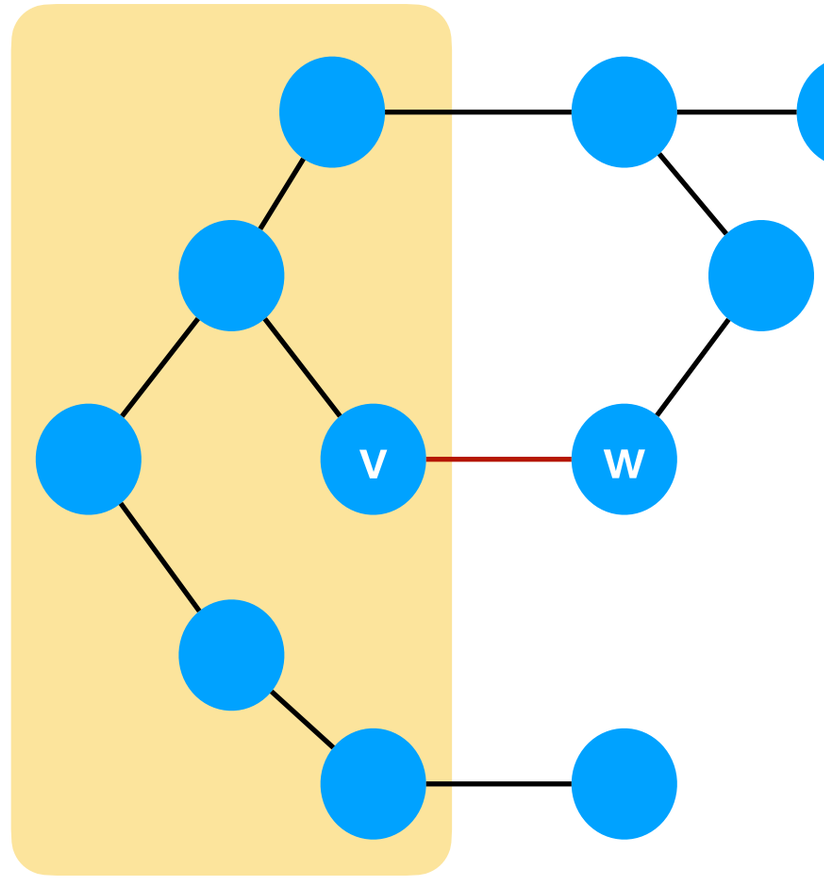

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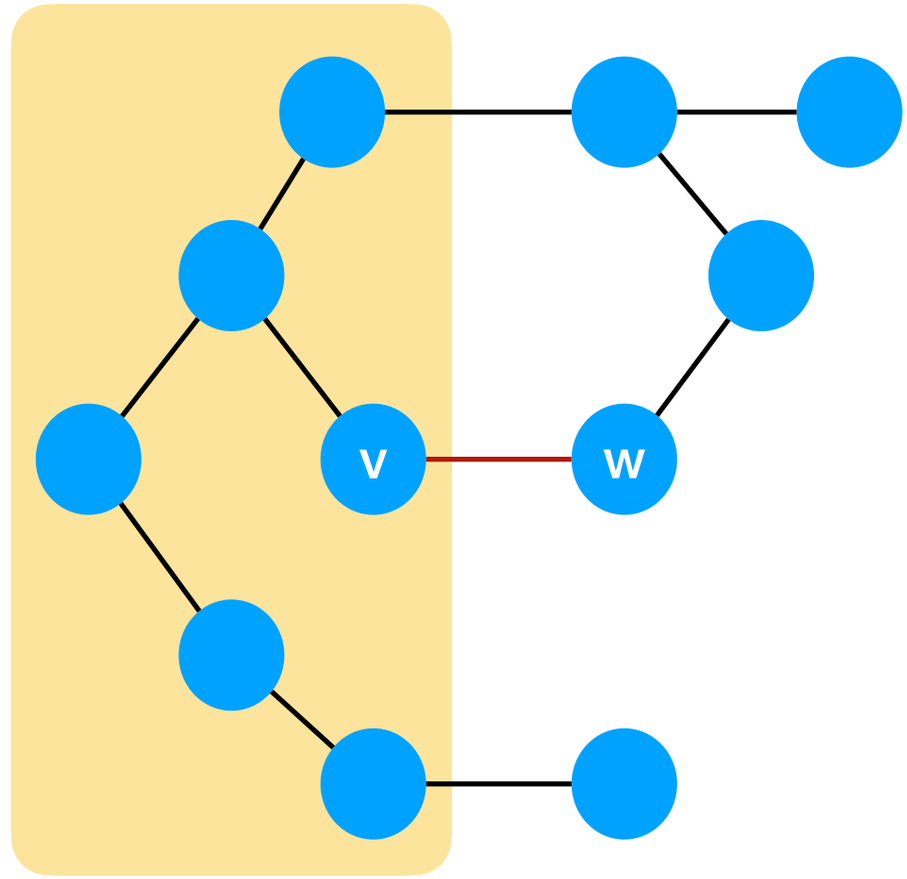

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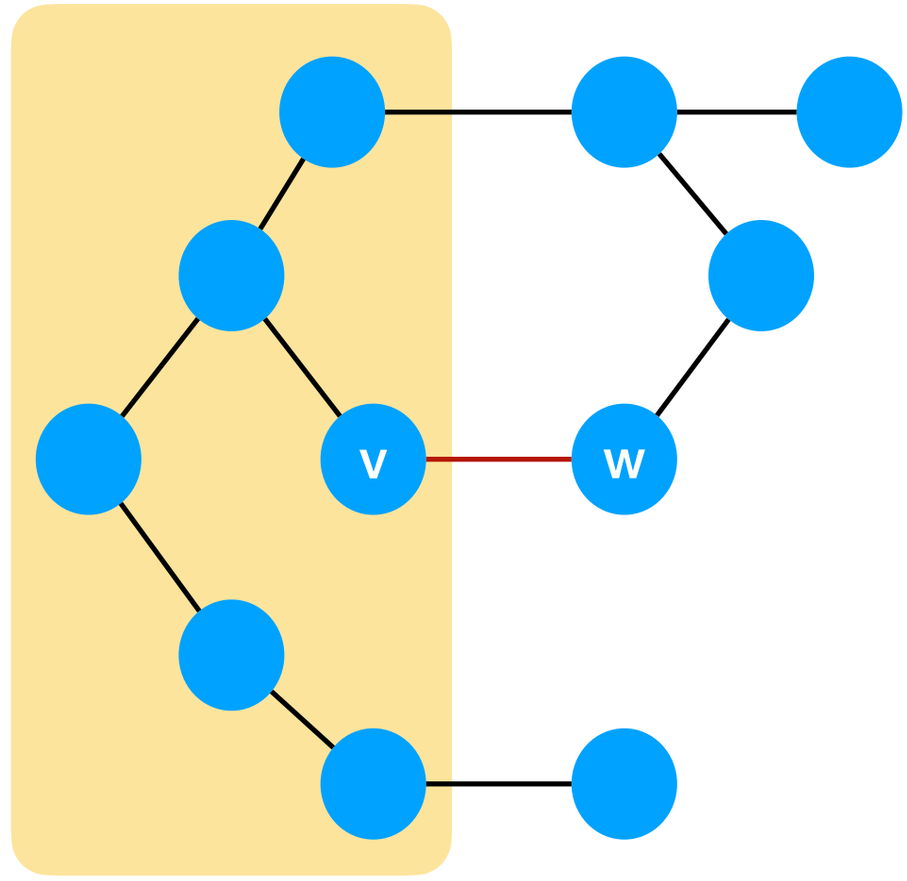

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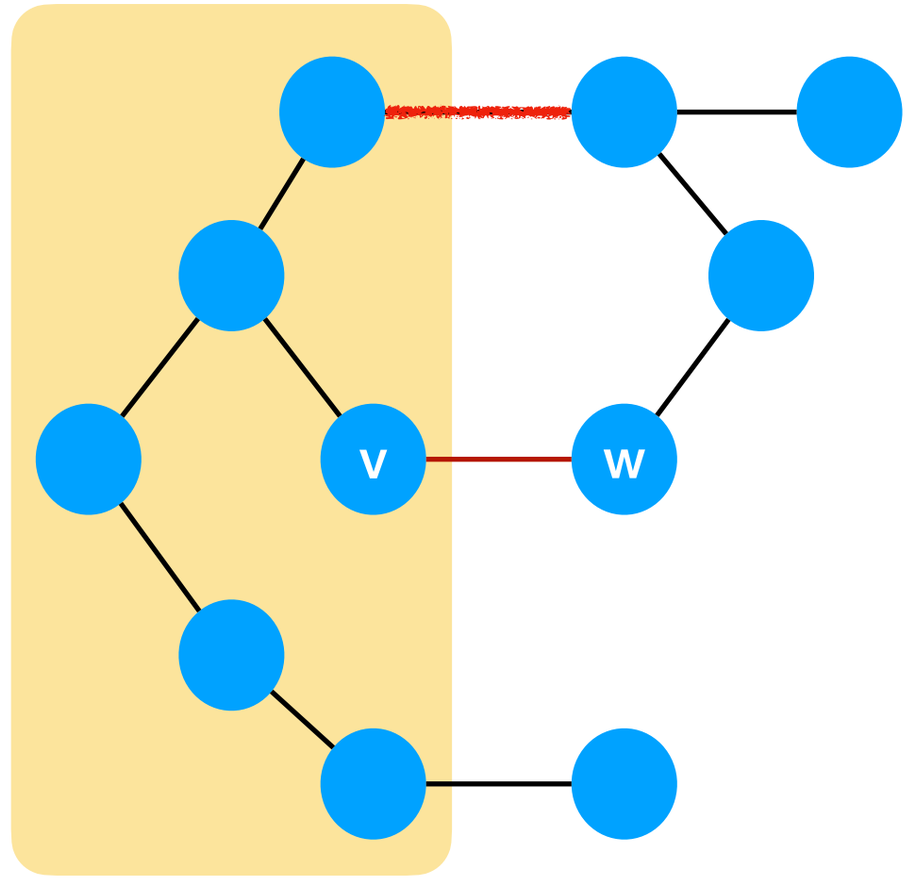

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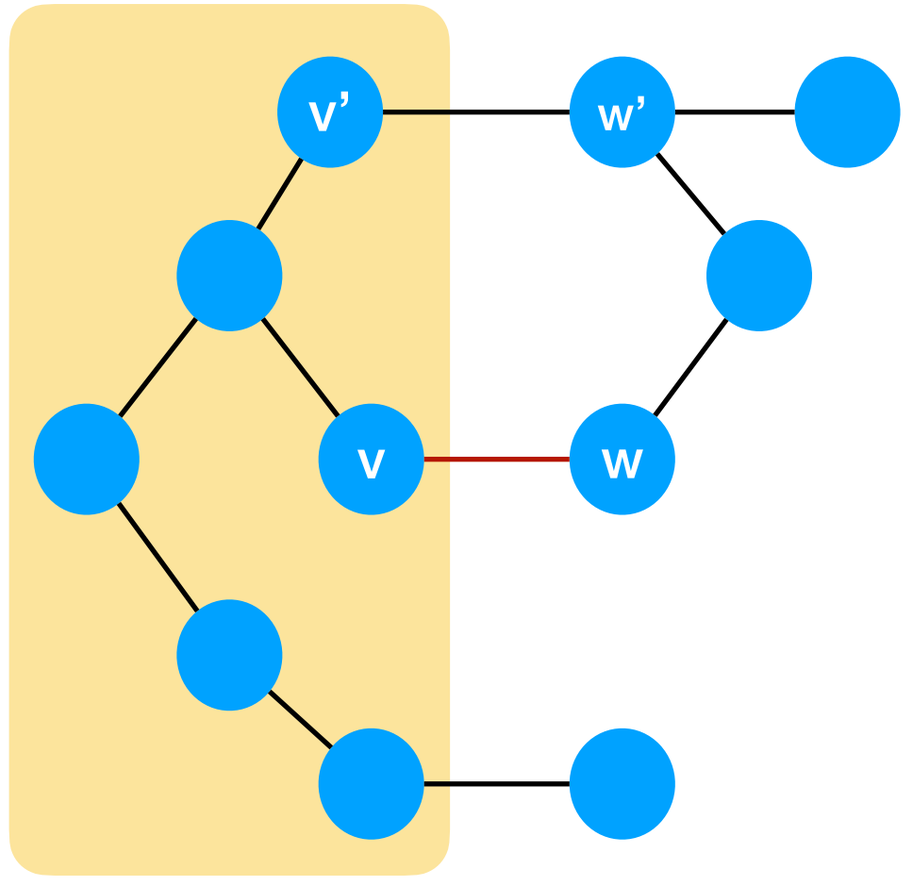

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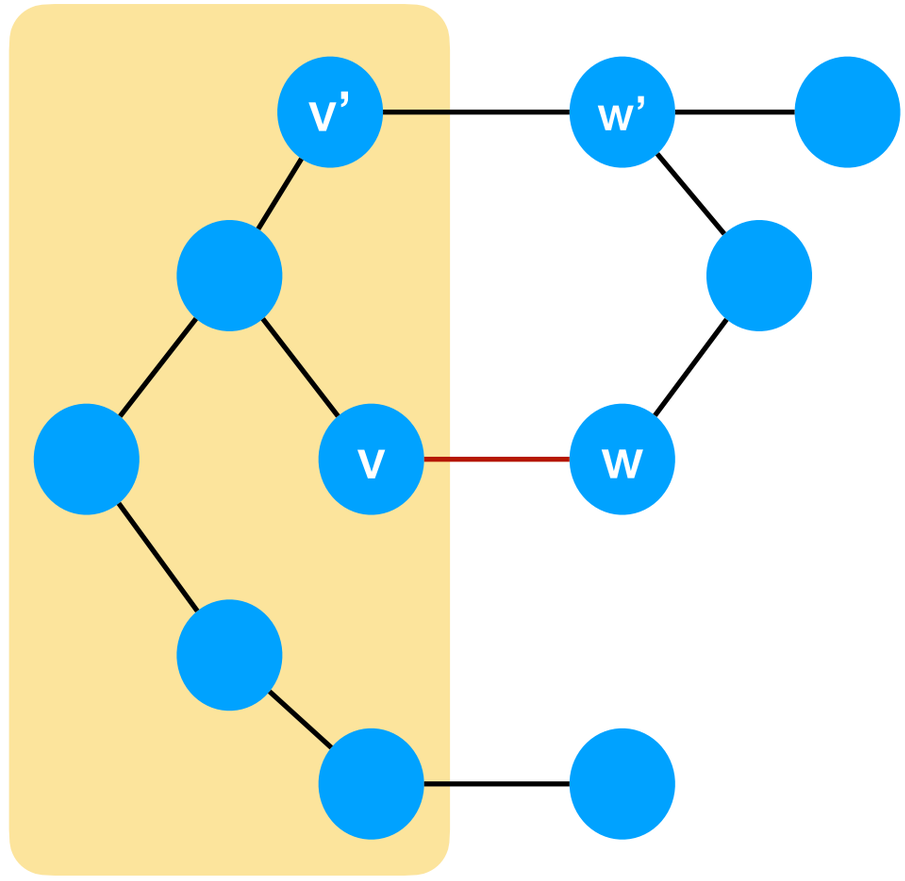

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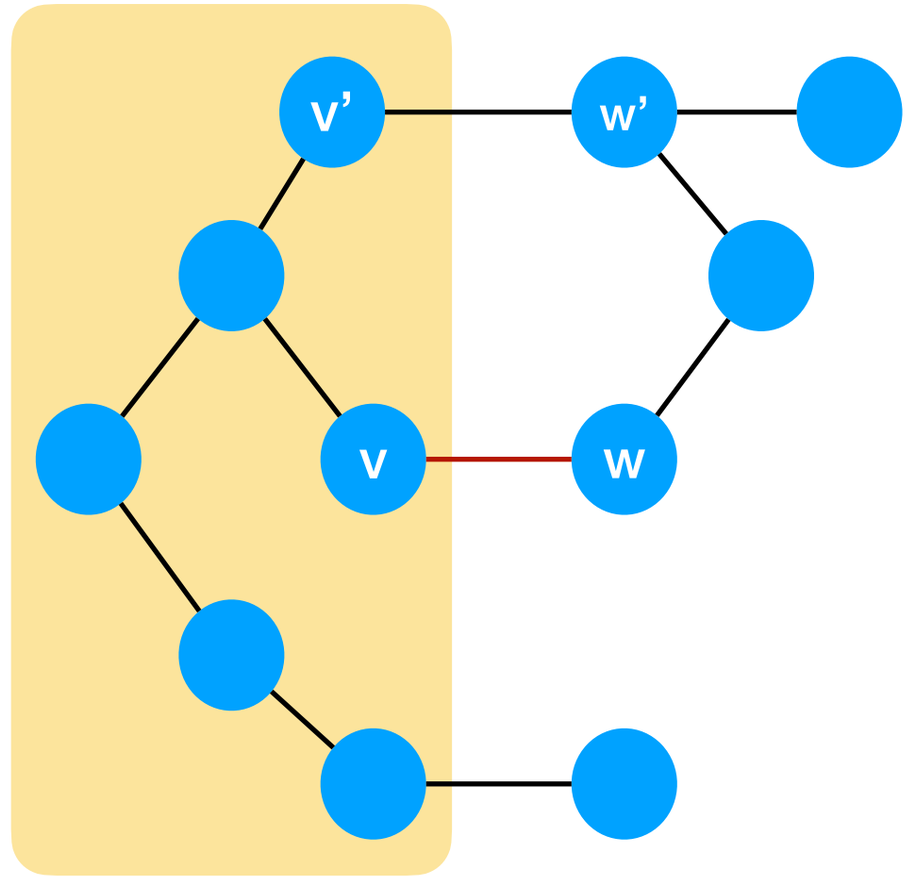

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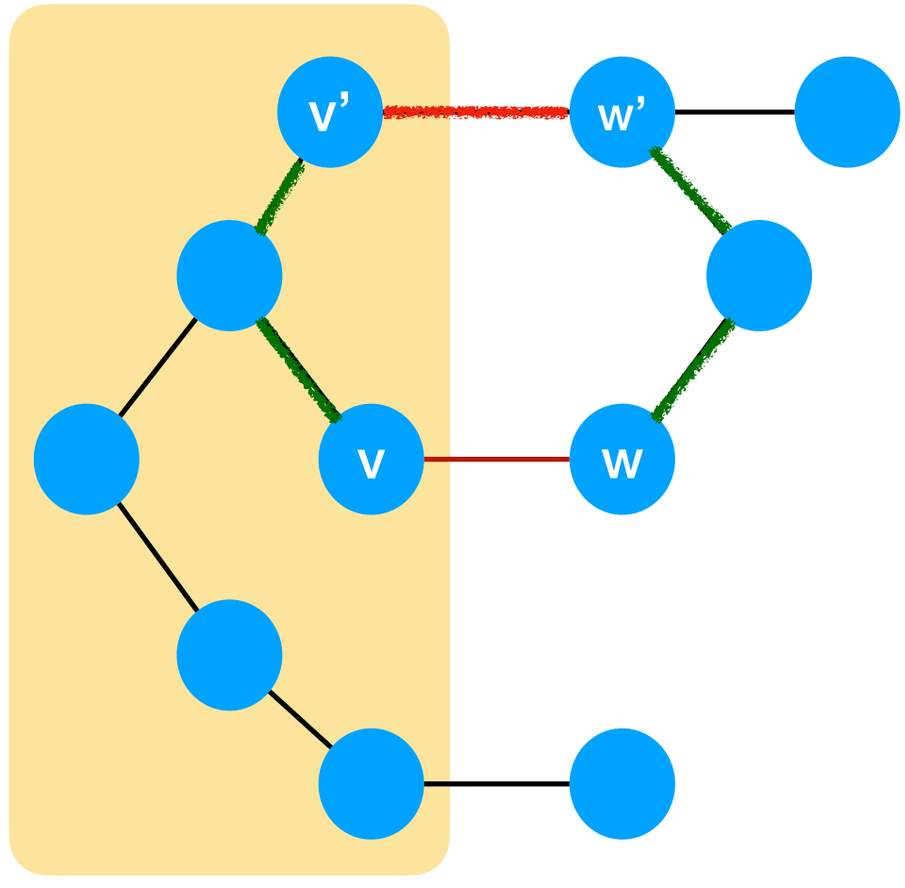

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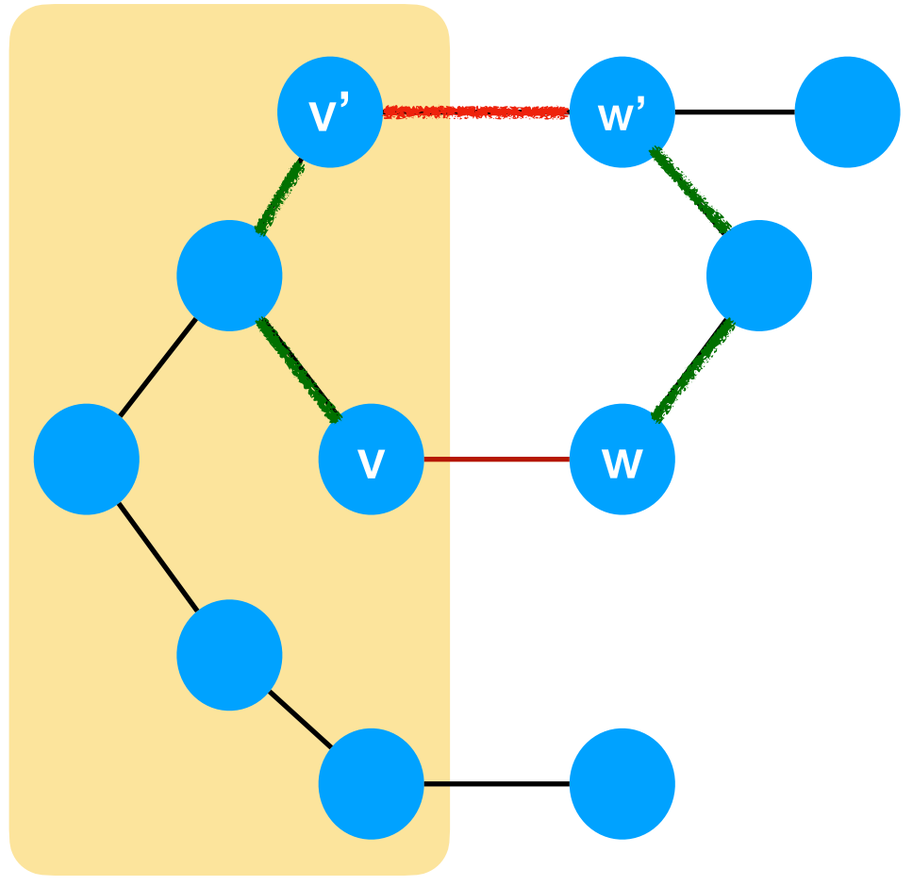

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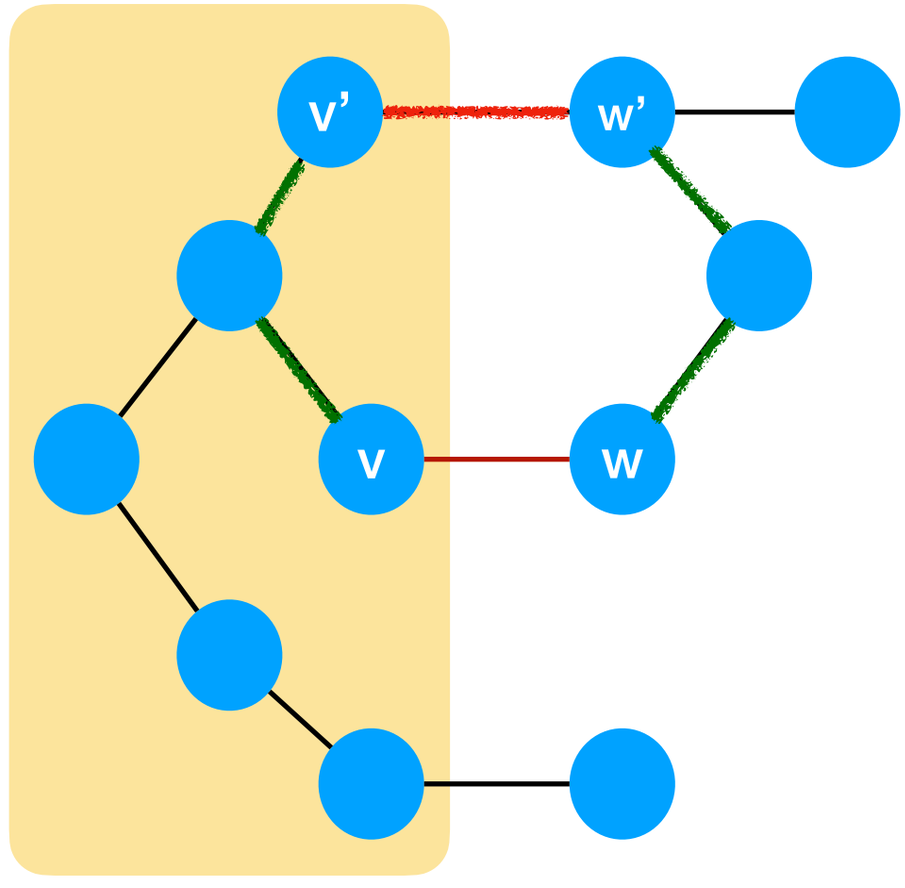
Consider $G'_T = G_T - \{ e' \} \cup \{ e \}$.



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By the cut property, it belongs to every minimum spanning tree.

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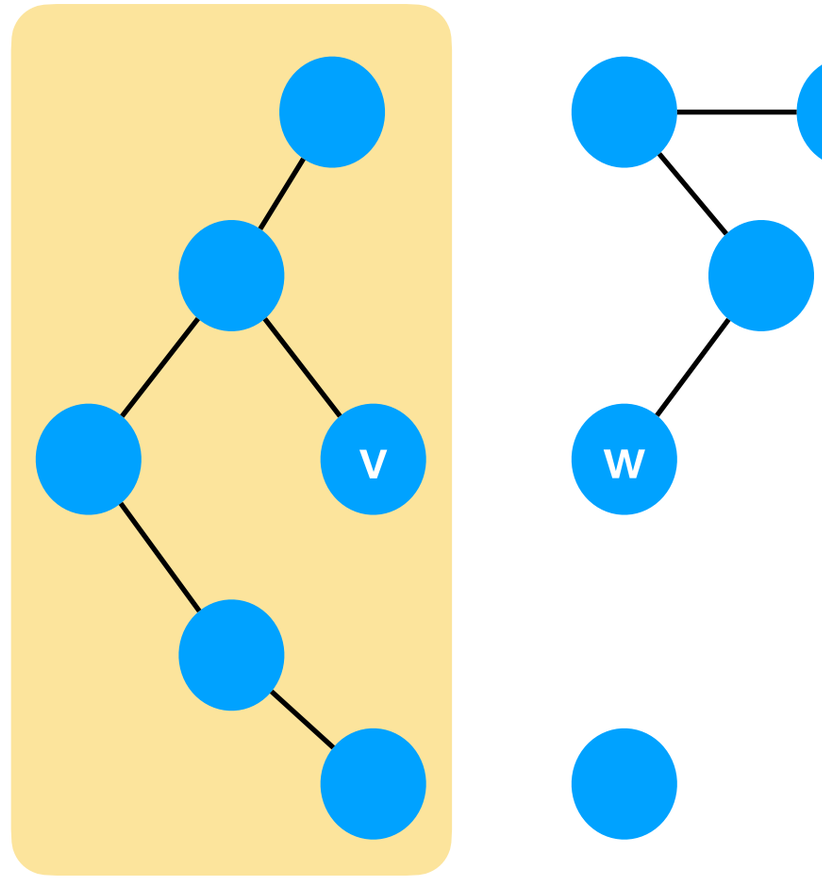
i.e., does it always produce a spanning tree?



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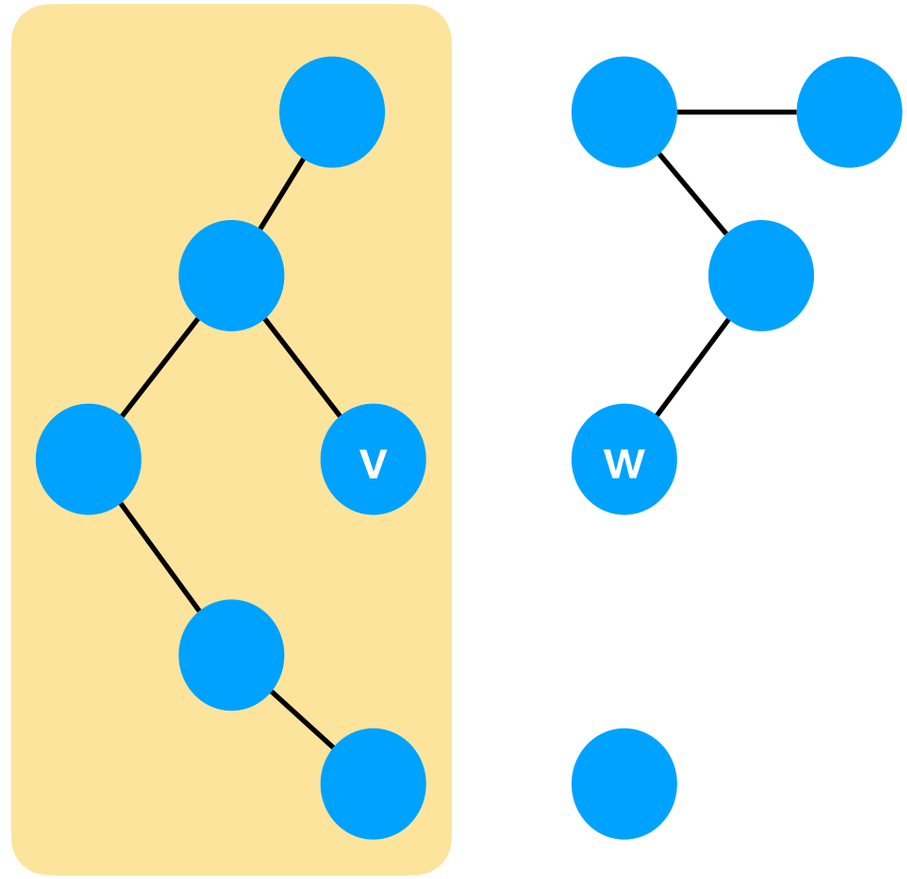

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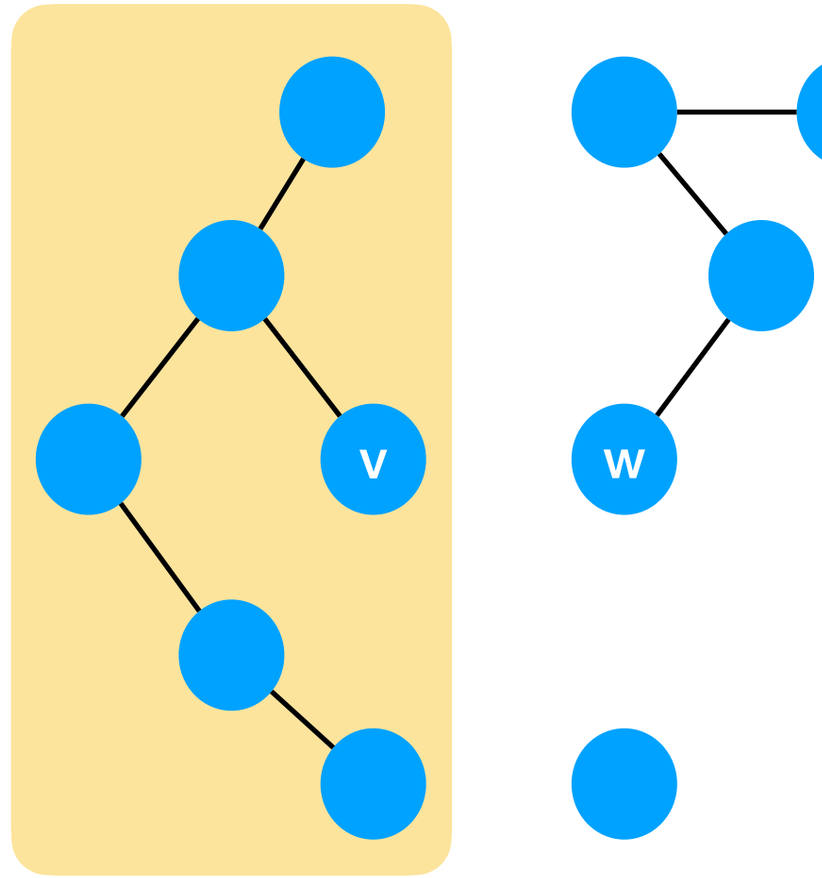

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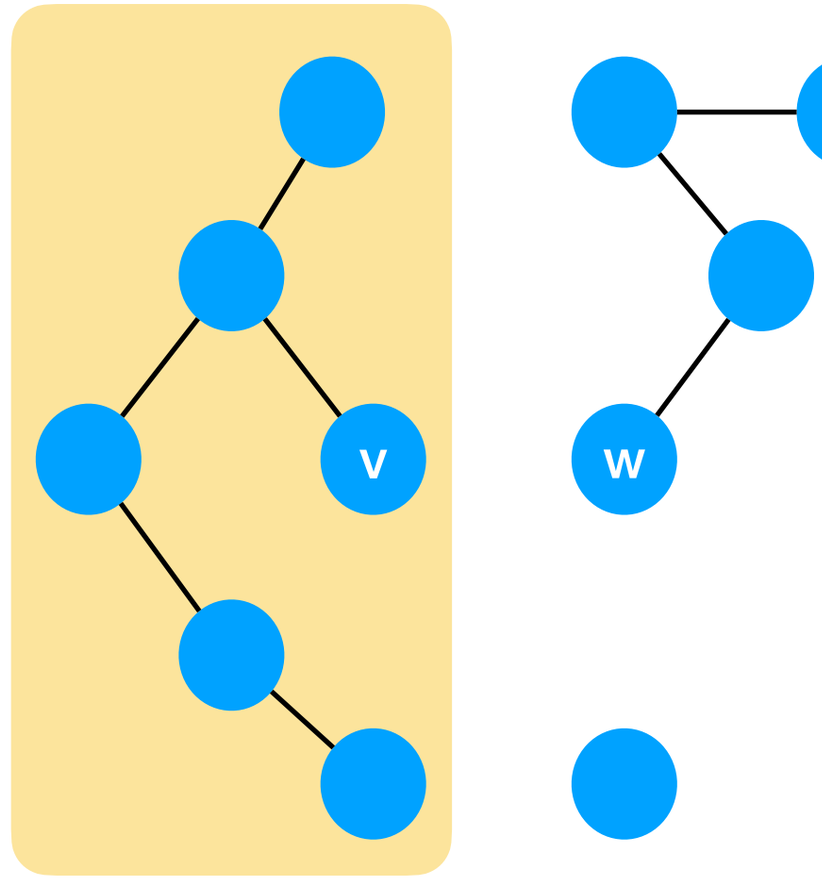

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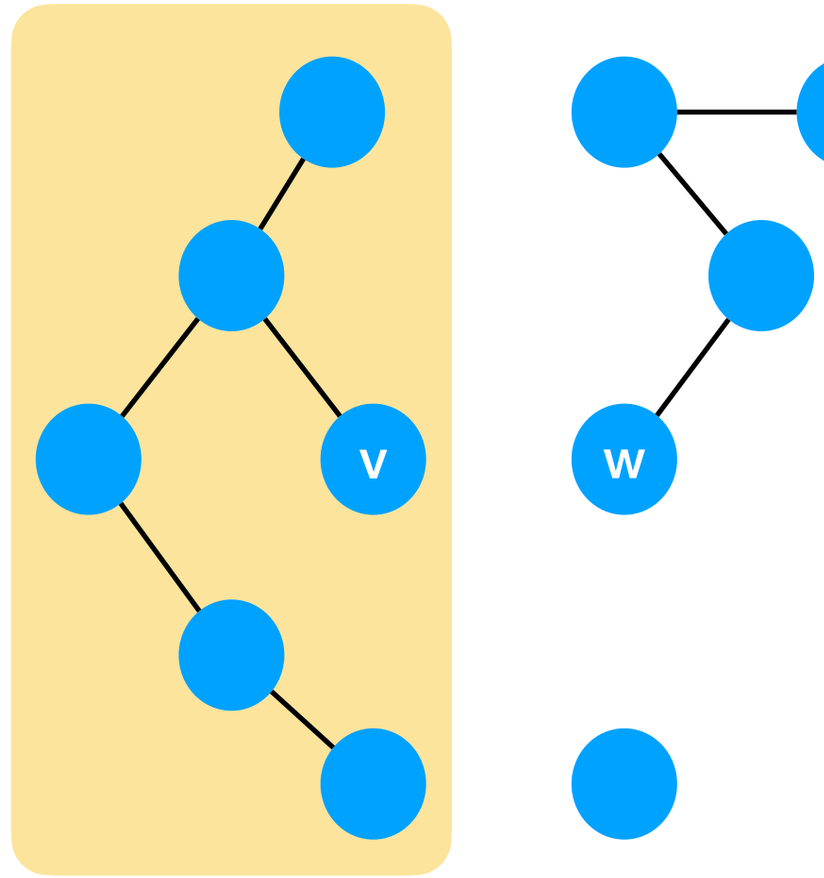

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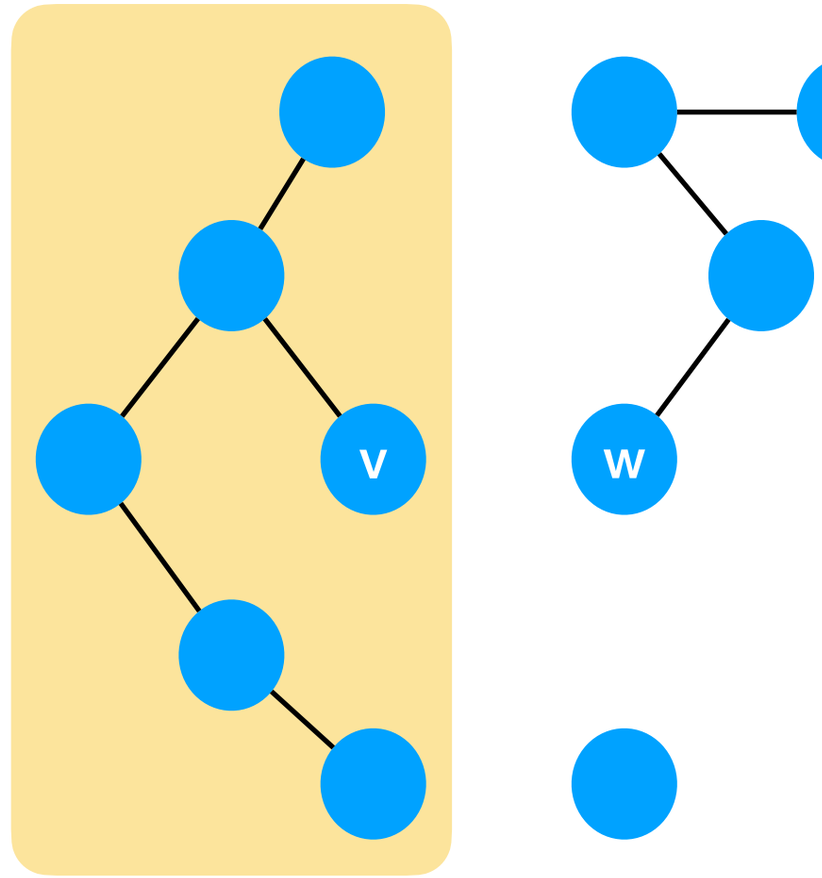

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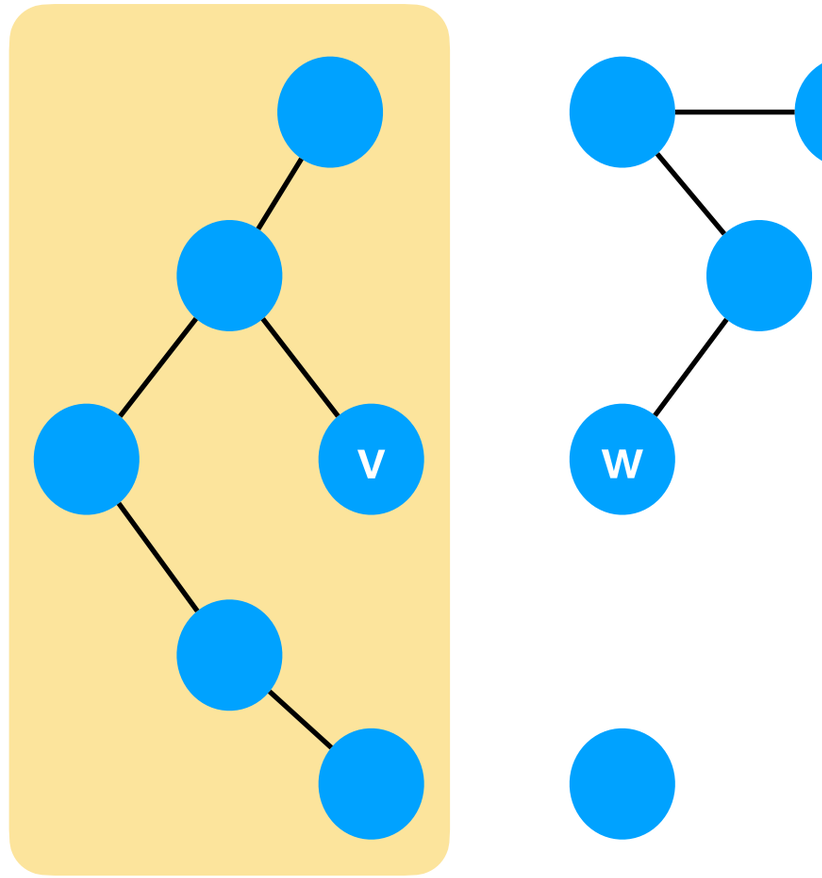

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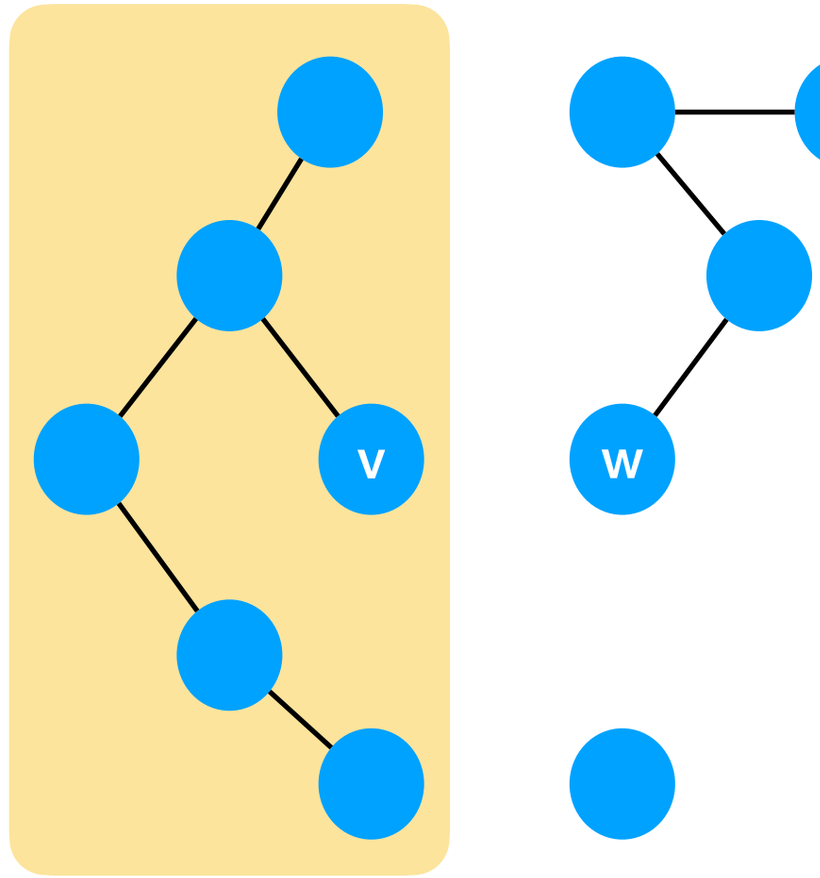
The algorithm would have added an edge crossing the two components.



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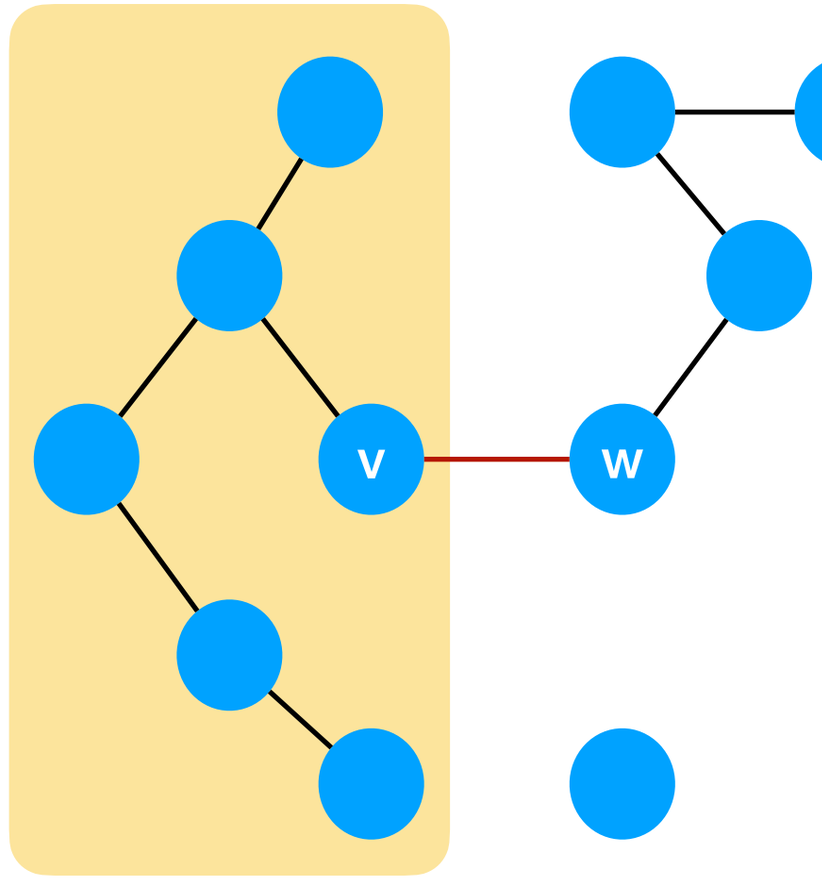

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So it must be part of every minimum spanning tree, by the cut property.

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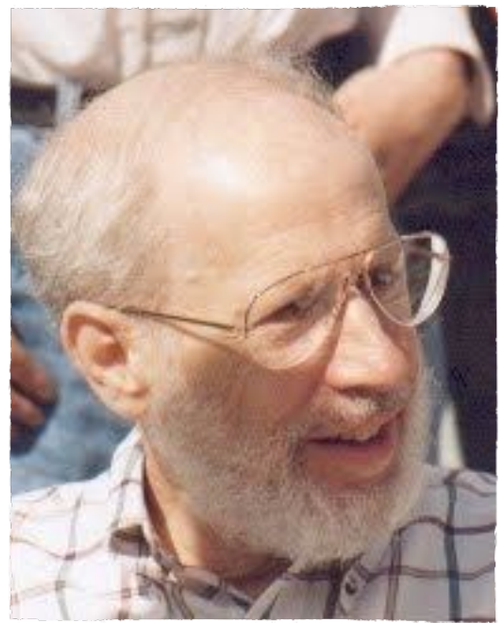

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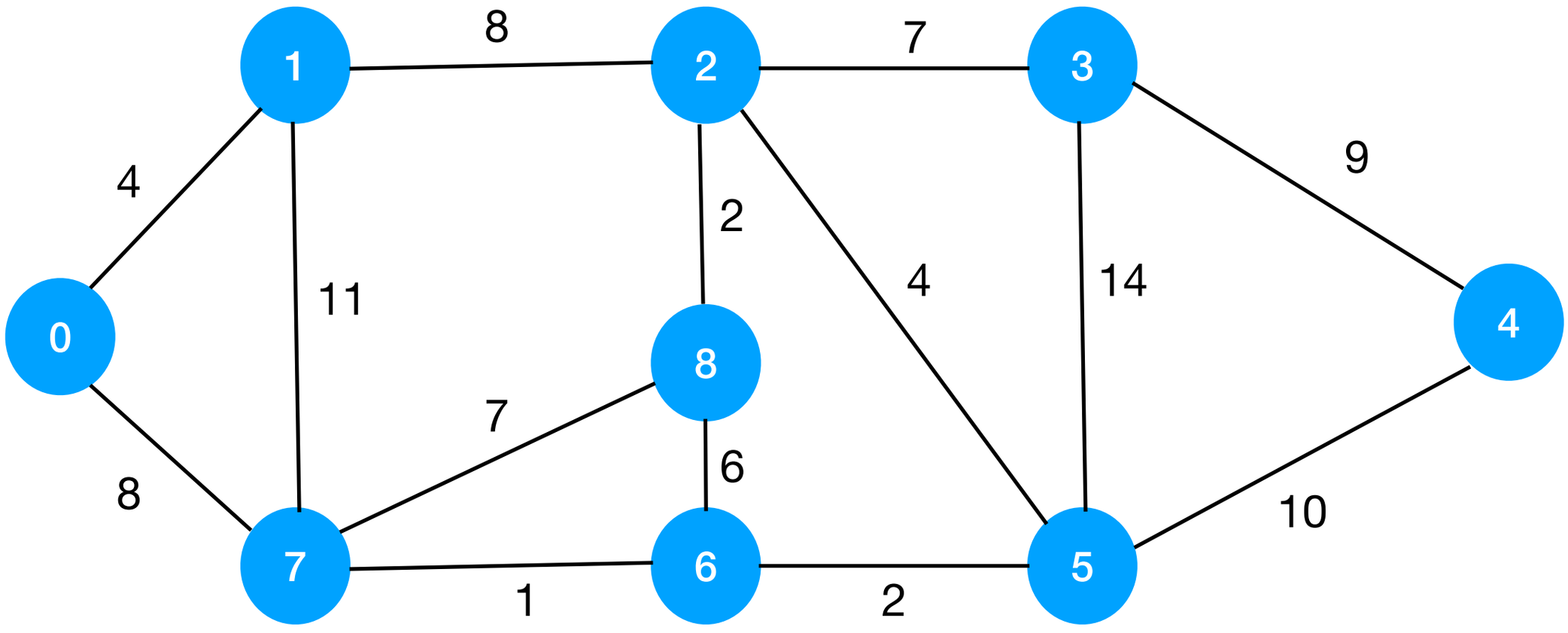

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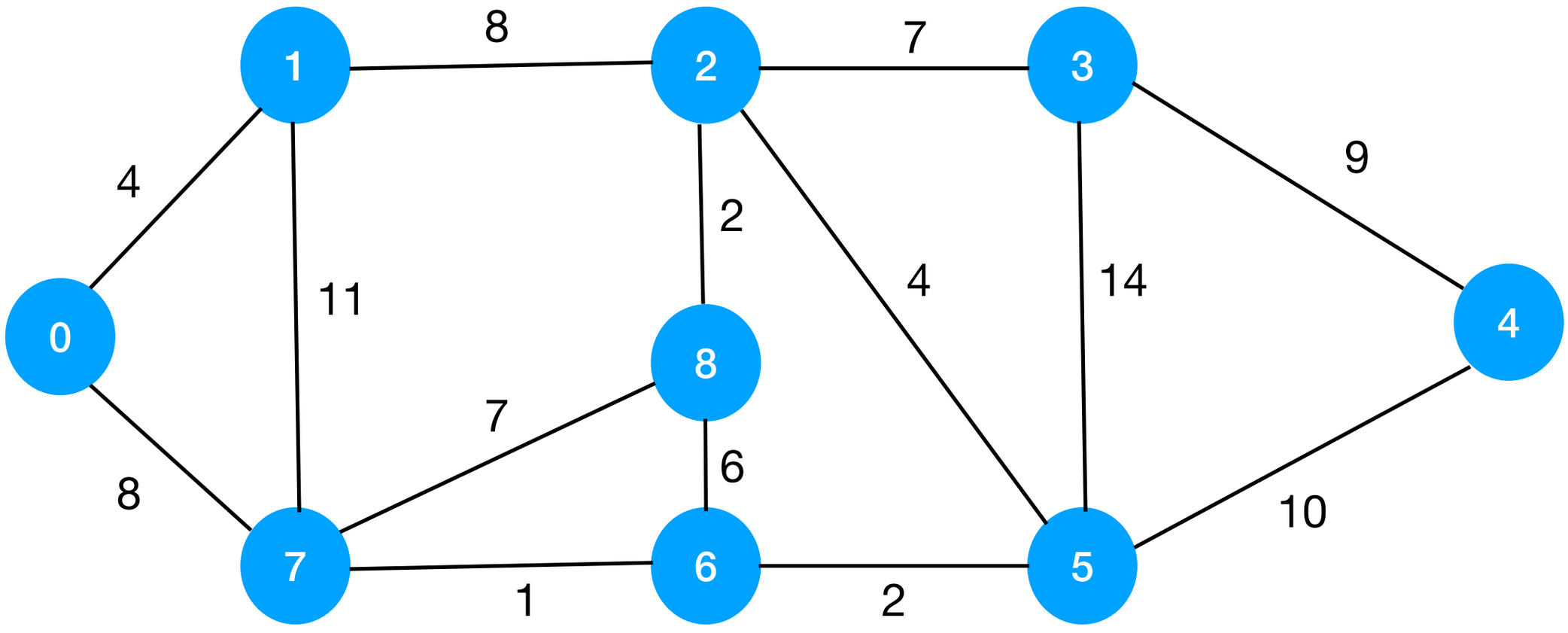

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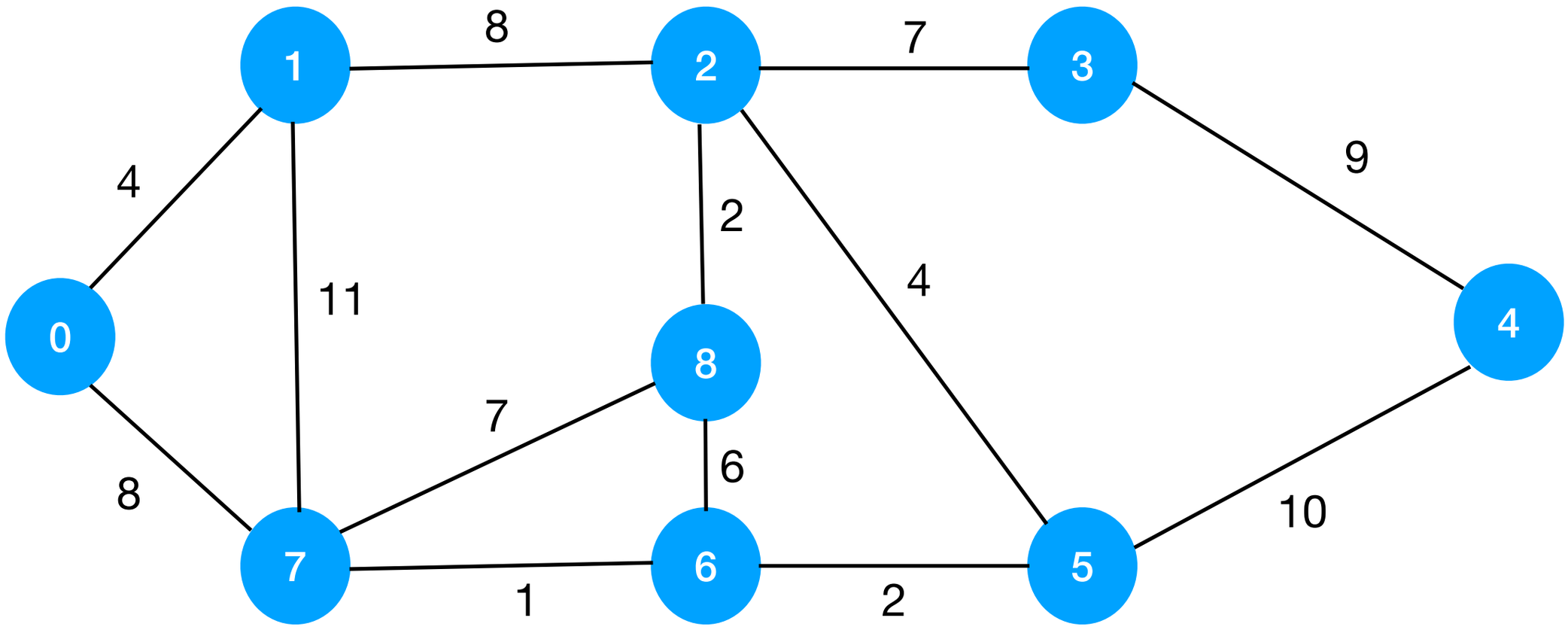

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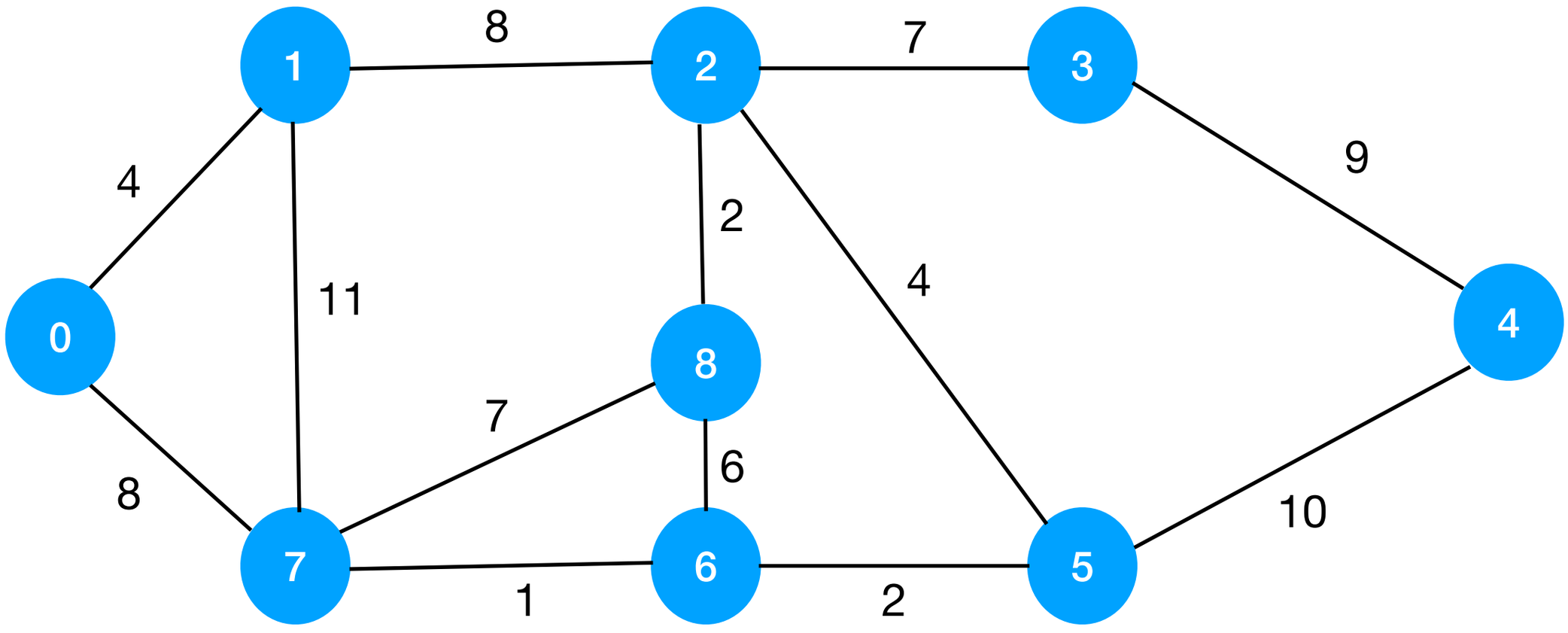

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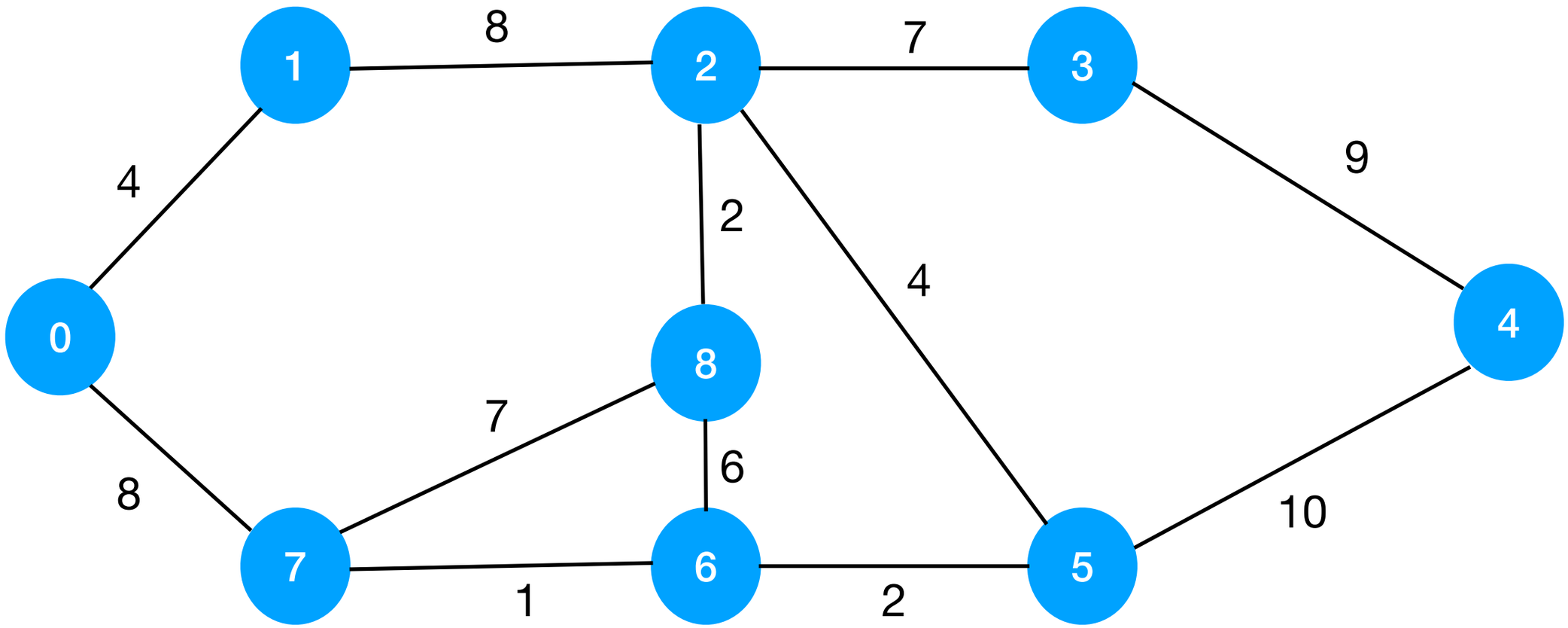

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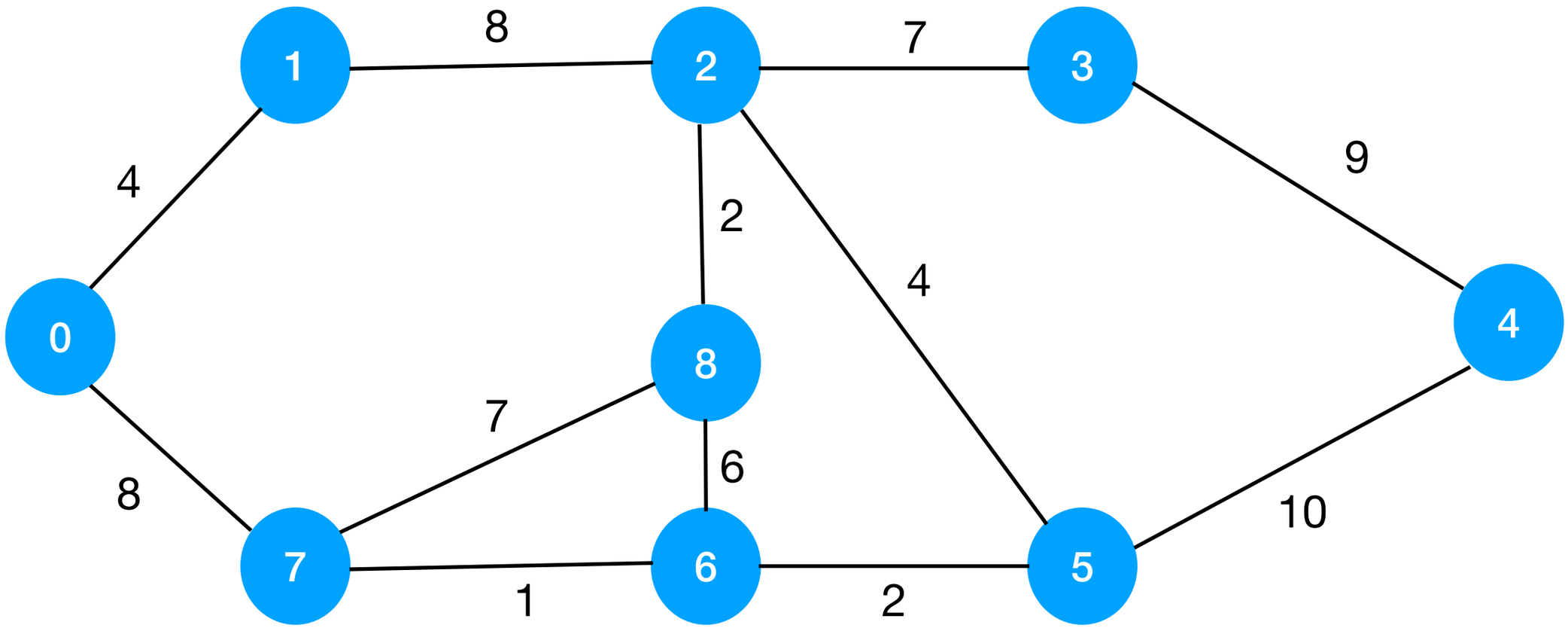

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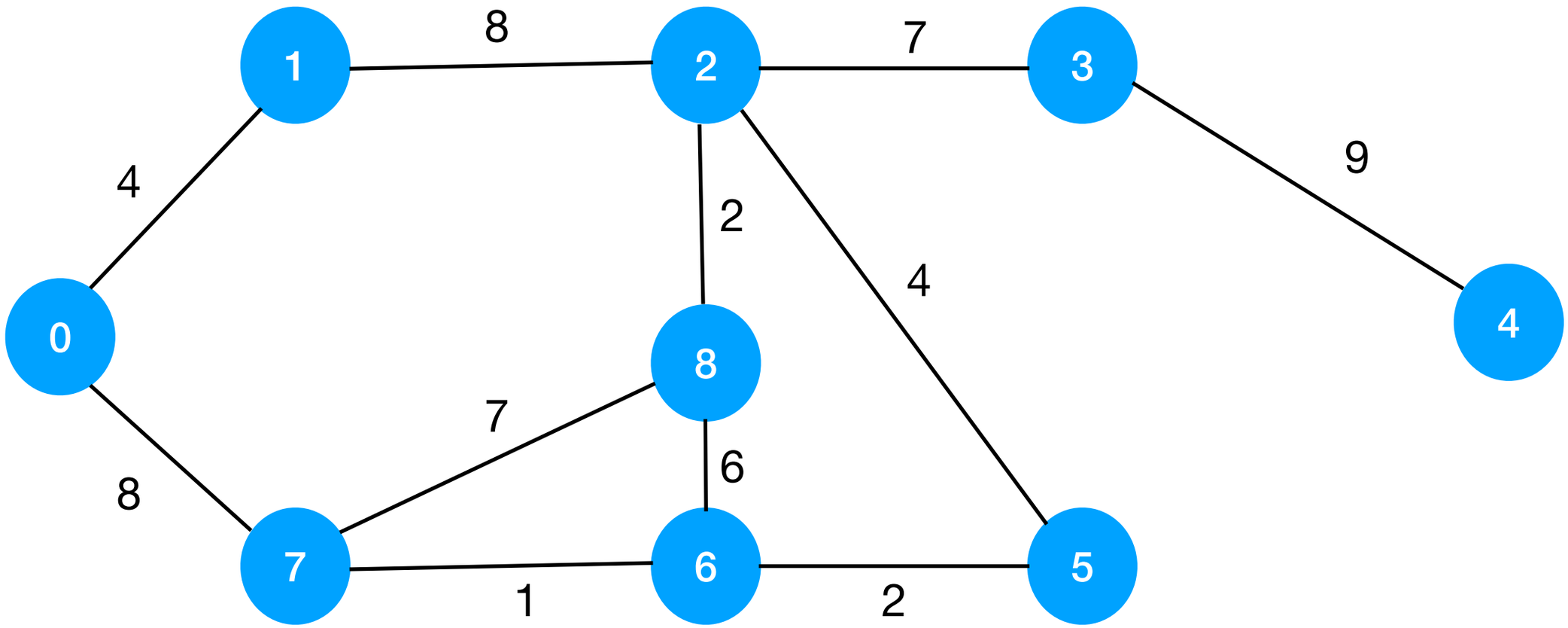

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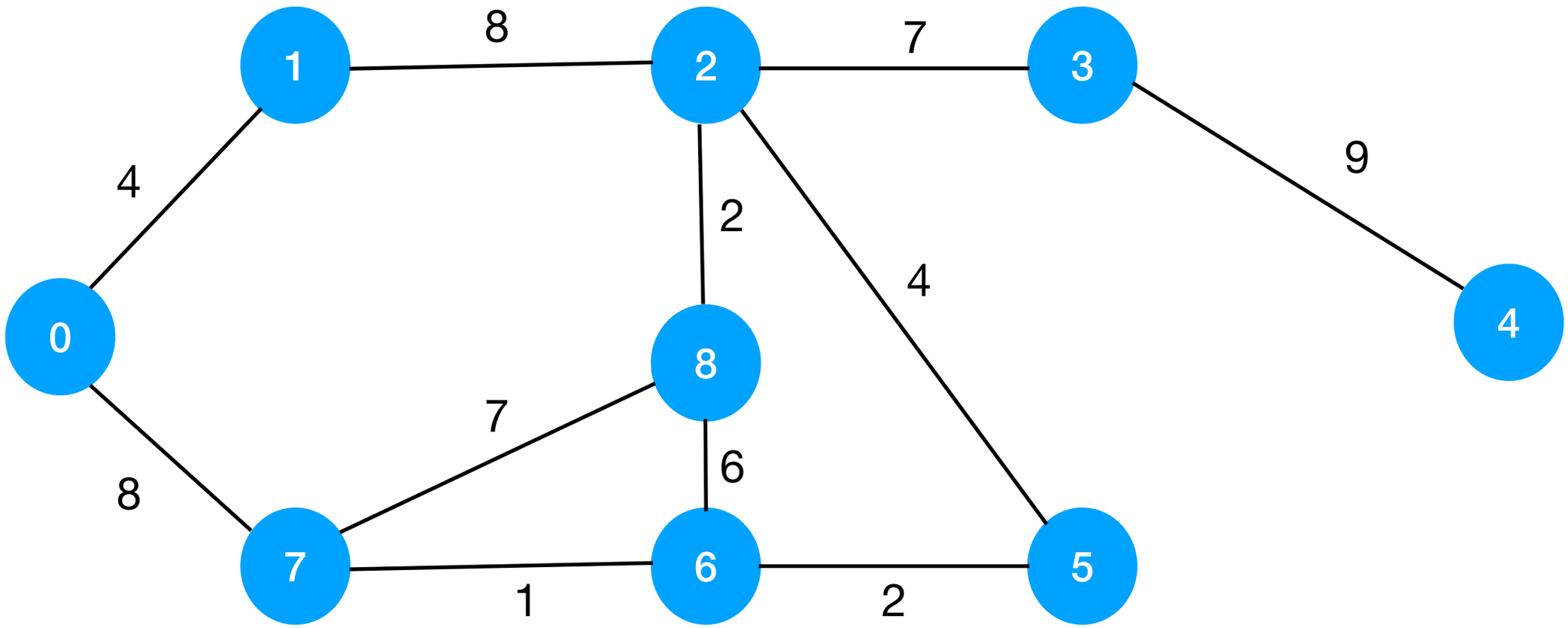

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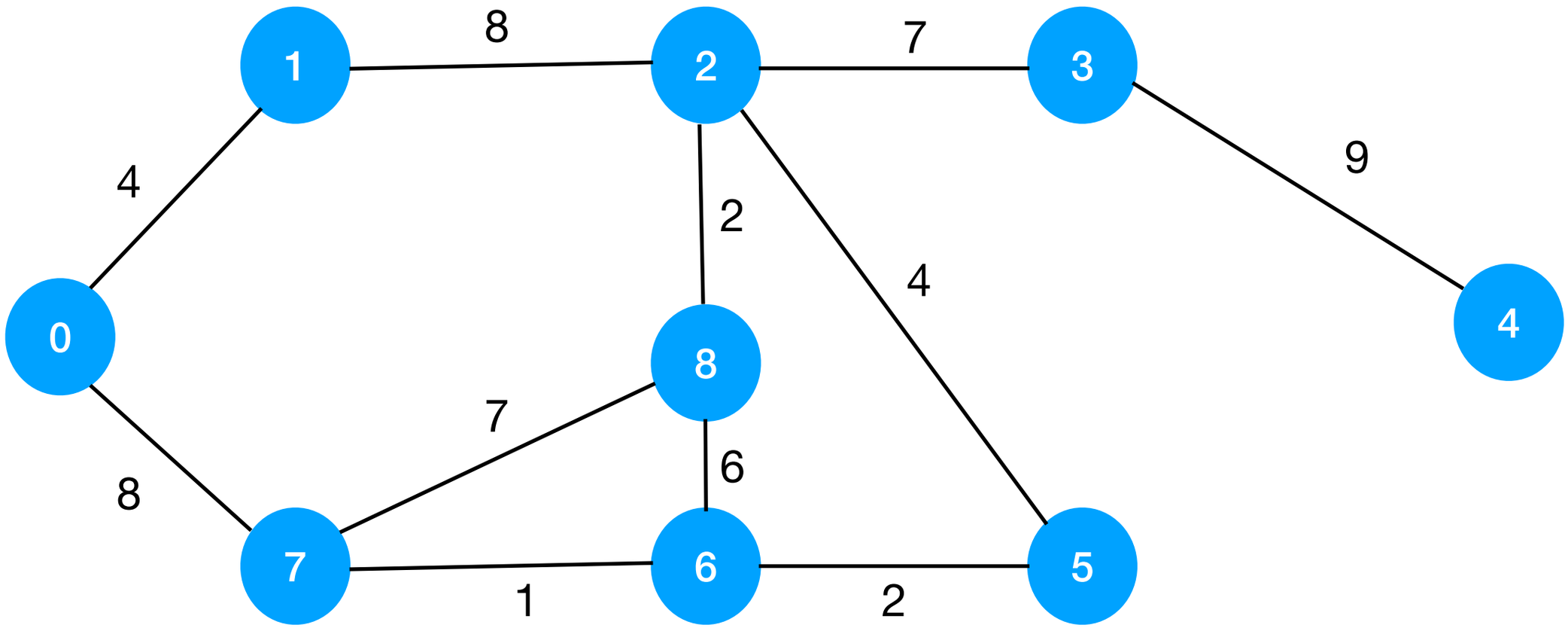

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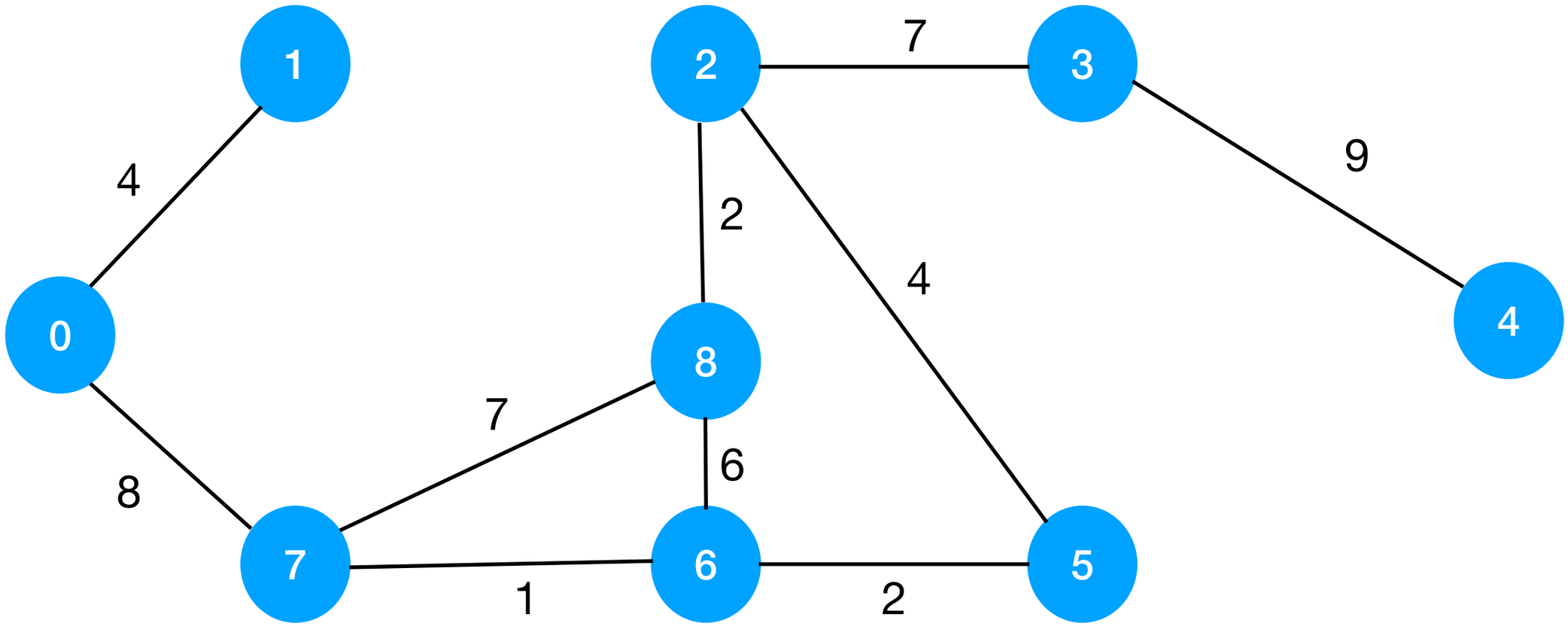

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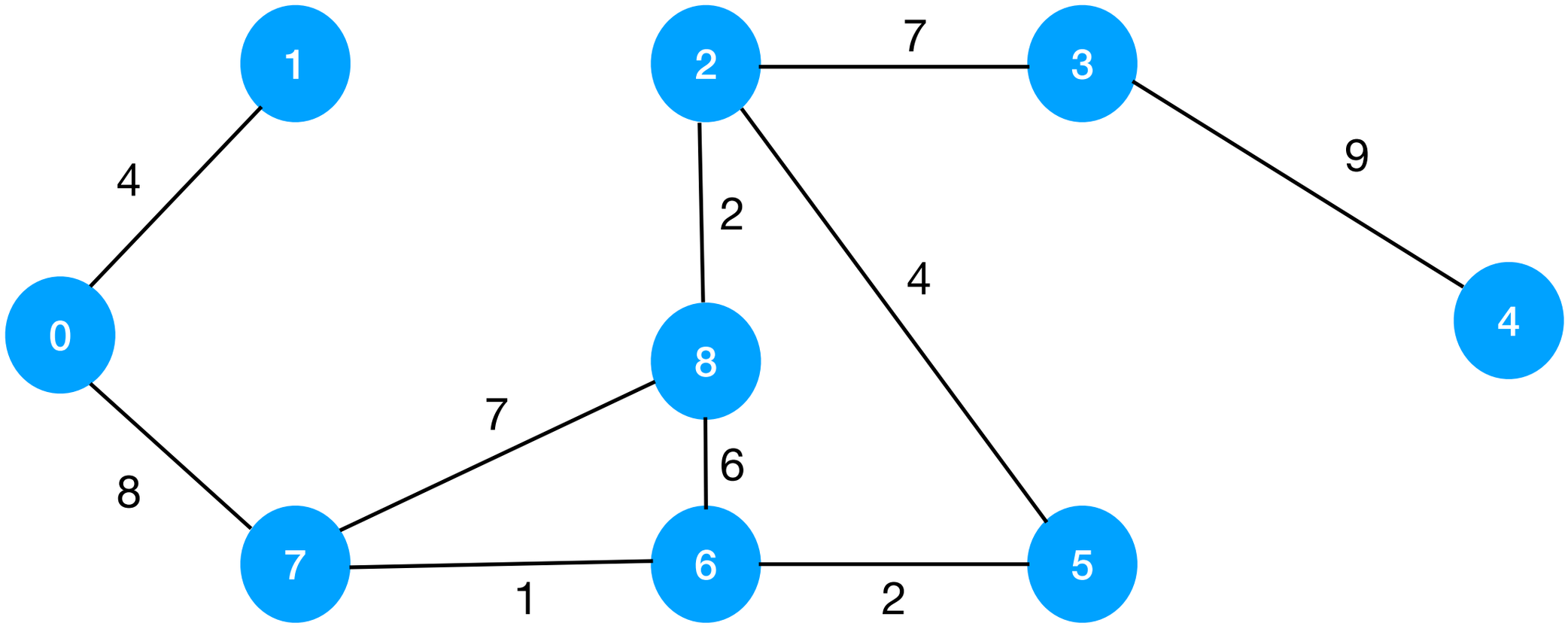

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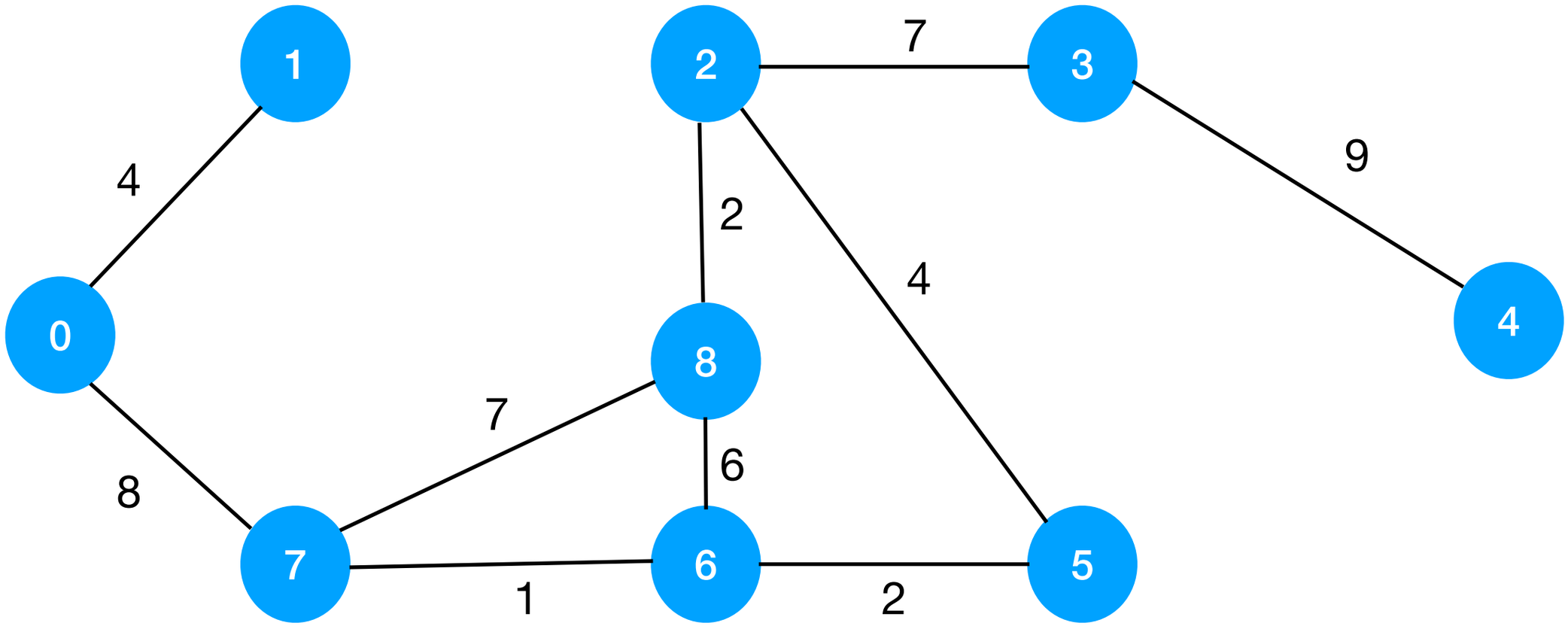

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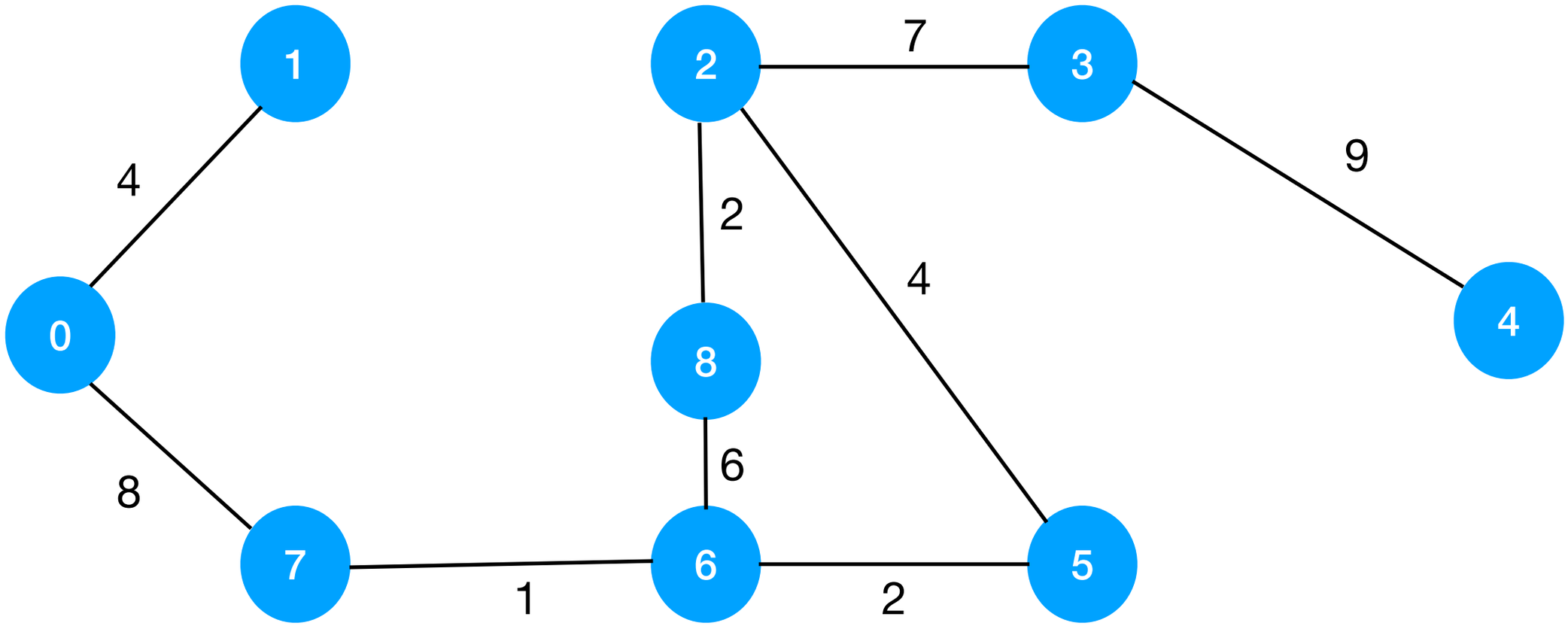

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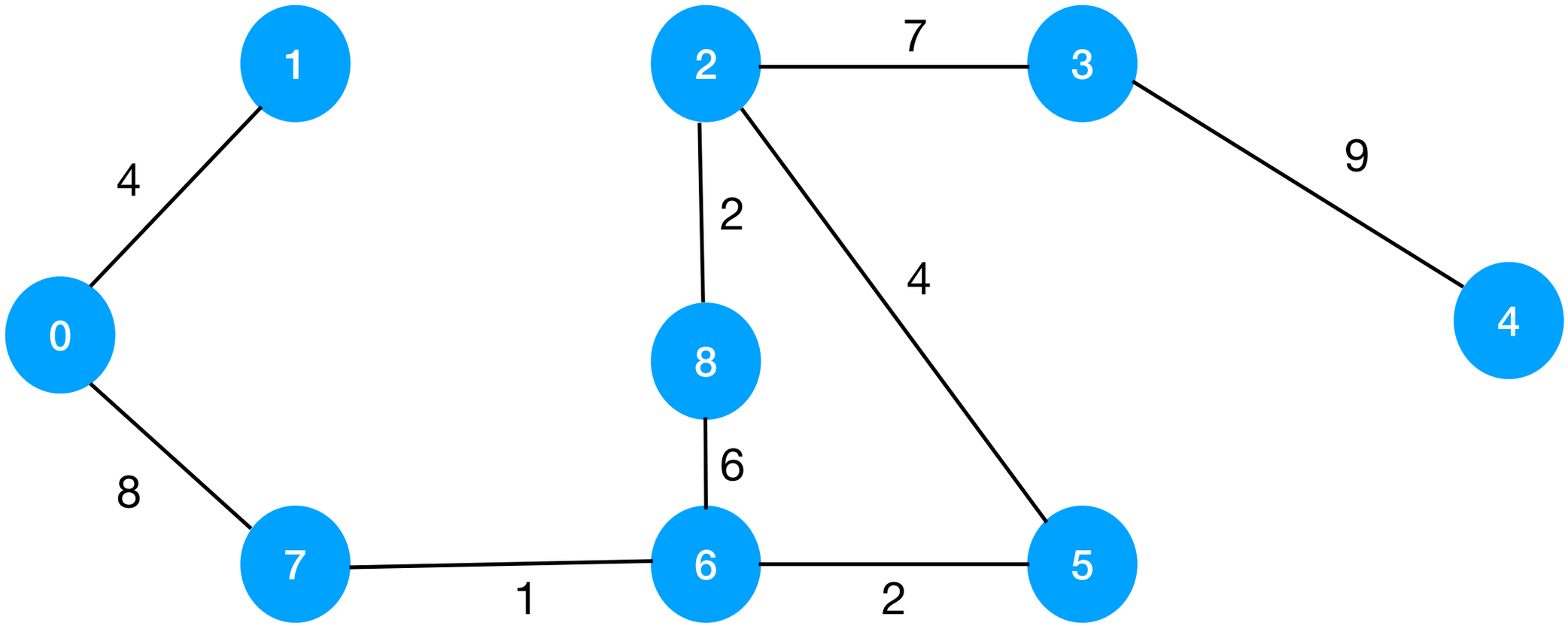

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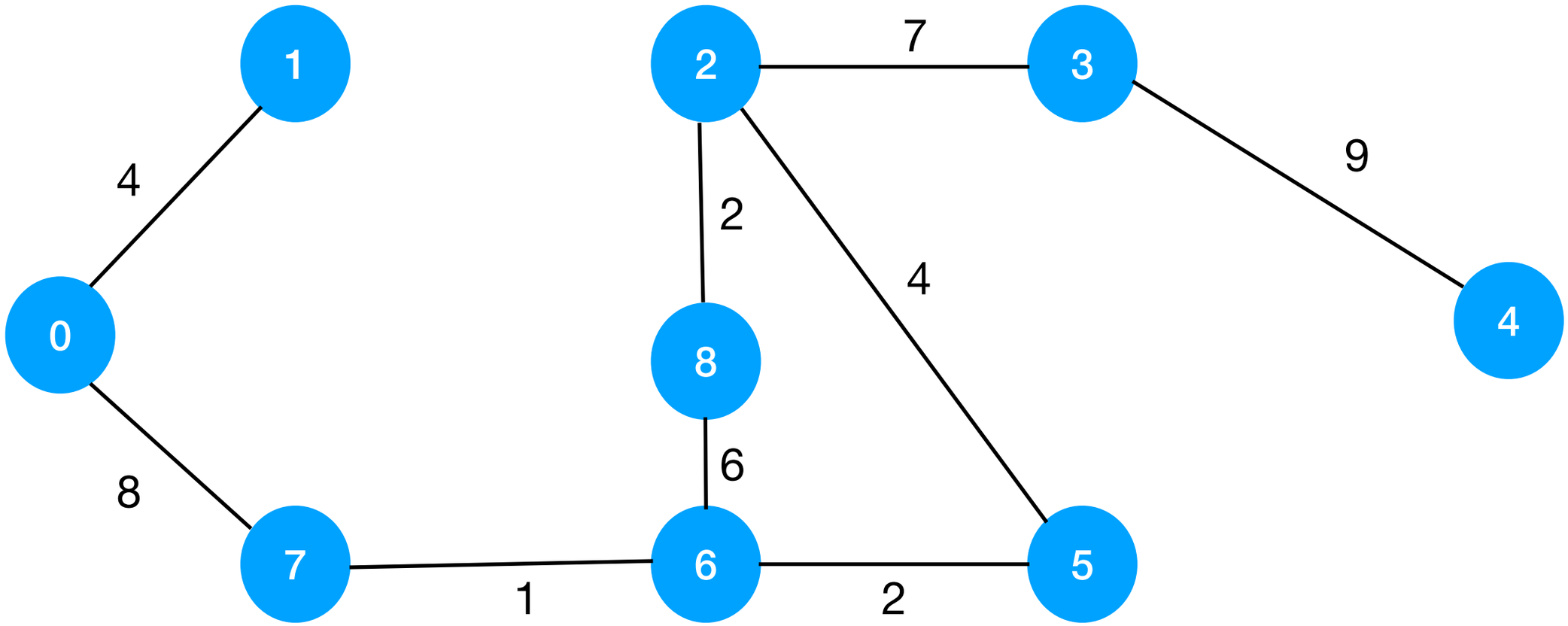

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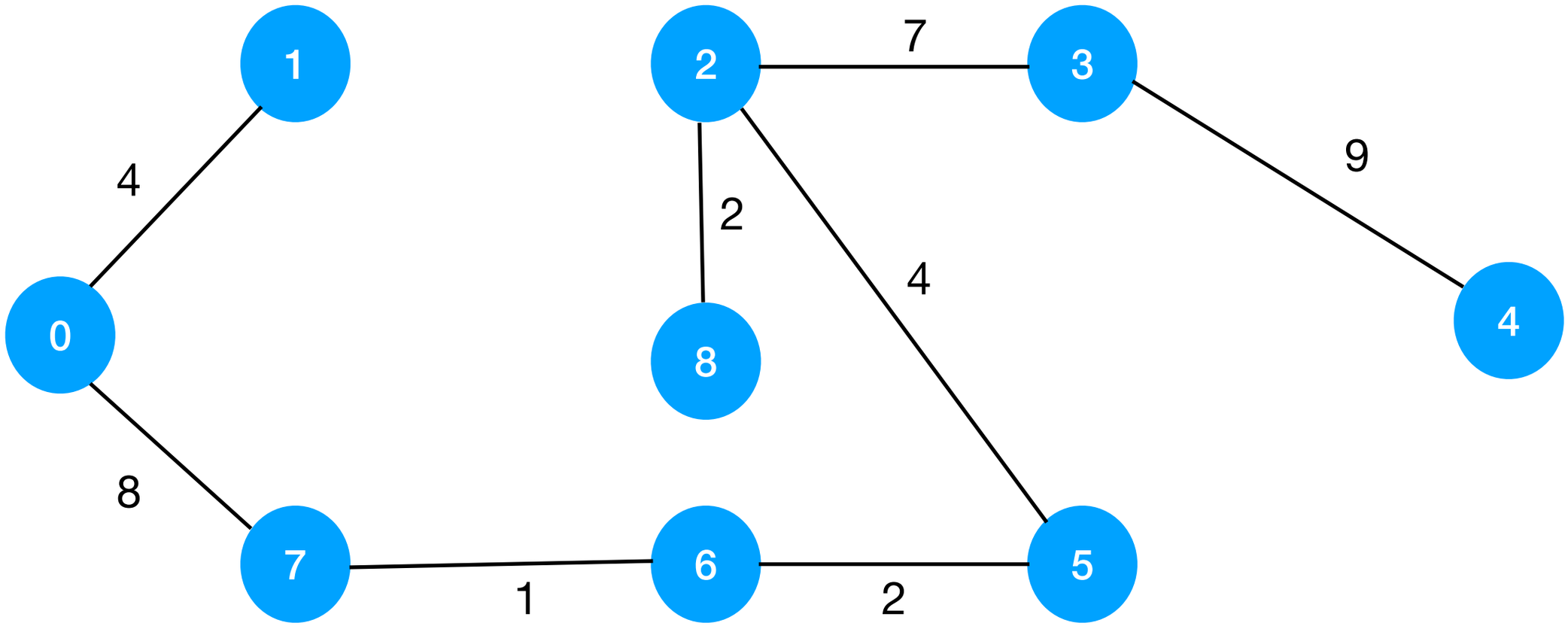

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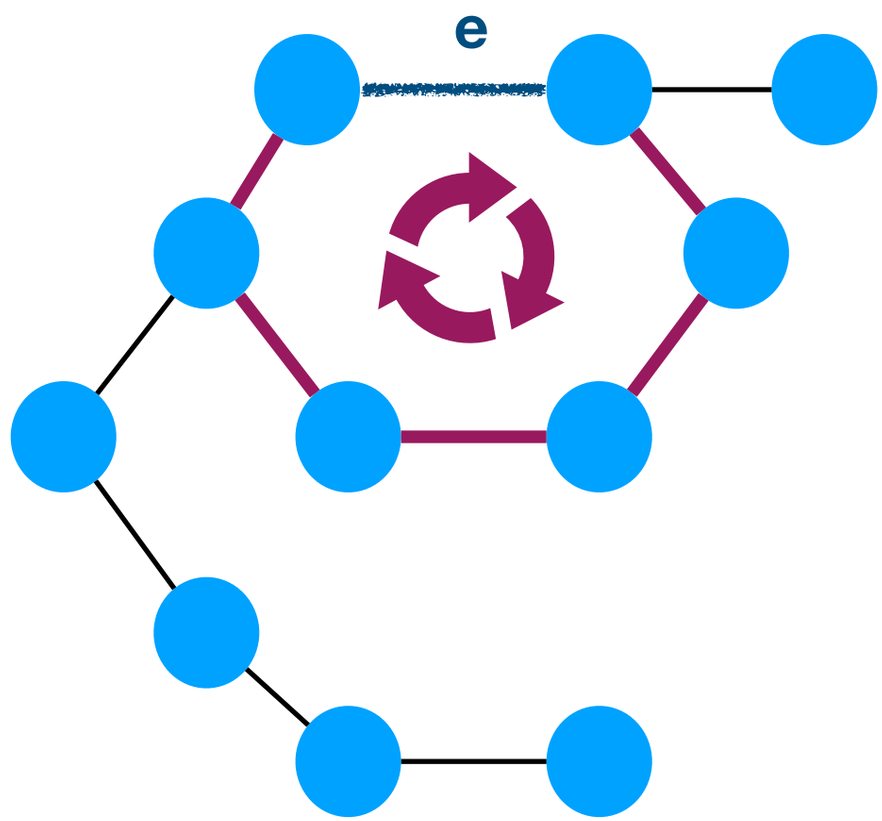

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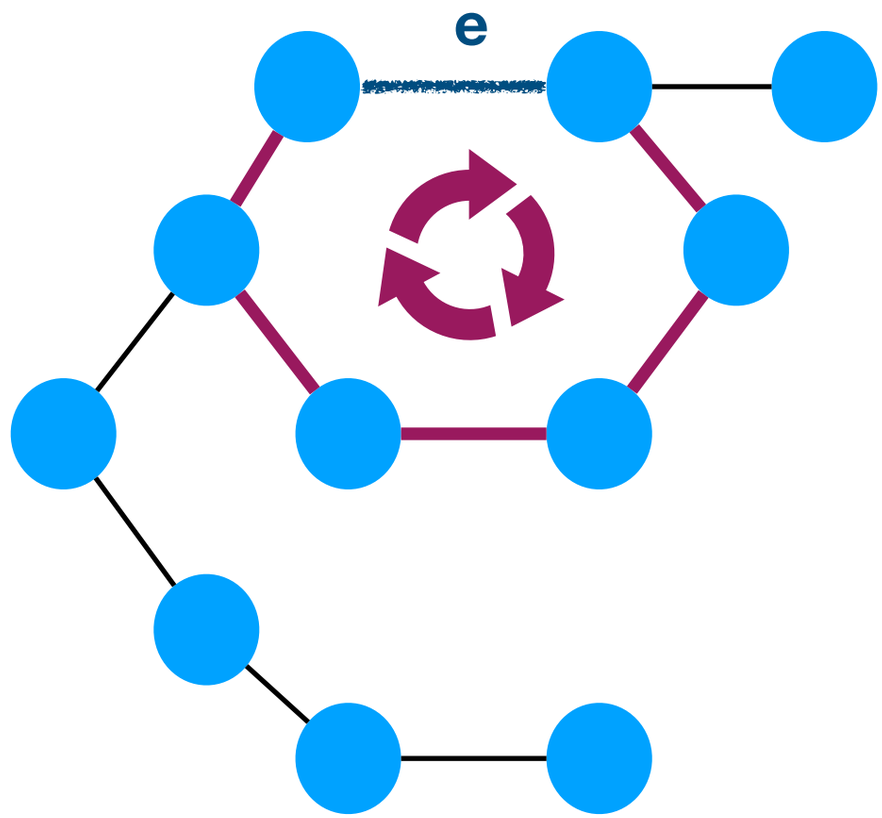

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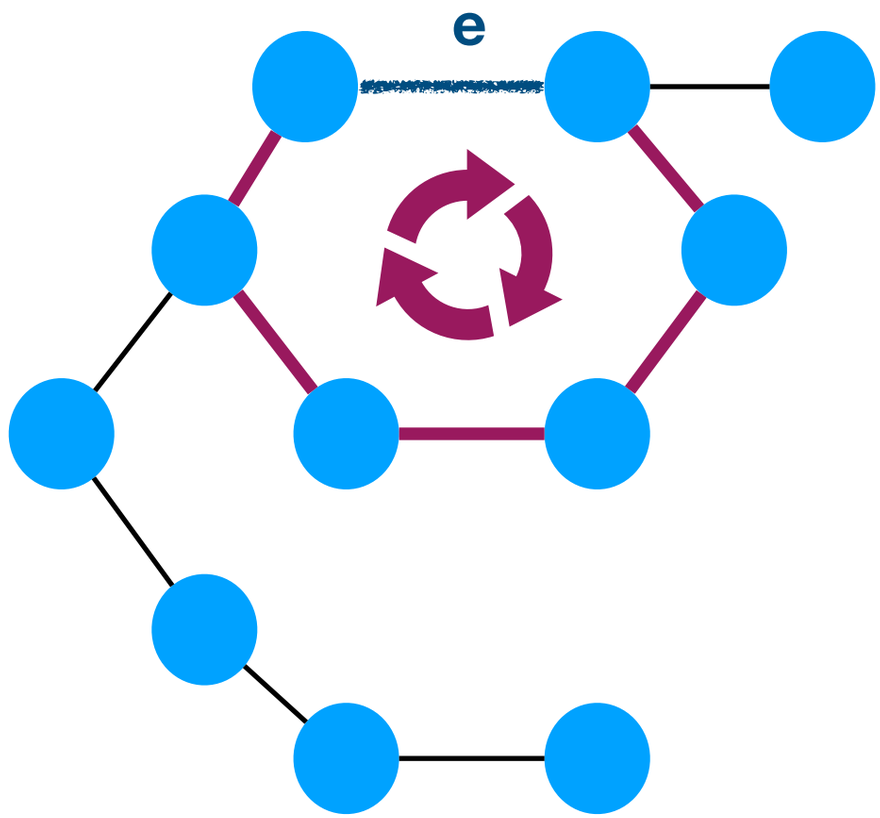
- Let T be a spanning tree that contains e.



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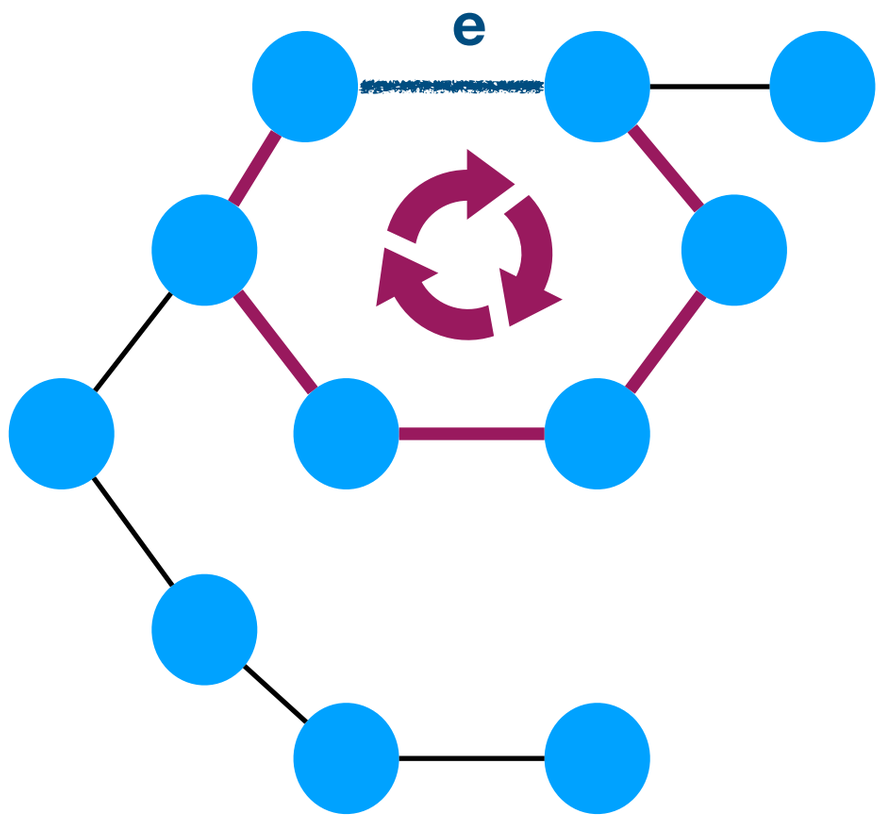

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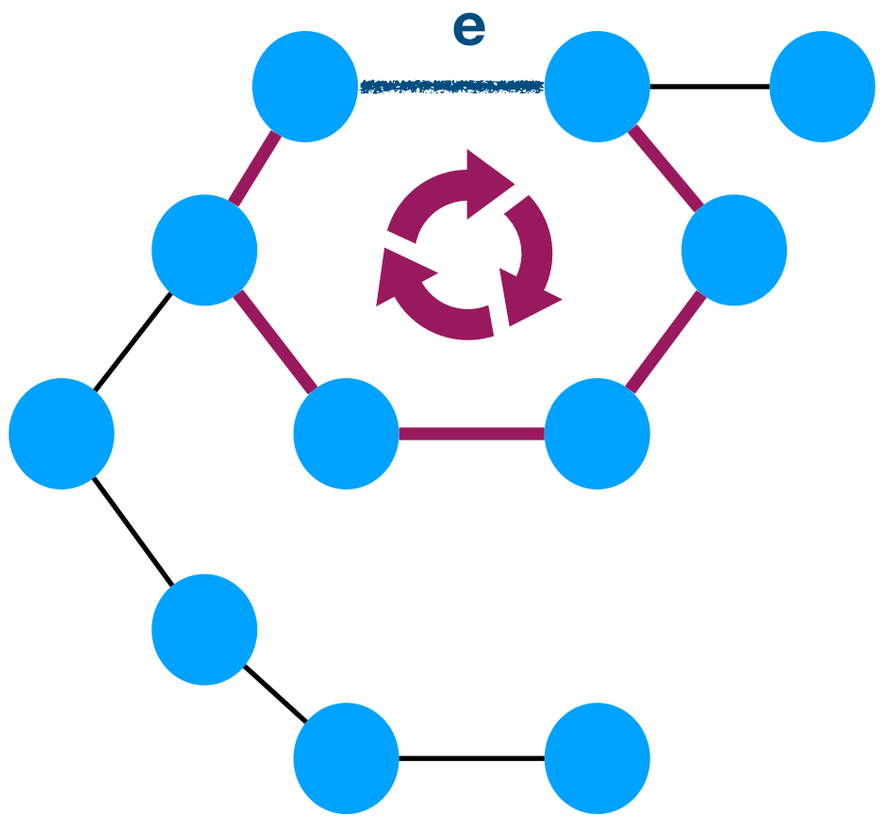

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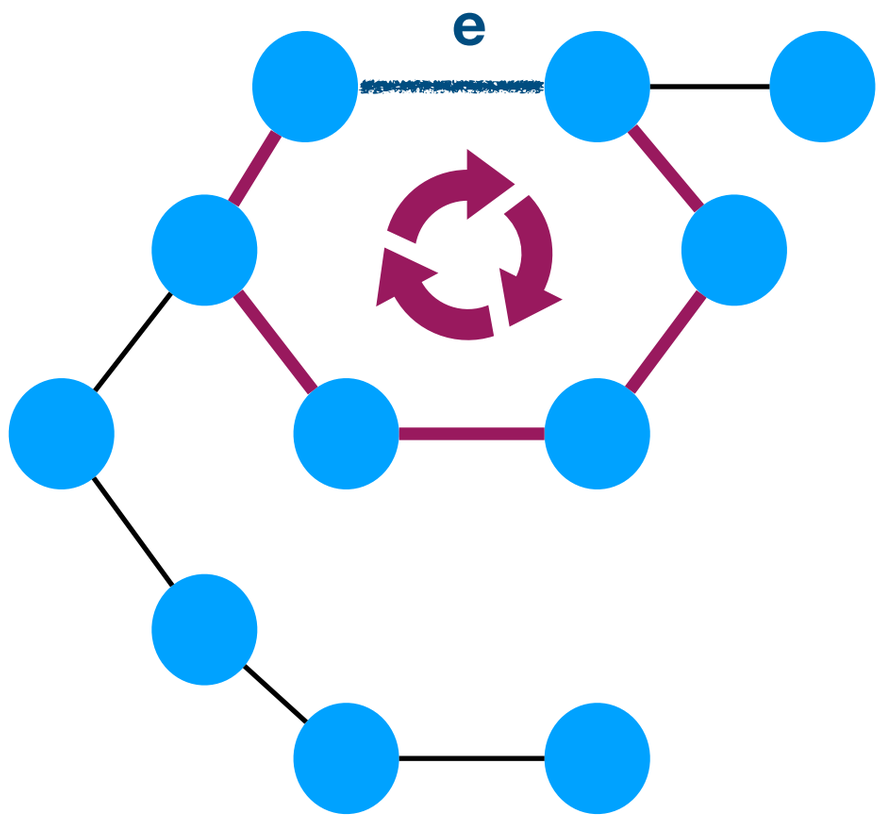

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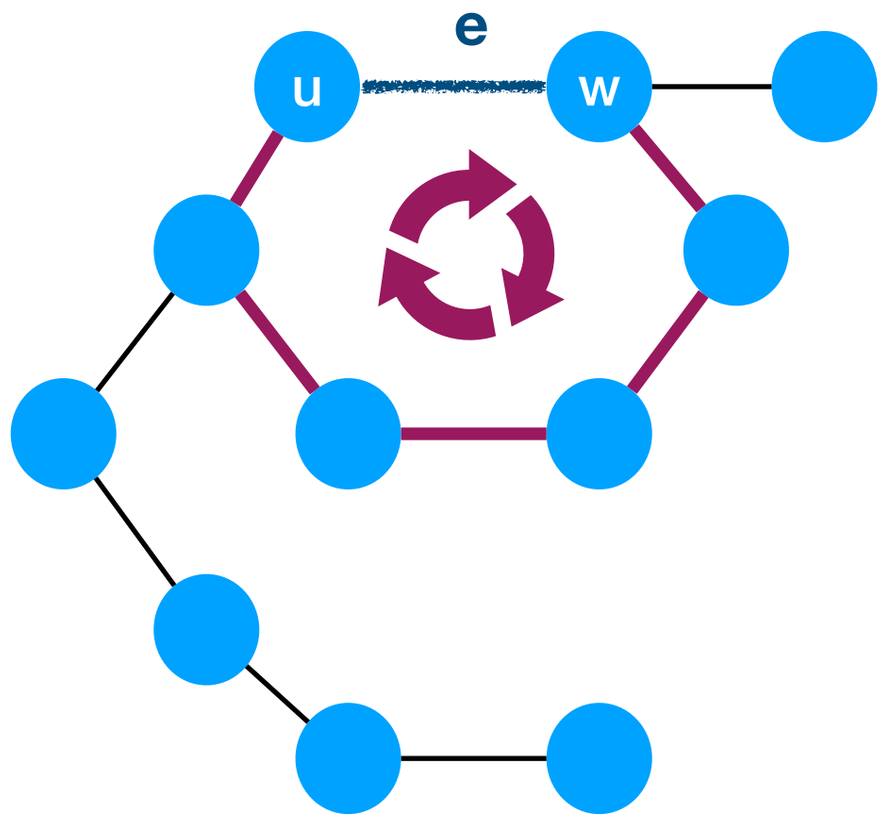

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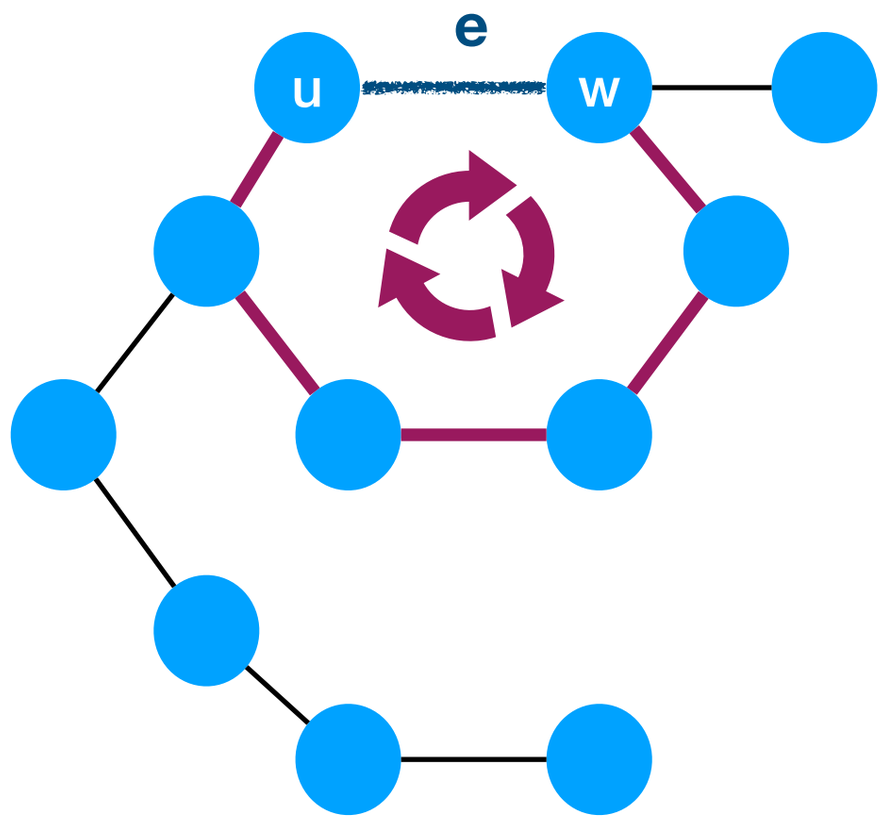

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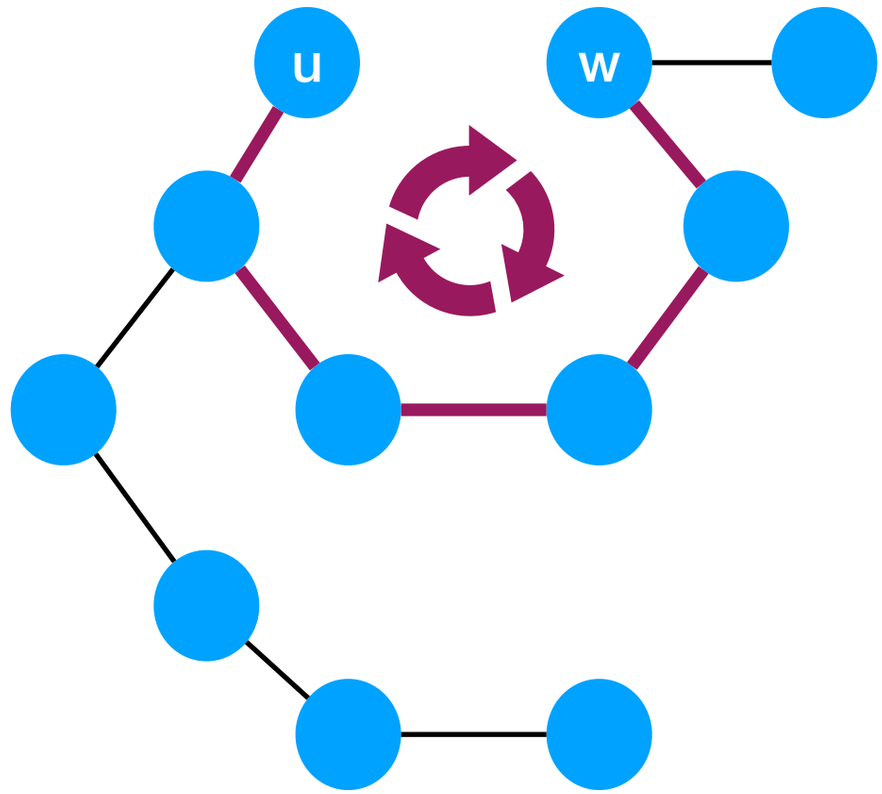

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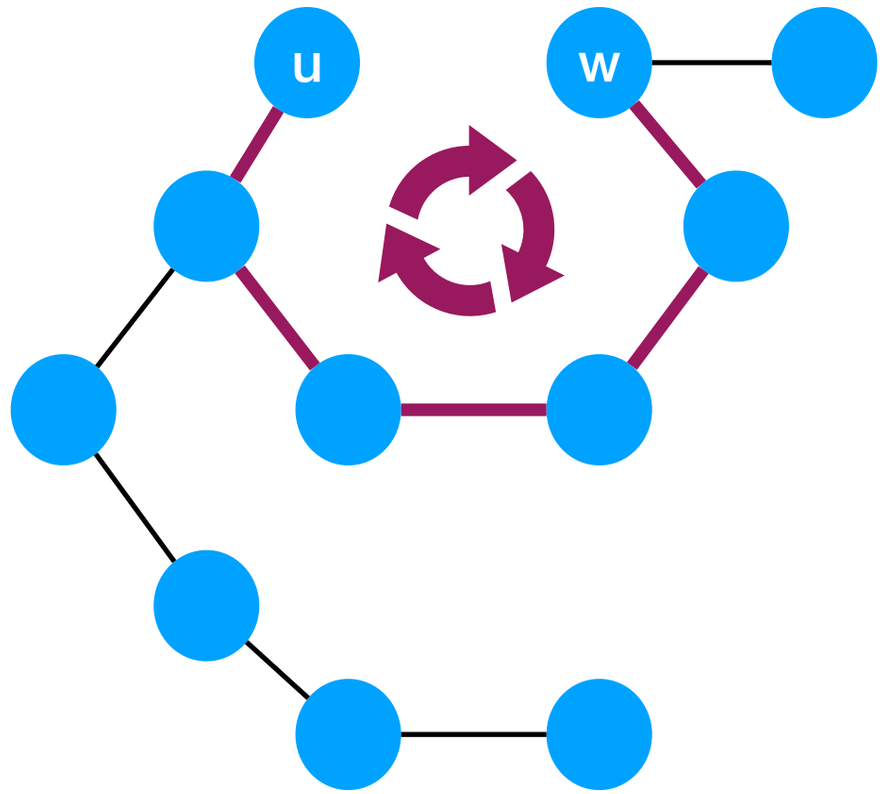

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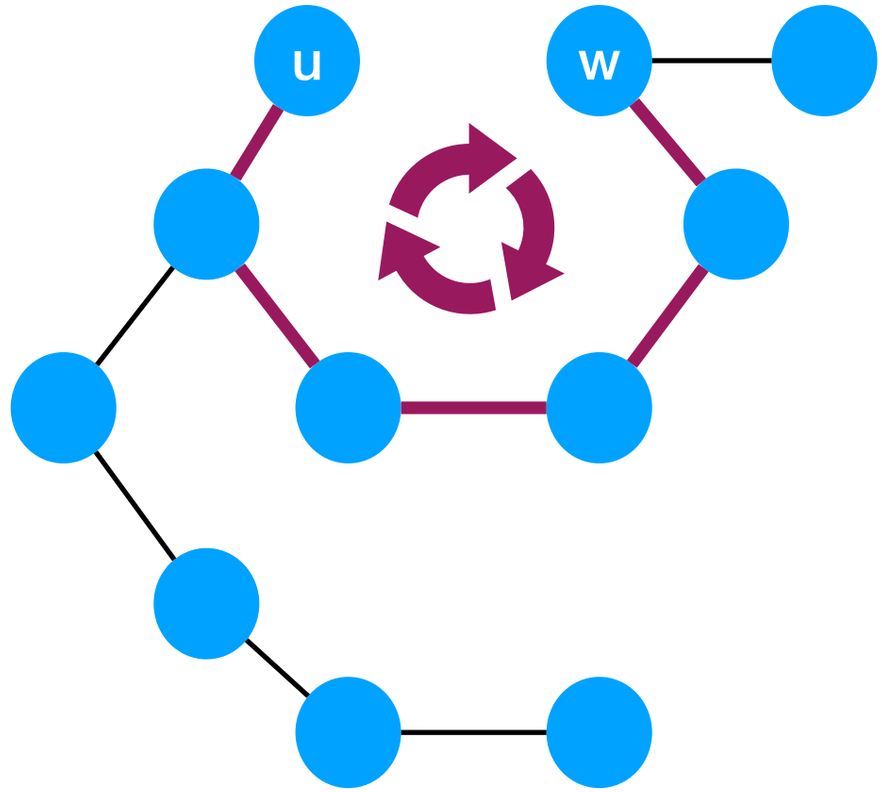

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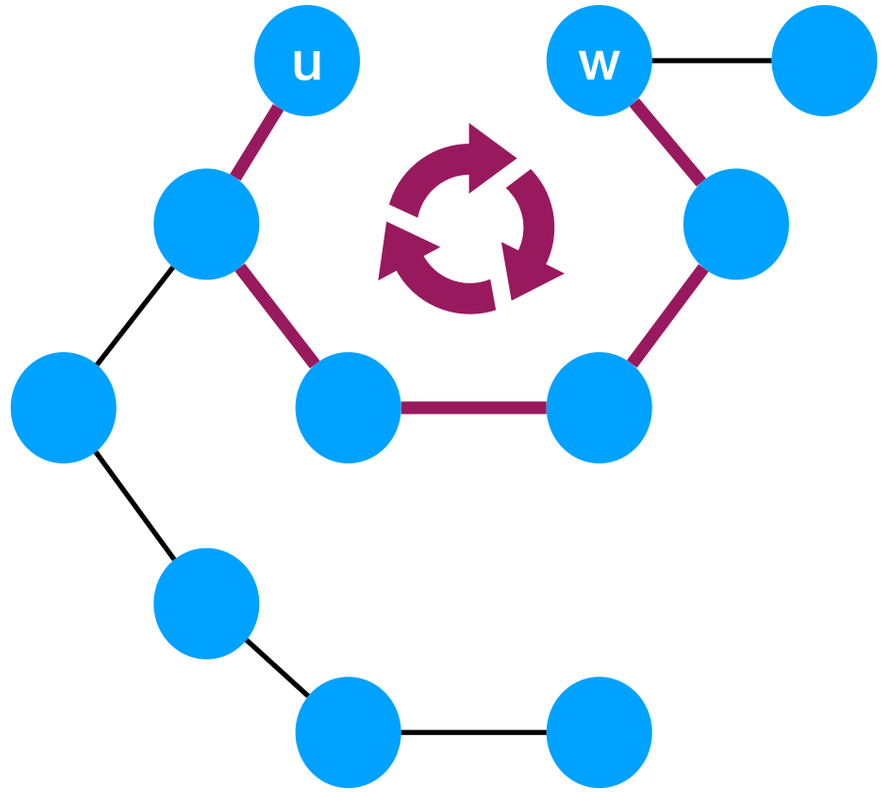

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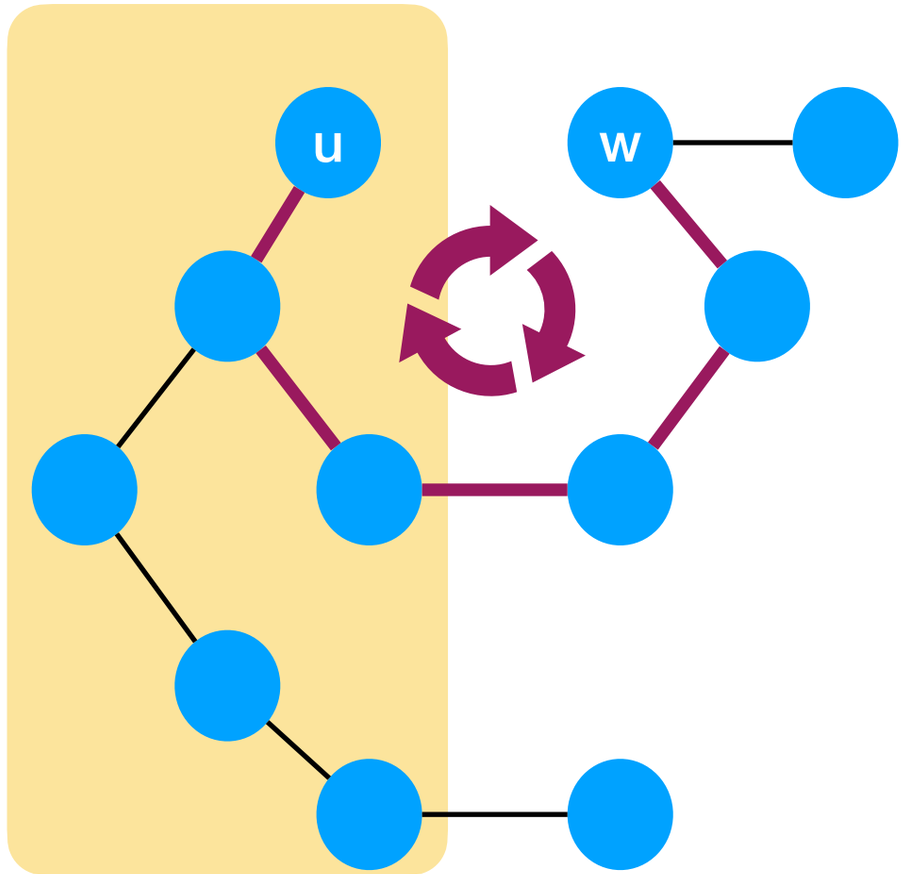

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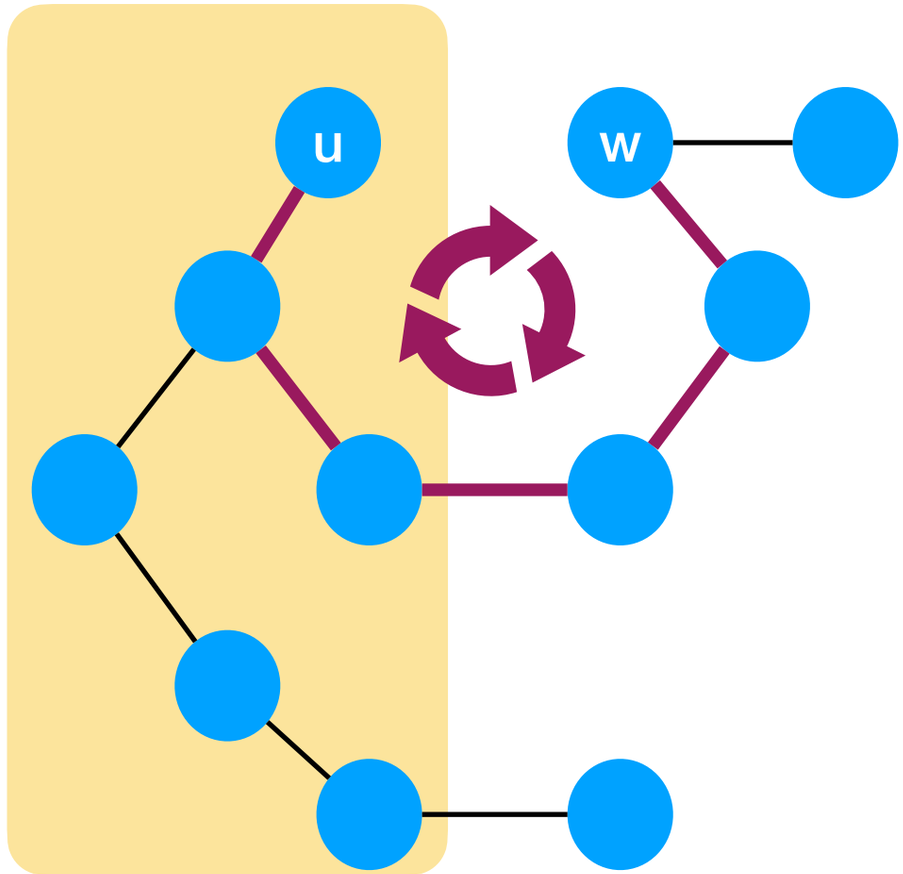
We follow the other path the cycle from $u$ to $w$.



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At some point we cross from $S$ to $V$ - $S$,
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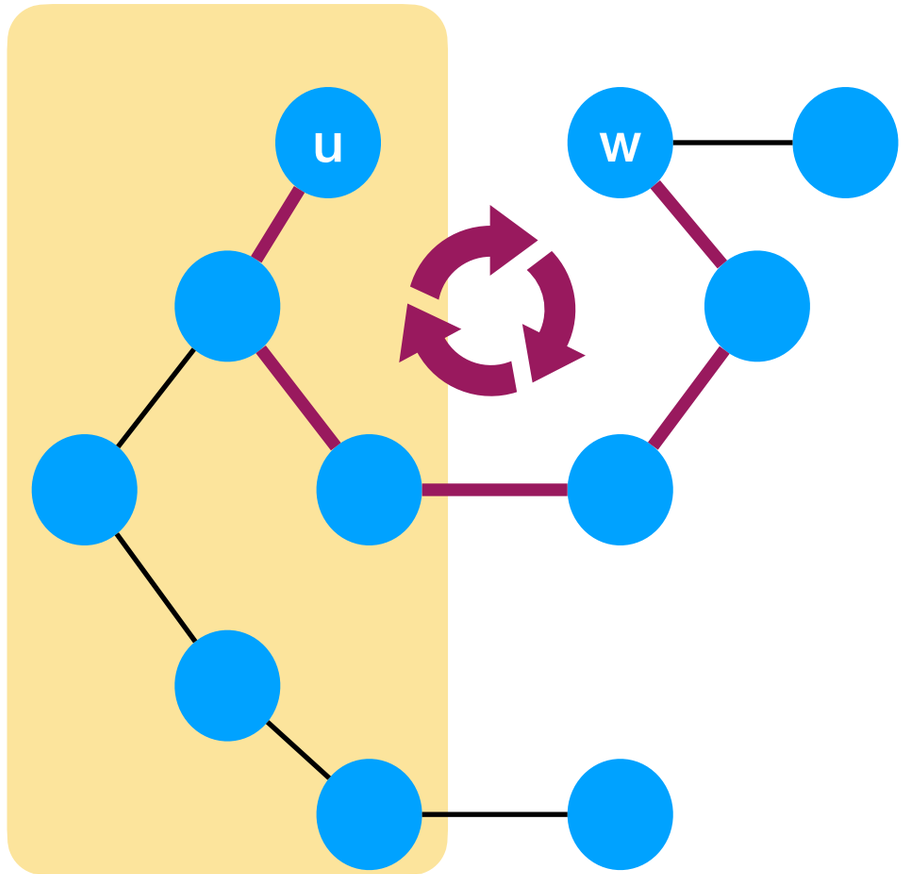

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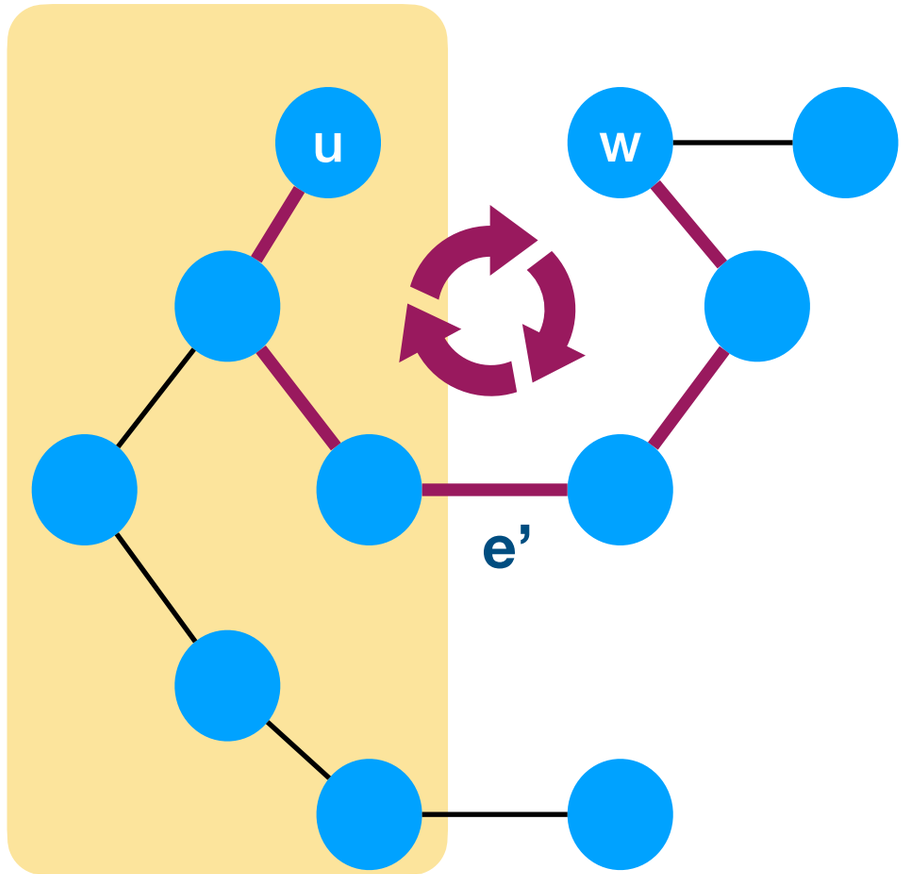

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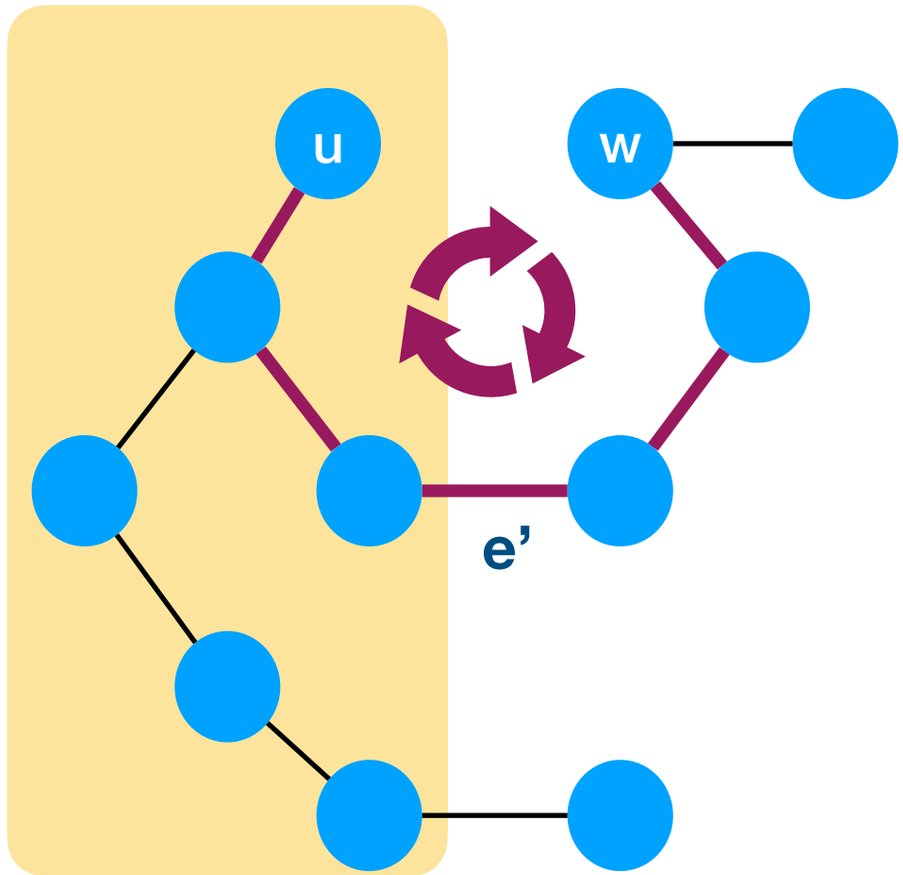
We follow the other path the cycle from $u$ to $w$.

At some point we cross from $S$ to $V$ - $S$, following edge $e'$.

The resulting graph is a tree with smaller cost.



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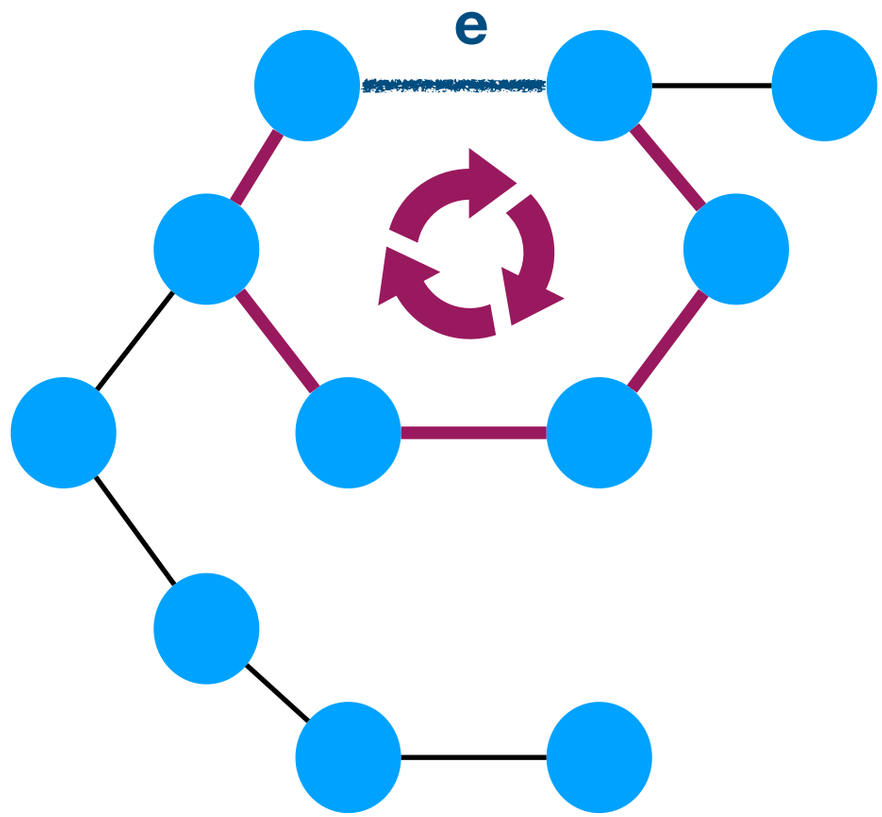
Just before deleting, it lies on some cycle $C$.

It has the maximum cost among edges, so it cannot be part of any minimum spanning tree.

# Is it feasible?

i.e., does it always produce a spanning tree?



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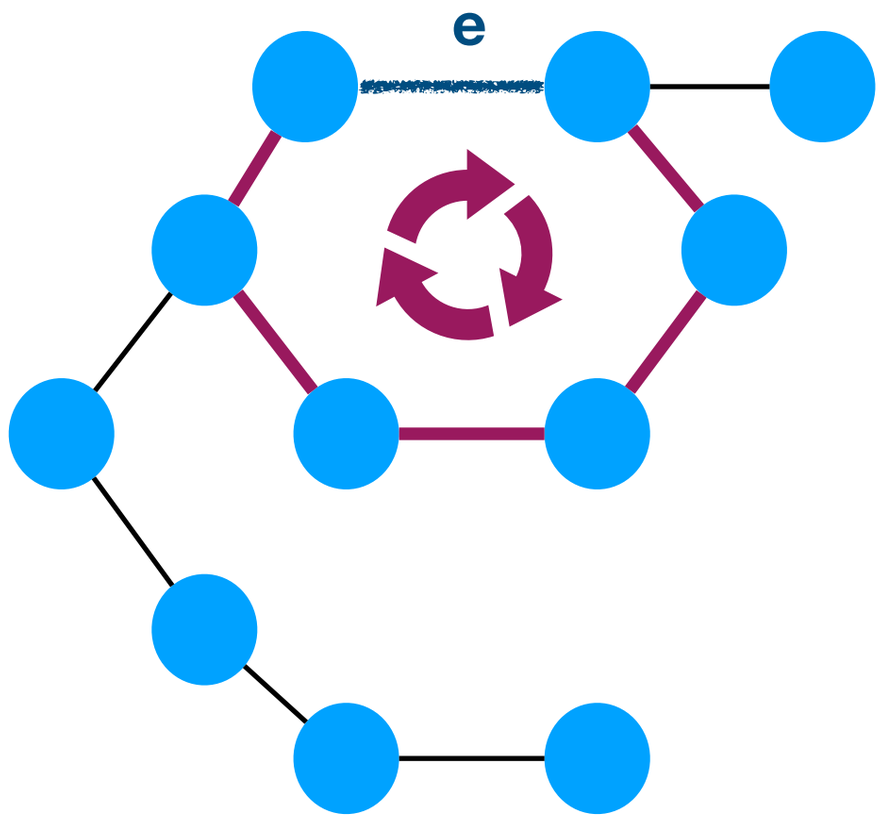

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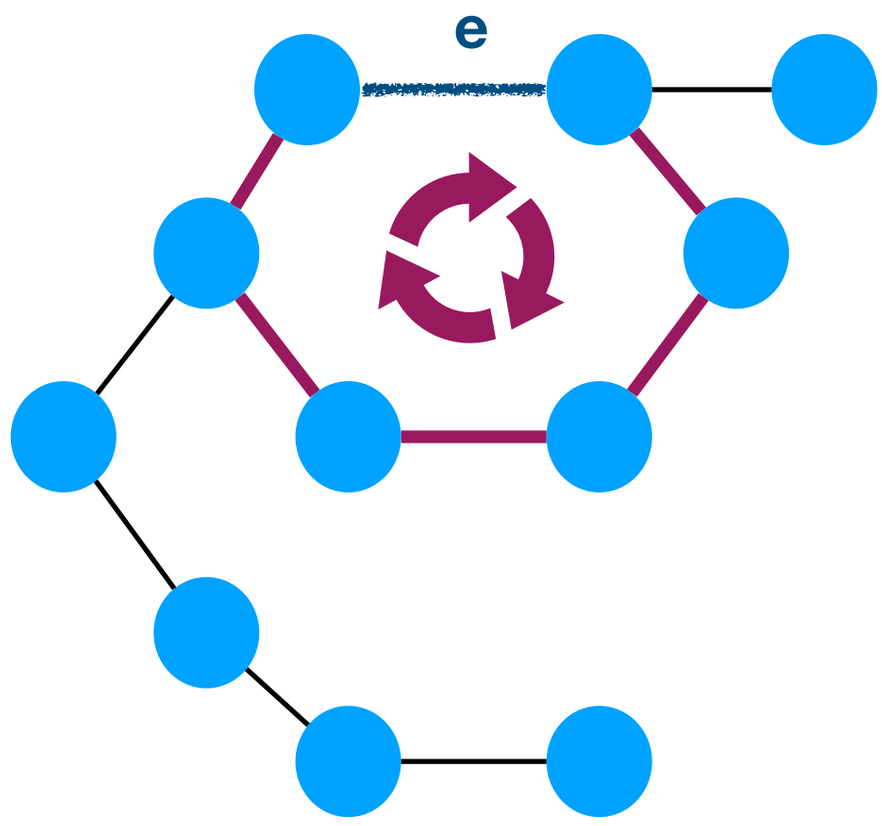

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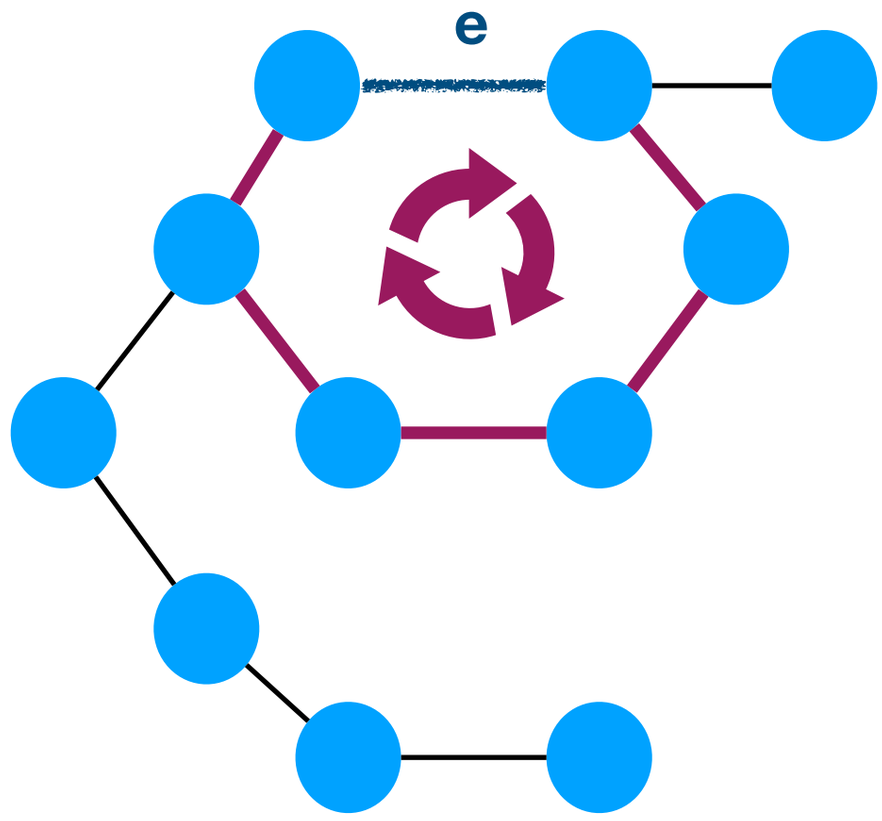

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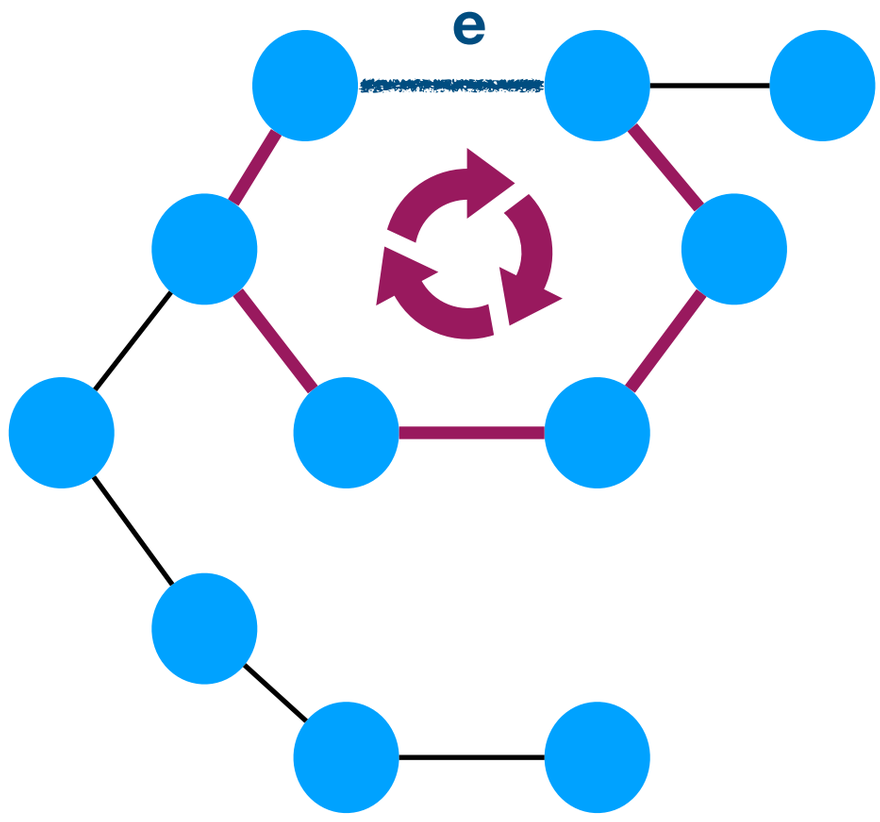
Suppose that it’s not.



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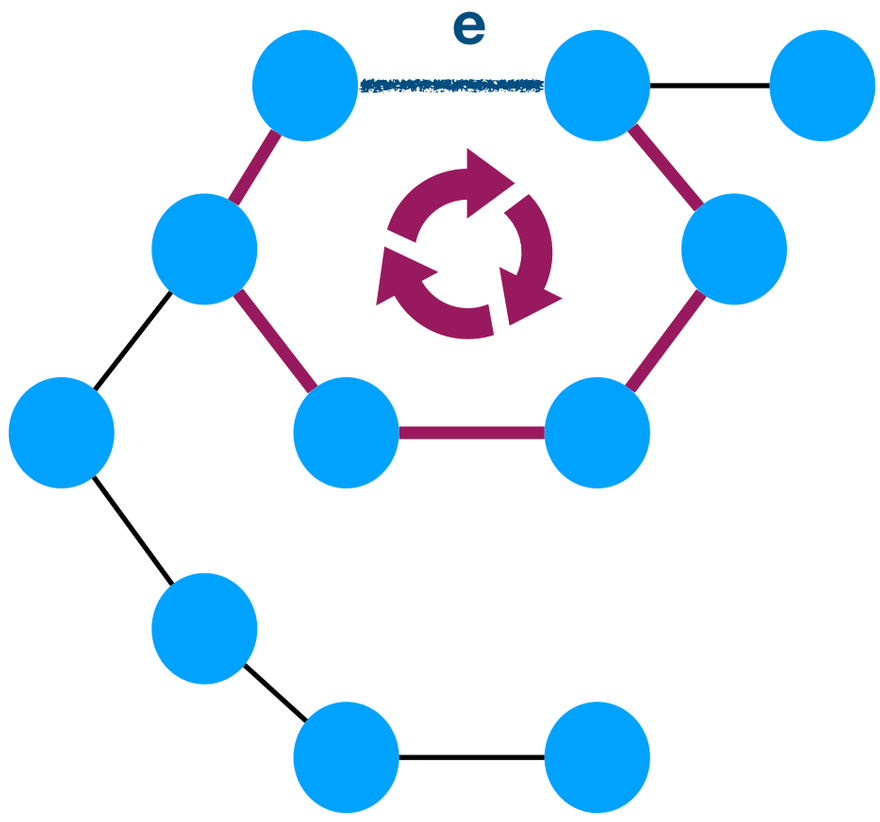

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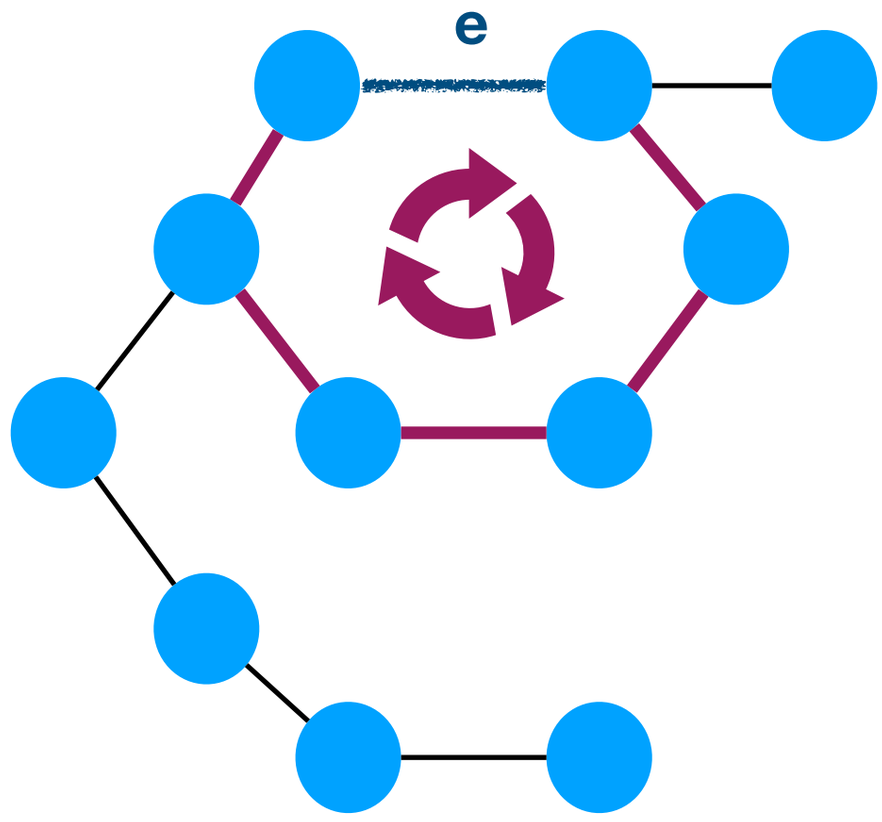

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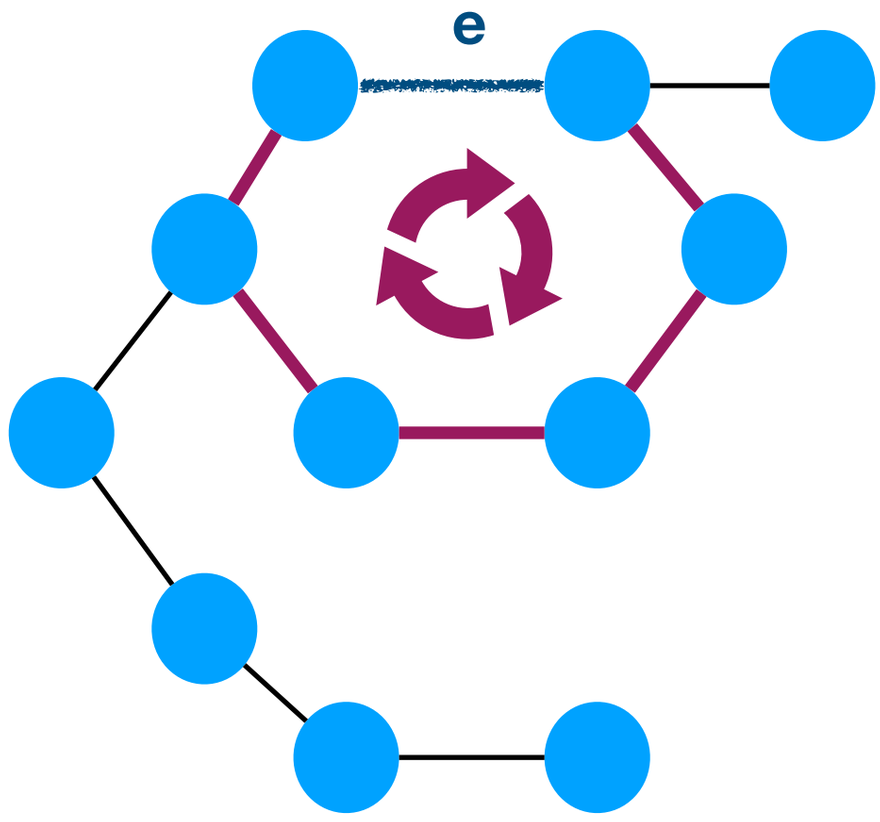
Consider the most expensive edge $e$ on that cycle.

The algorithm would have removed that edge.



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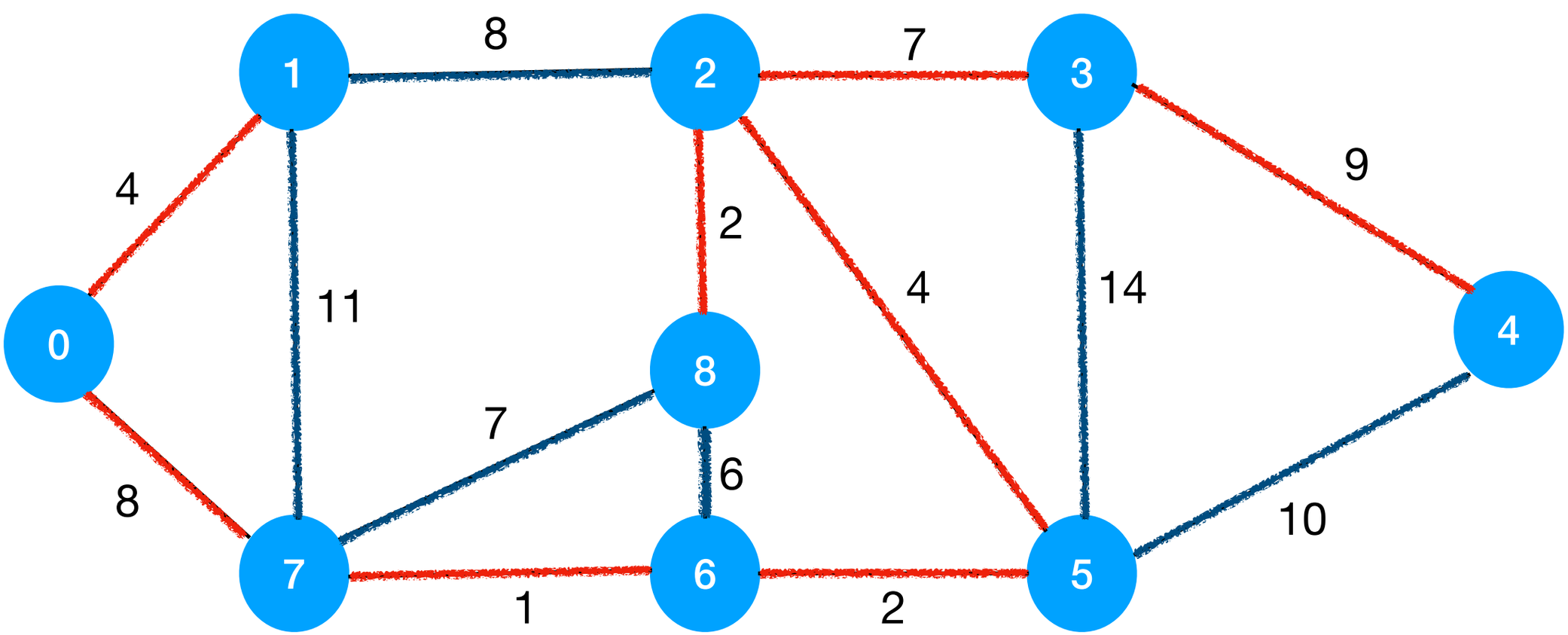

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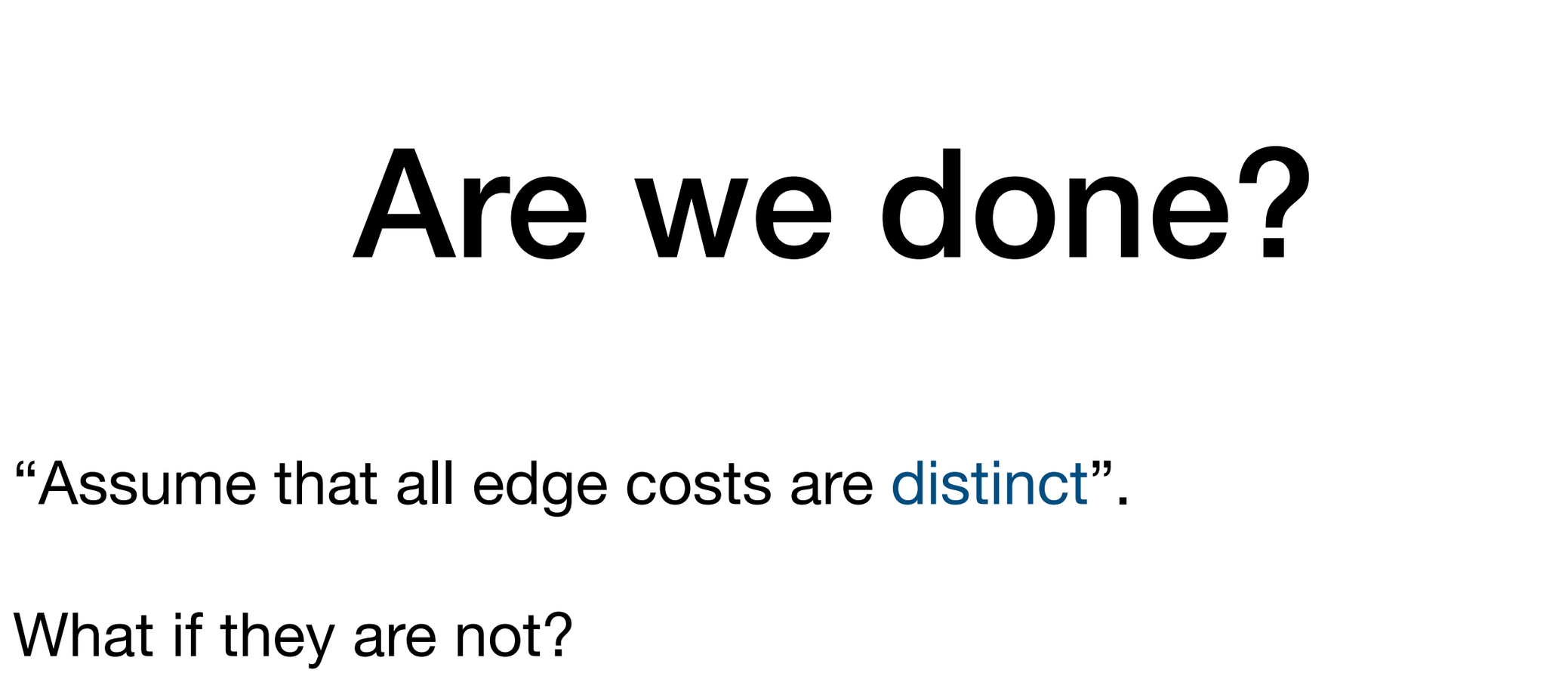

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$G_T$ is a minimum spanning tree on the original instance.

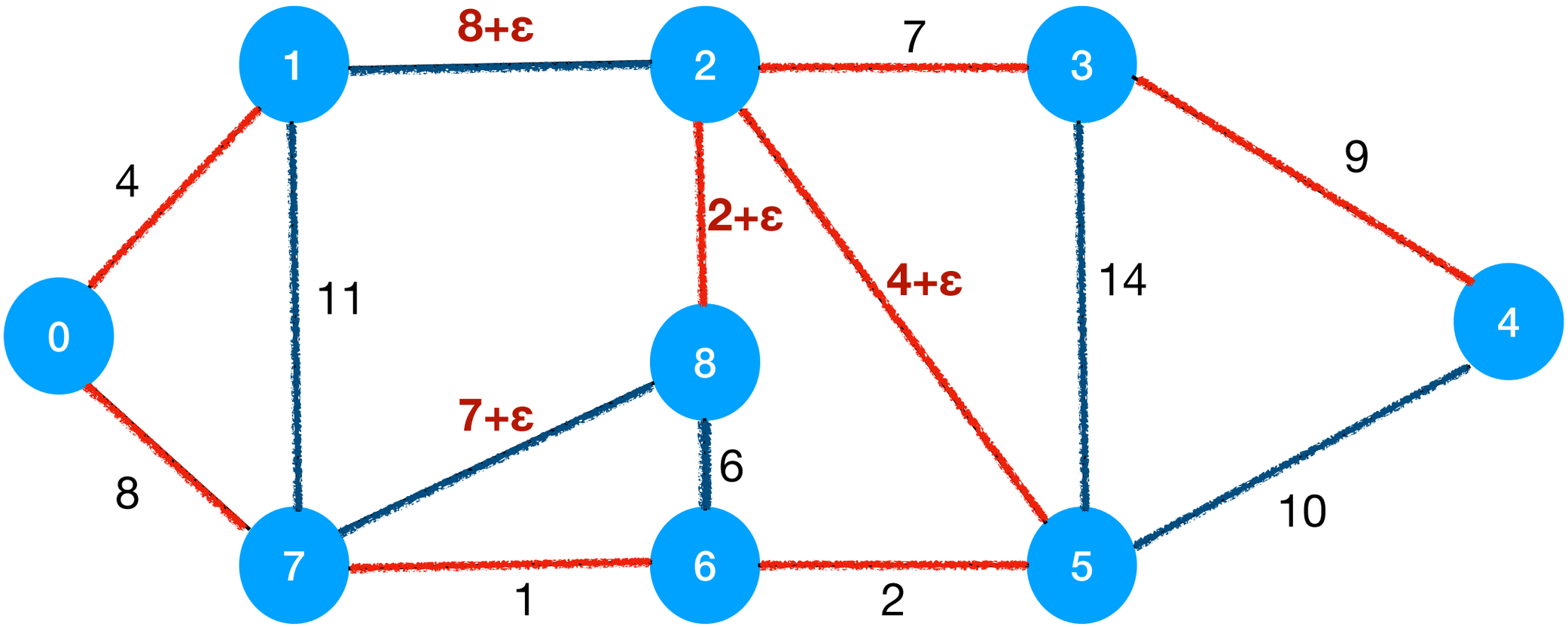
# $G_T$ in the original instance

Suppose that there was a cheaper spanning tree $G_T^*$ on the original instance.

If $G_T^*$ contains different edges with the same costs, it is not cheaper than $G_T^*$ on the original instance.

If $G_T^*$ contains different edges with different costs, we can make $\varepsilon$ small enough to make sure the ones we selected are still cheaper.

# Perturbing the costs



# Perturbing the costs

1, 2, 2, 4, 4, 6, 7, 7, 8, 8, 9, 10, 11, 14

1, 2, 2+ε, 4, 4+ε, 6, 7, 7+ε, 8, 8+ε, 9, 10, 11, 14

# Running time?

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Next lecture!