Algorithms and Data Structures

Modelling with Linear Programs

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In practice: There are many fantastic LP solvers (e.g., CPLEX, Gurobi, whatever-your-favourite-library-of-your-favourite-programming-language-uses, etc).

It suffices to formulate/model/express a problem as an LP and then ask one of those solvers for the solution.

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Product j can be sold in the market for σ_j pounds per unit.

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What is the revenue in total? $\sum_{i=1}^{n} c_j x_j$

Our LP formulation

Maximise $\sum_{i=1}^{n} c_j x_j$

subject to $?$

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Maximise $$\sum_{i=1}^{n} c_j x_j$$

subject to $$\sum_{j=1}^{n} \alpha_{ij} \cdot x_j \leq b_i \quad \textbf{for all } i = 1, \ldots, m$$

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Linear Regression

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This formula does not fit our observations 100%, so we would like to find the values of a and b that best approximate our observations.

In particular, we would like to minimise the worst-case error of the prediction, i.e., to minimise

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Variables?

A first attempt

minimise e

subject to $a, b \in \mathbb{R}$ (i.e., no constraint)

A first attempt

minimise $\max_{i=1}^{100} |w_i - (ah_i + b)|$

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A first attempt

What is wrong with this?

minimise $\max_{i=1}^{100} |w_i - (ah_i + b)|$

subject to $a, b \in \mathbb{R}$ (i.e., no constraint)

Removing the max

Let $e = \max_{i=1}^{100} |w_i - (ah_i + b)|$

This means that for each $i = 1,\ldots,100$, we have that

$$|w_i - (ah_i + b)| \leq e$$

Our new attempt

minimise e

subject to $|w_i - (ah_i + b)| \leq e$ for all $i = 1, \ldots, 100$

Removing the absolute value

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Let's look at the constraint $|x| \leq e$

This means that $\max\{x, -x\} \leq e$

Which means that both $x \leq e$ and $-x \leq e$

Our new attempt

minimise e

subject to $|w_i - (ah_i + b)| \leq e$ for all $i = 1,\ldots,100$

Our new LP

minimise e

subject to $w_i - (ah_i + b) \leq e$ for all $i = 1,\ldots,100$

$-w_i + (ah_i + b) \leq e$ for all $i = 1,\ldots,100$

Linear Regression

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What if we wish to minimise the average error of the prediction, i.e., to minimise

$$e = \frac{1}{100} \sum_{i=1}^{100} | w_i - (ah_i + b) |$$

A first attempt

minimise $\frac{1}{100}\sum_{i=1}^{100} |w_i - (ah_i + b)|$

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We can add those as constraints.

Our new LP

minimise $\frac{1}{100}\sum_{i=1}^{100} x_i$

subject to $w_i - ah_i + b \leq x_i$ for all $i = 1,\ldots,100$

$-w_i + ah_i + b \leq x_i$ for all $i = 1,\ldots,100$

Our new LP

Should we worry that the LP will choose some x_i such that
$x_i > \max\{-w_i + ah_i + b, w_i - ah_i + b\}$?

minimise $\frac{1}{100}\sum_{i=1}^{100} x_i$

subject to $w_i - ah_i + b \leq x_i$ for all $i = 1,\ldots,100$

$-w_i + ah_i + b \leq x_i$ for all $i = 1,\ldots,100$

Integer Linear programming

$$\text{maximise} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1 \in}^{n} \alpha_{ij} x_j \leq b_i, \quad i = 1, ..., m$$

$$x_j \geq 0, \quad j = 1, ..., n$$

$$x_j \text{ is integer}$$

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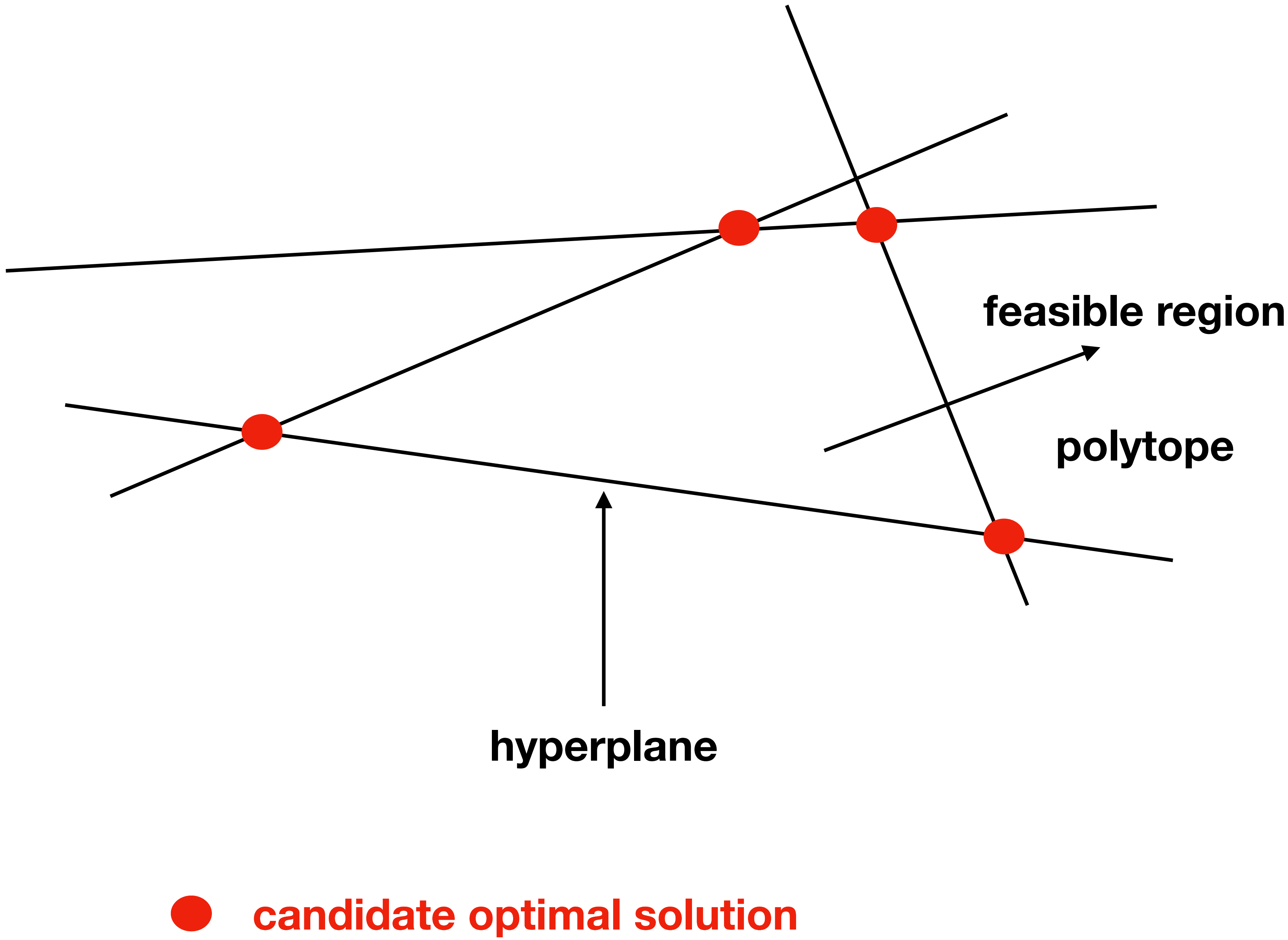

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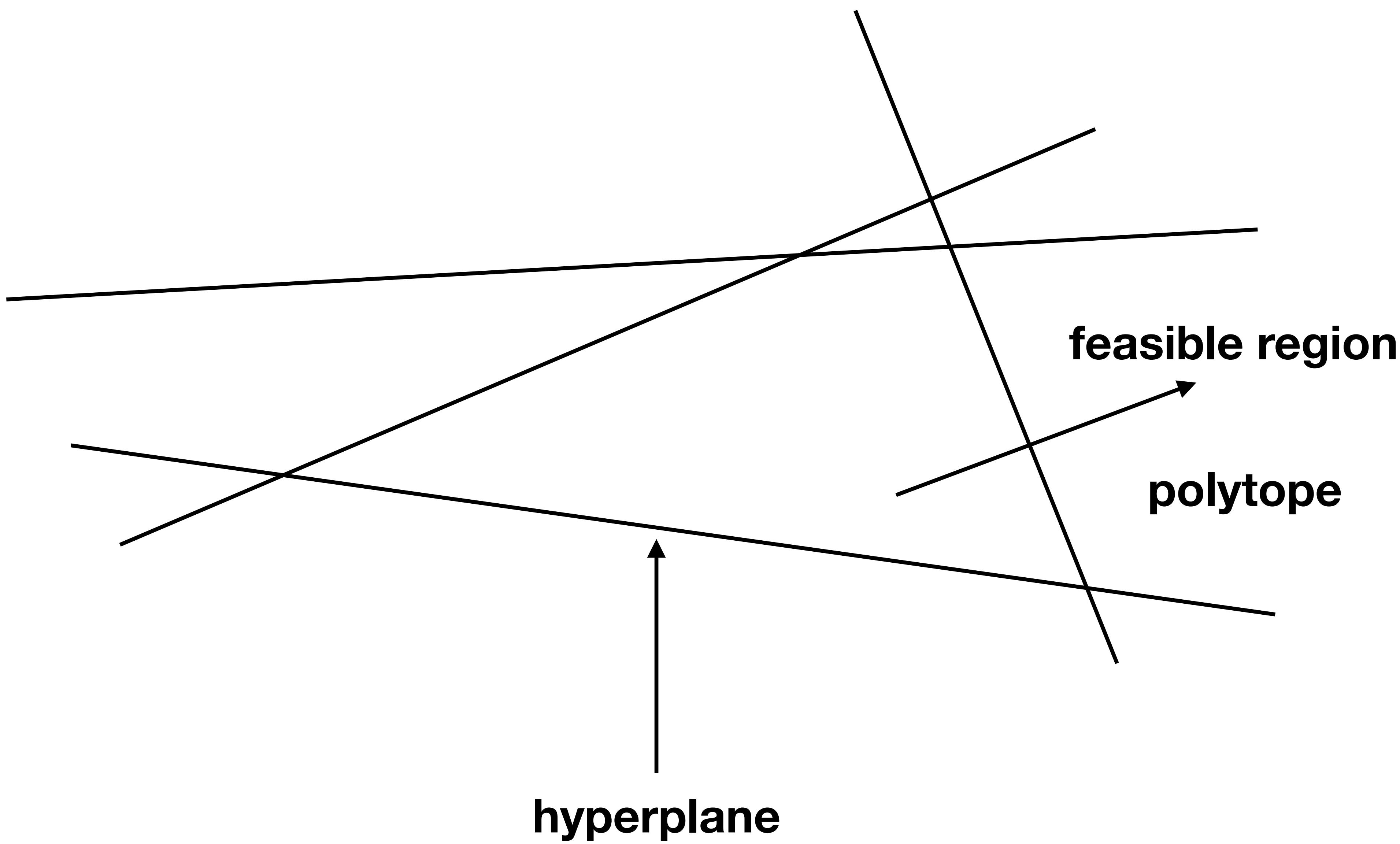

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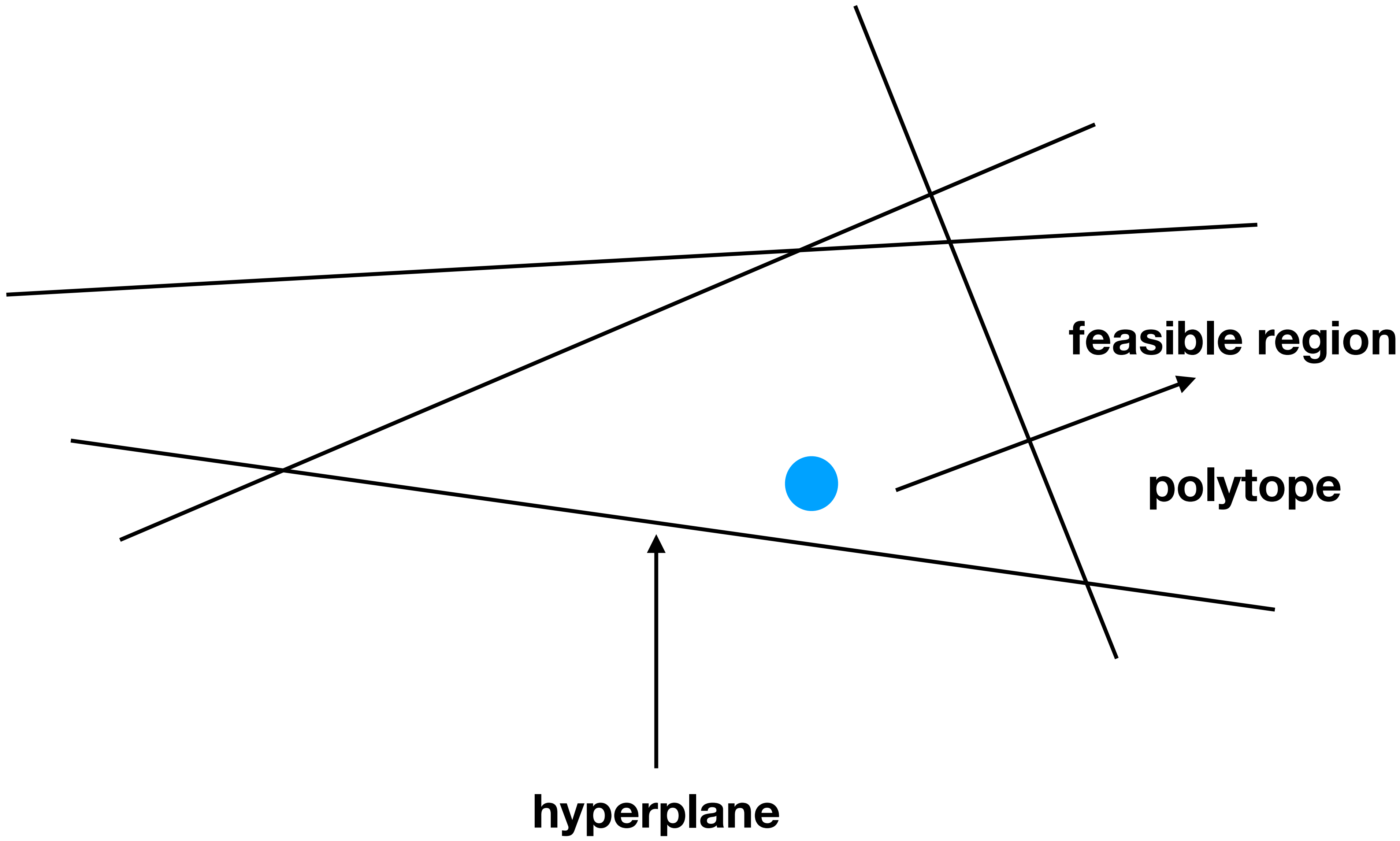

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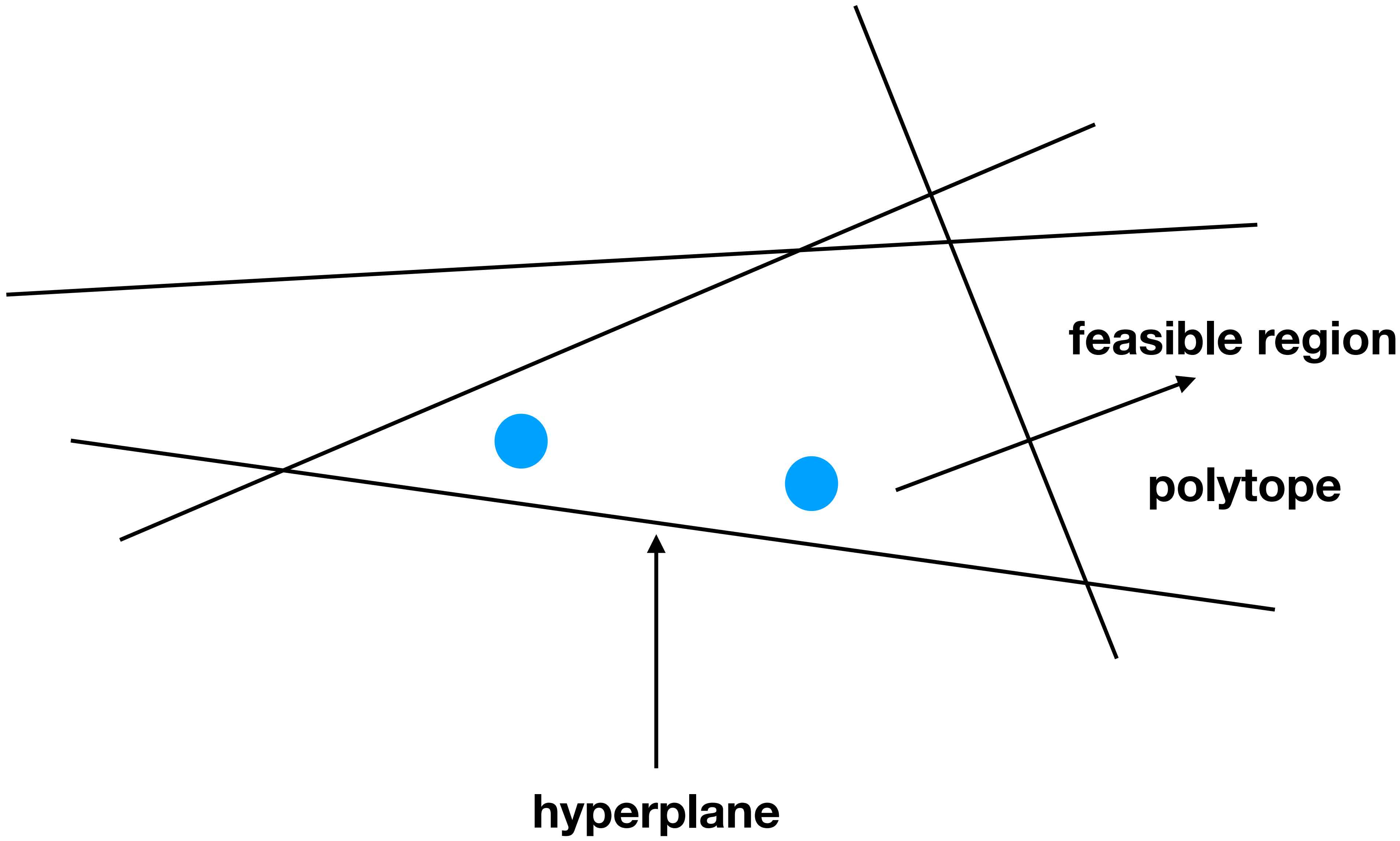

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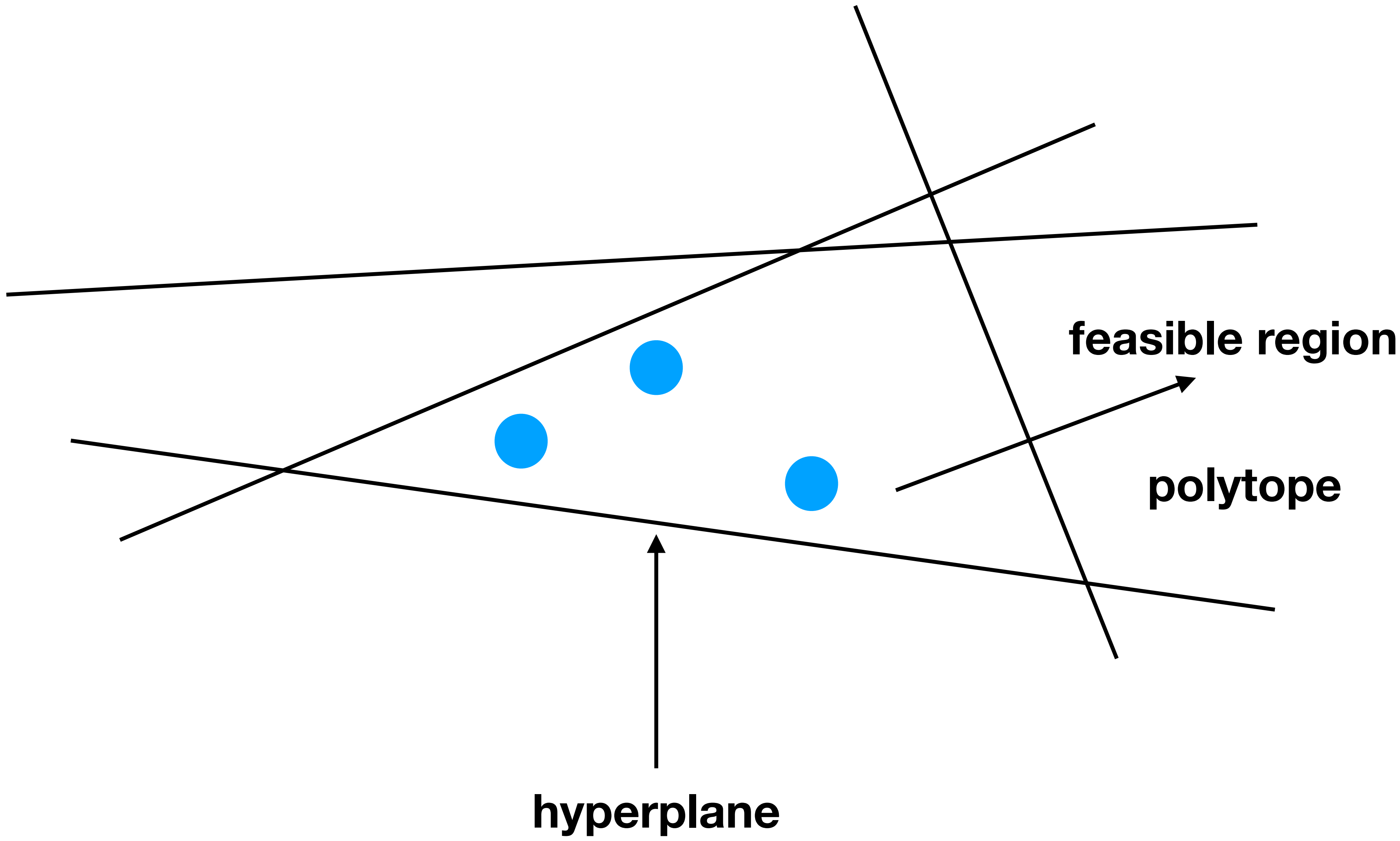

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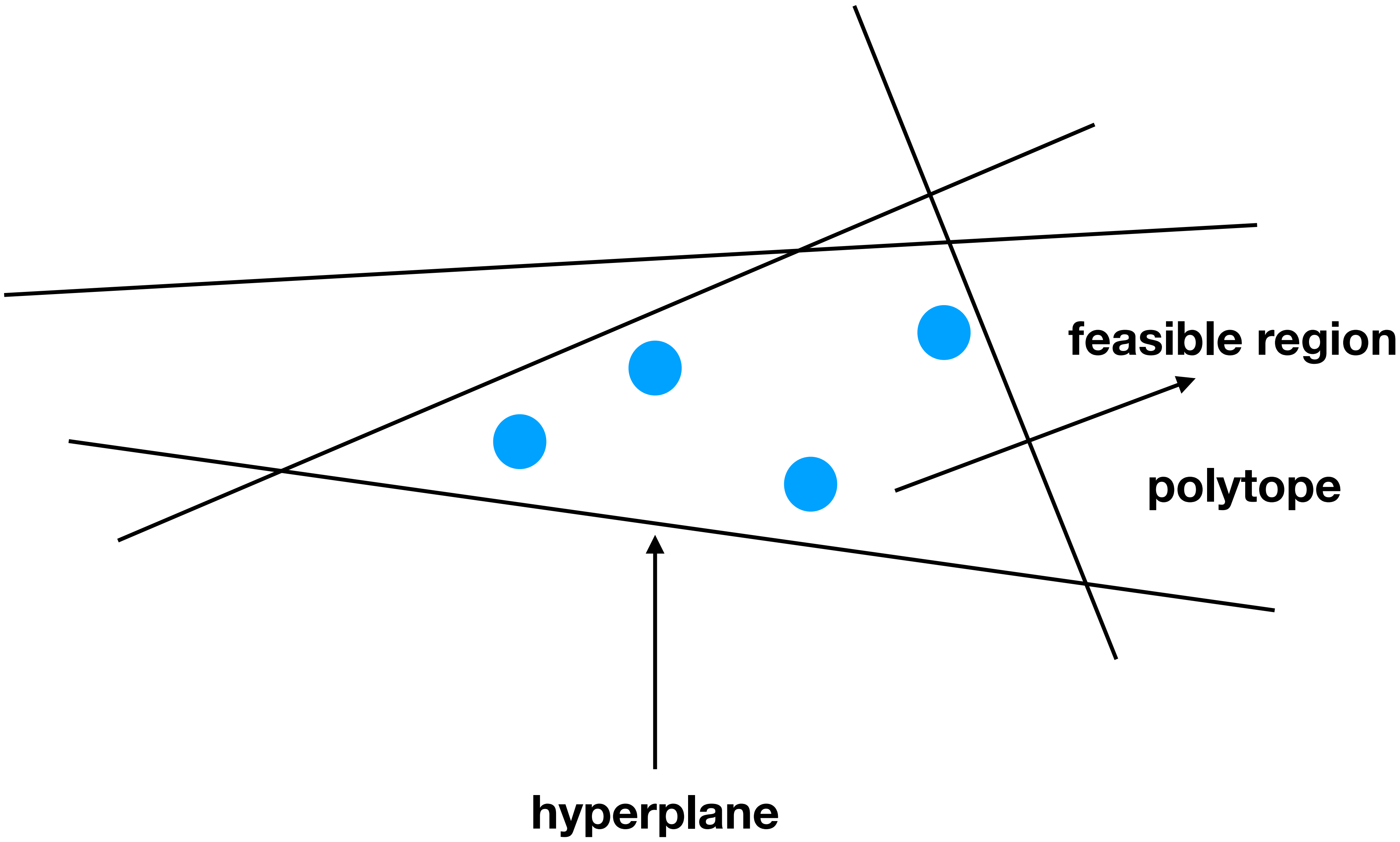

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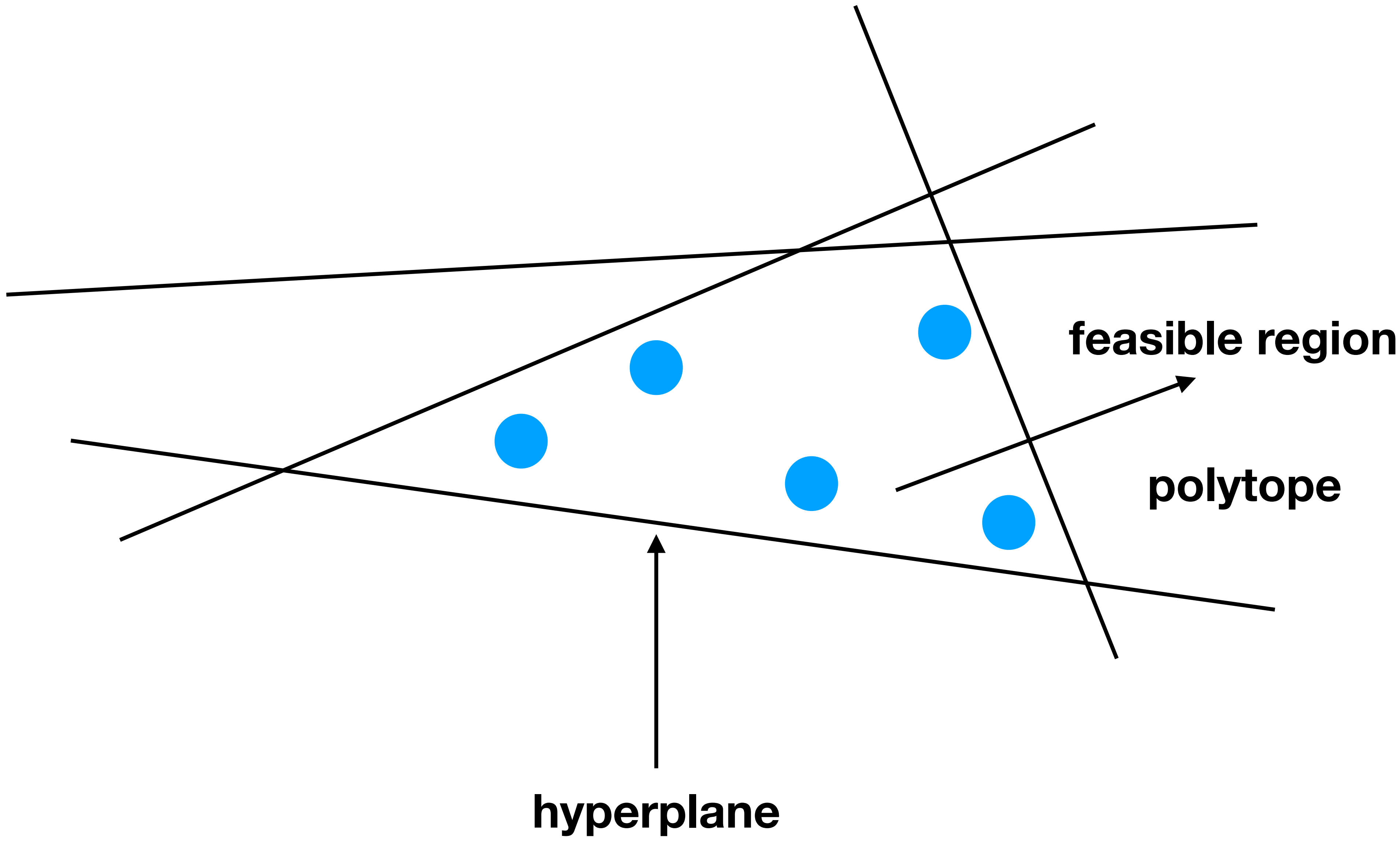

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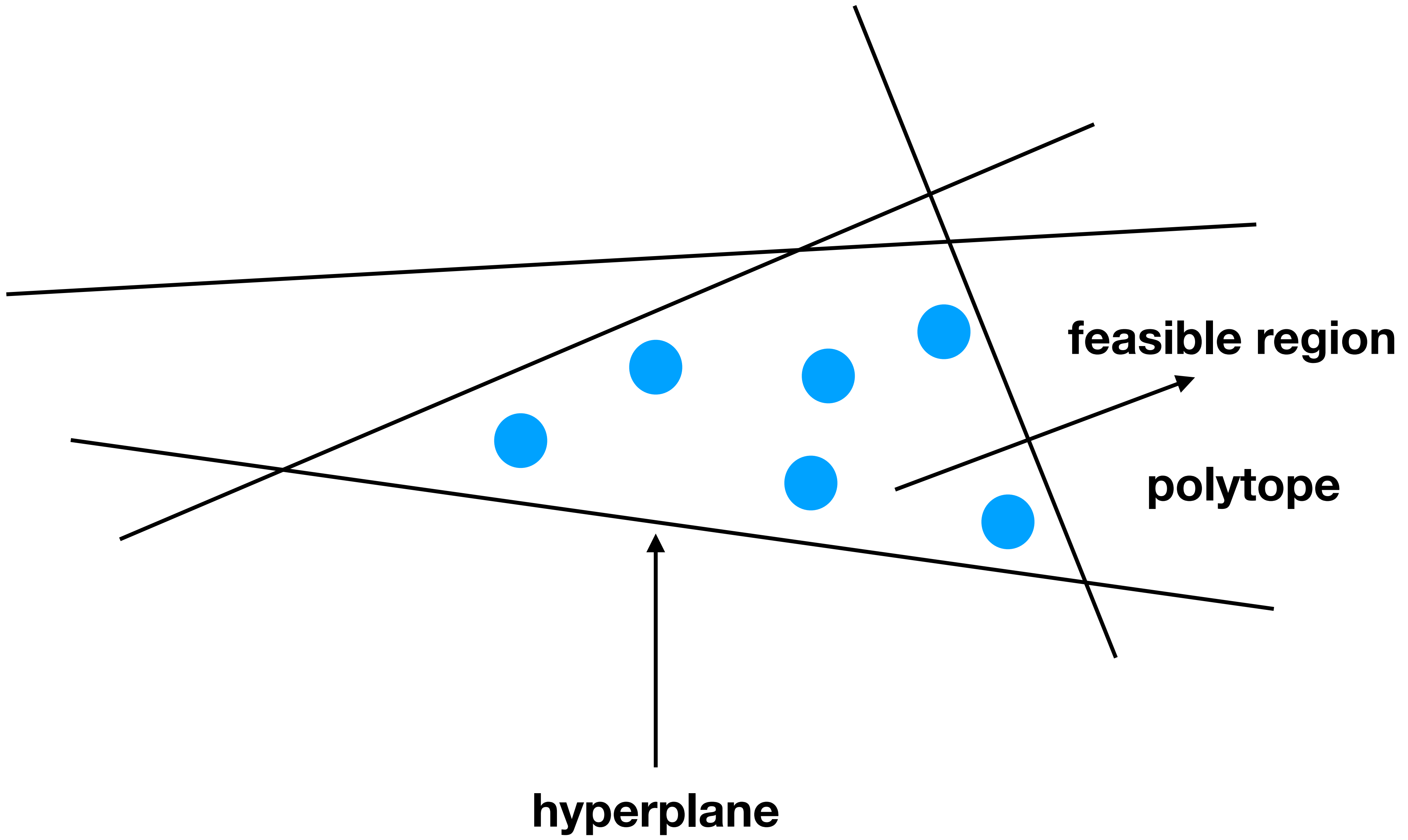

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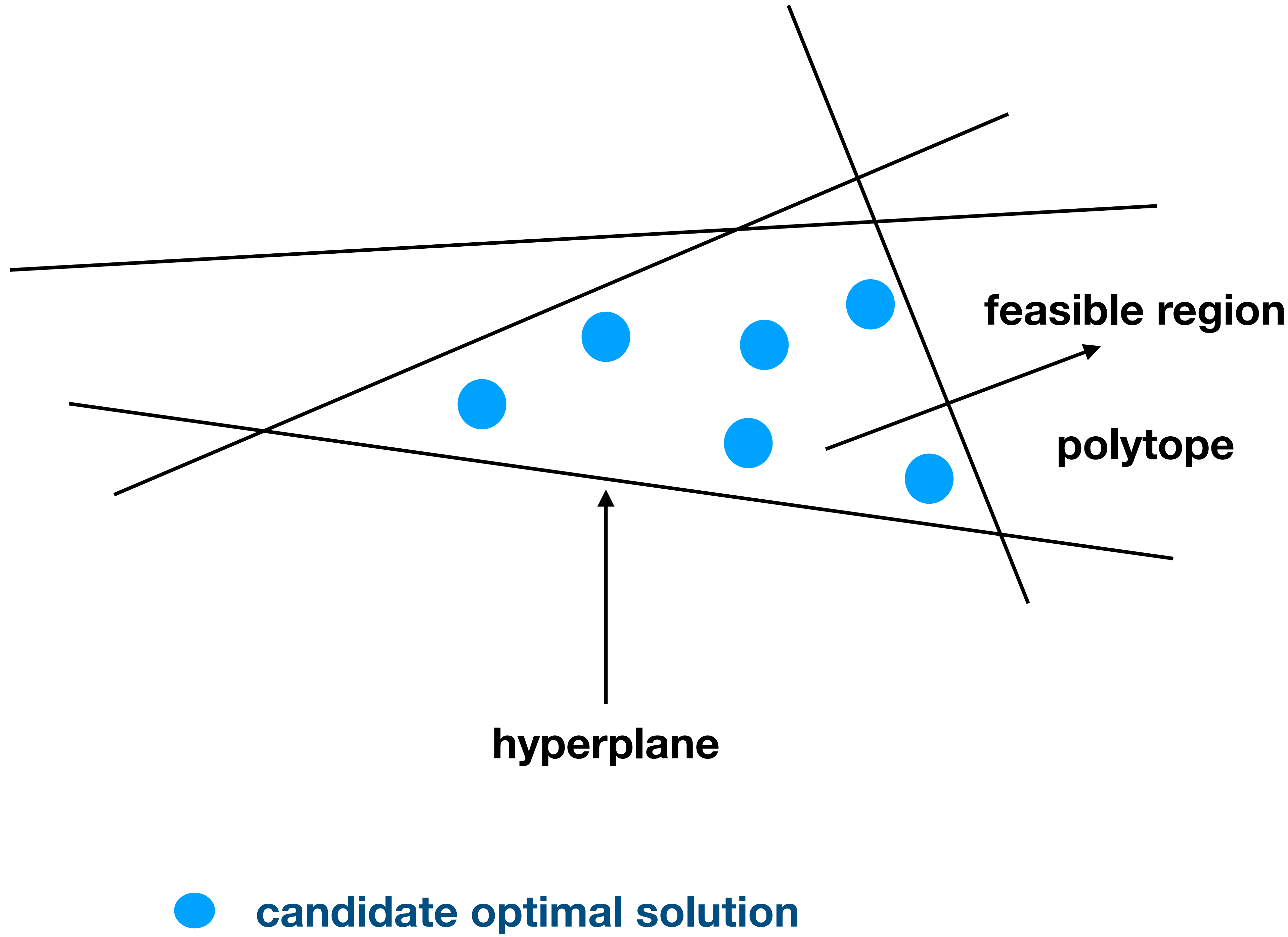

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We would like to find a subset of the routes such that each leg is included in exactly one route.

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Here, we will let $x_j = 1$ if we select route j and $x_j = 0$ otherwise.

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$$\sum_{j=1}^{n} c_j x_j$$

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What is the total number of routes that i is a part of?

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Our ILP formulation

Minimise $\sum_{i=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j$ **for** $i = 1, \ldots, m$

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The salesman would like to minimise the total distance travelled.

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Enumeration is obviously too slow. We will use an ILP formulation approach instead and rely on our clever solvers to be faster than enumeration.

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Alternative interpretation: Think of the map as a fully connected graph with a node for every city and an edge between every two cities. Then $x_{ij} = 1$ if and only if the edge (i, j) is being used by the tour.

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e.g. $t_3 = 4$ means that city 3 was visited 4th during the tour.

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Putting them together: $t_j \geq t_i + 1 - n(1 - x_{ij})$

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Assume that we instead have two disjoint subtours.

Consider one of these subtours that does not include city 0, and let r be the number of cities visited by this subtour.

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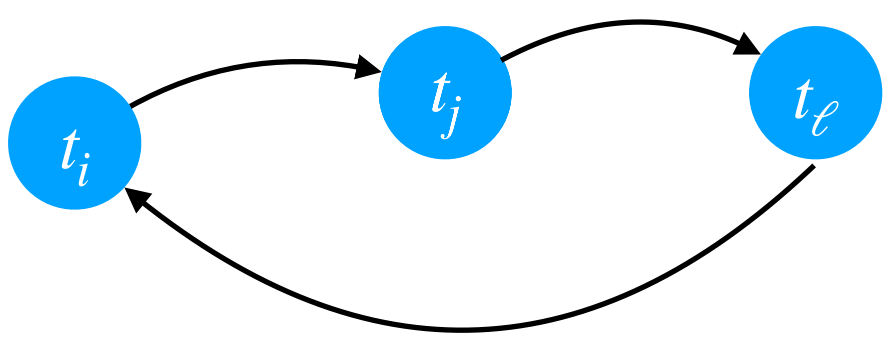

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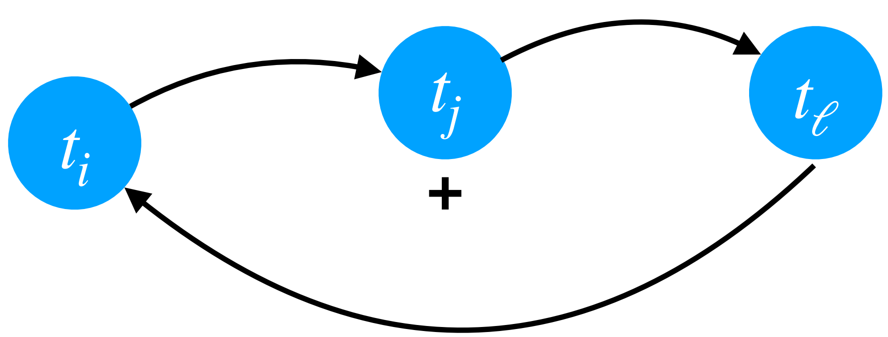

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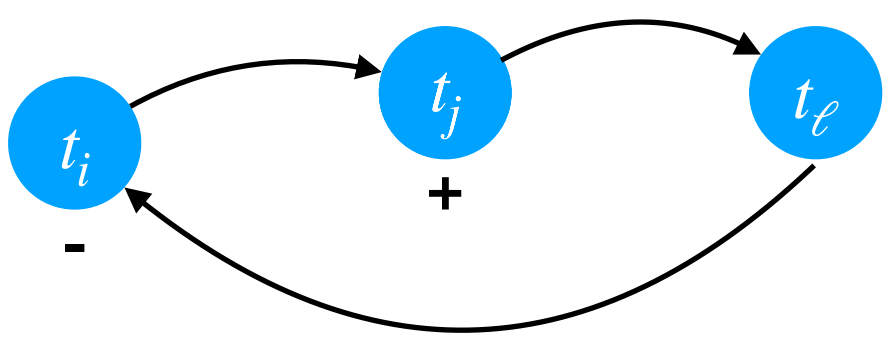

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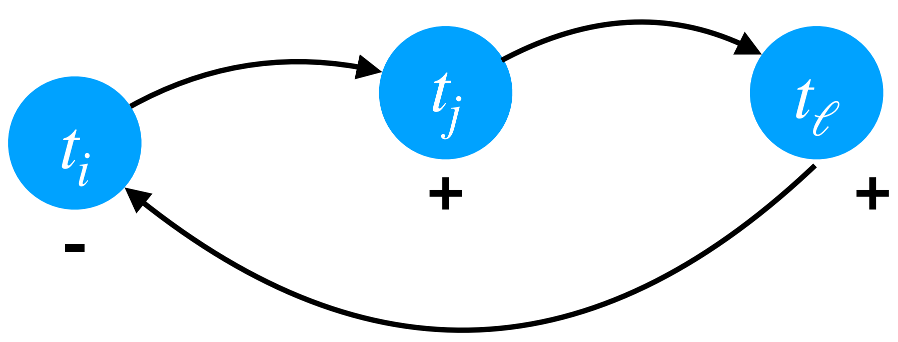

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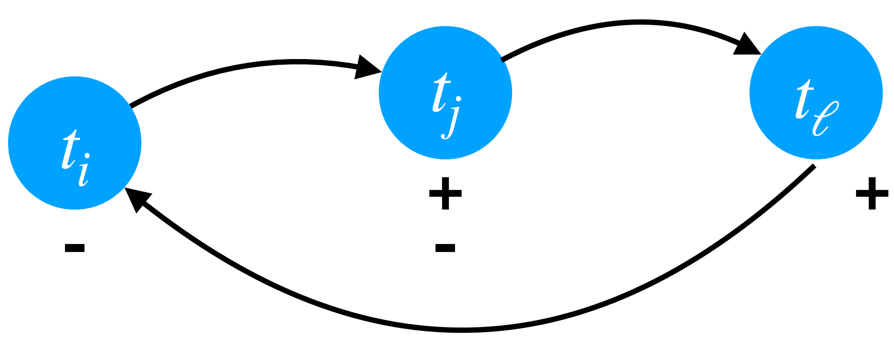

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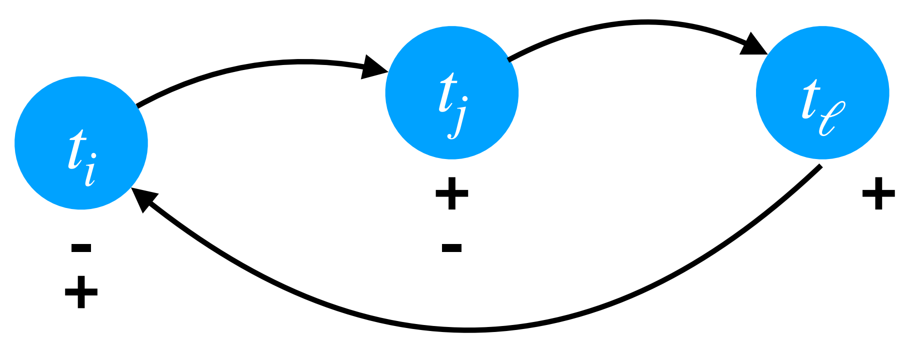

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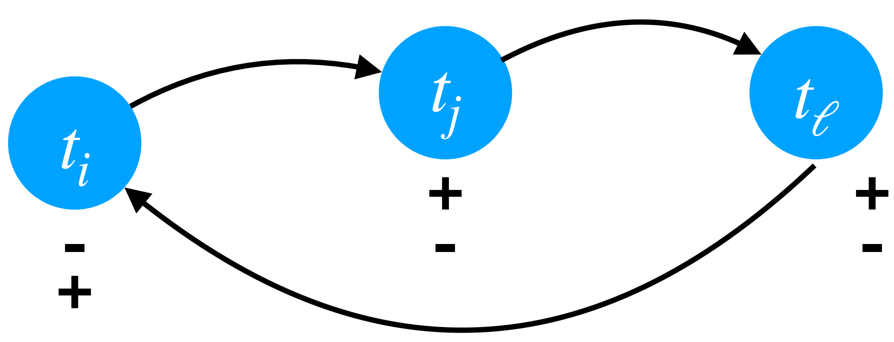

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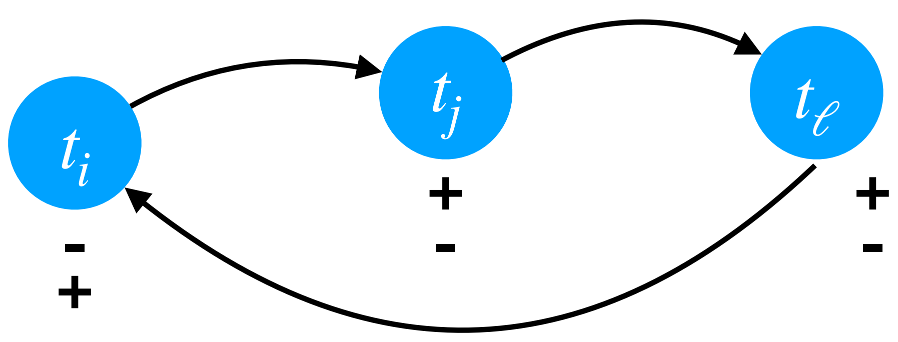

LHS = X

RHS = $X + r$

contradiction