Algorithms and Data Structures

Introduction to Dynamic Programming and Chain Matrix Multiplication

Dynamic Programming

An technique for solving optimisation problems.

Term attributed to Bellman (1950s).

“Programming” as in “Planning” or “Optimising”.

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Solve the subproblems from the smallest to the largest. When you solve a subproblem, store the solution (e.g., in an array) and use it to solve the larger subproblems.

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For that we will use the *standard* algorithm for matrix multiplication:

RECTANGULAR-MATRIX-MULTIPLY(A, B, C, p, q, r)

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for i = 1 to p
    for j = 1 to r
        for k = 1 to q
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The running time is the number of scalar multiplications, i.e., pqr.

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If instead we do $(A_1 \cdot (A_2 \cdot A_3))$ we have $100 \cdot 5 \cdot 50 = 25000$ scalar multiplications + $10 \cdot 100 \cdot 50 = 50000$ scalar multiplications for a total of 75000.

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We can think of the input as a sequence of dimensions $\langle p_0, p_1, p_2, \ldots, p_n \rangle$.

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If $i < j$, then there is some $k \in [i, j)$ such that the product is split between A_k and A_{k+1}.

In other words, there is a k for which the product $A_{i:j}$ can be written as the product of $A_{i:k}$ and $A_{k+1:j}$.

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If not, there is a cheaper parenthesisation of $A_i \cdot \ldots \cdot A_k$. We can use that one in the optimal parenthesisation of $A_i \cdot \ldots \cdot A_j$ instead to obtain an overall lower cost, *a contradiction*.

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Similarly for the the way to parenthesise the “suffix” $A_{k+1} \cdot \ldots \cdot A_j$.

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We must consider all the possible splits, i.e., choices of k.

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Notice: $M[i,j]$ gives us the optimal costs, but not the optimal splits. To find the optimal splits, we define $S[i,j]$ to be the value k such that $M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j$.

Computing a solution

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We could now define a recursive algorithm straightforwardly using this.

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RECURSIVE-MATRIX-CHAIN(p, i, j)
if i == j
    return 0
m[i, j] = ∞
for k = i to j − 1
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    if q < m[i, j]
        m[i, j] = q
return m[i, j]
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Running time?

The running time is given by the following recurrence relation:

$$T(n) \;\geq \begin{cases} 1, & \text{if } n = 1, \\ 1 + \sum_{k=1}^{n-1} \left(T(k) + T(n-k) + 1\right) & \text{if } n > 1. \end{cases}$$

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Computing the subproblems

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$M[1,4]$

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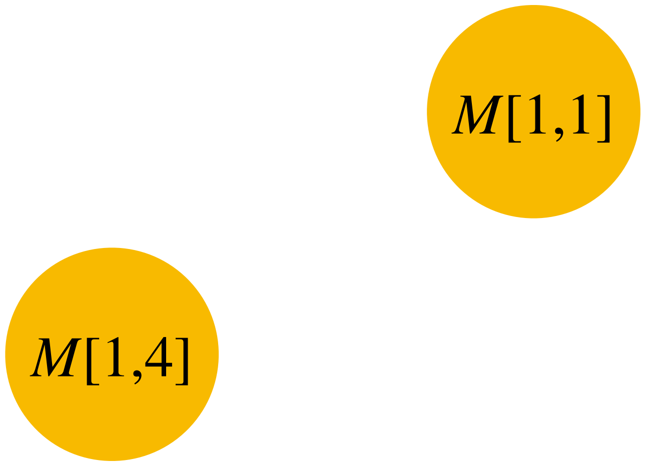

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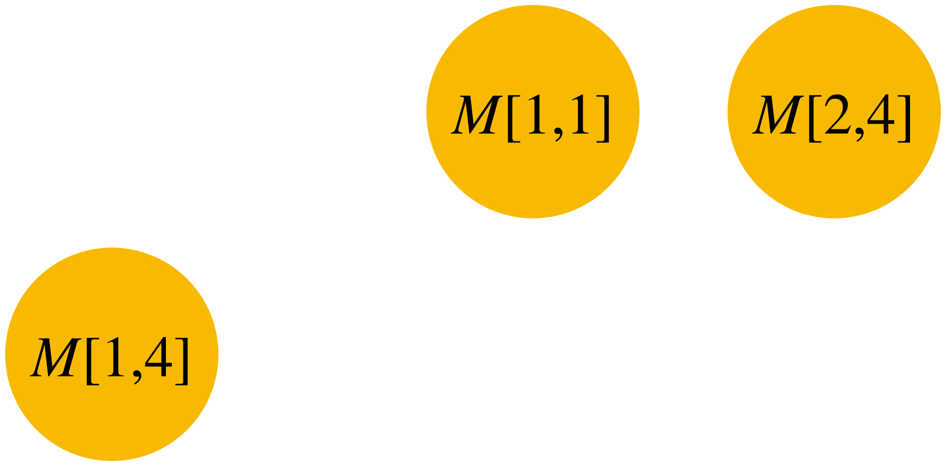

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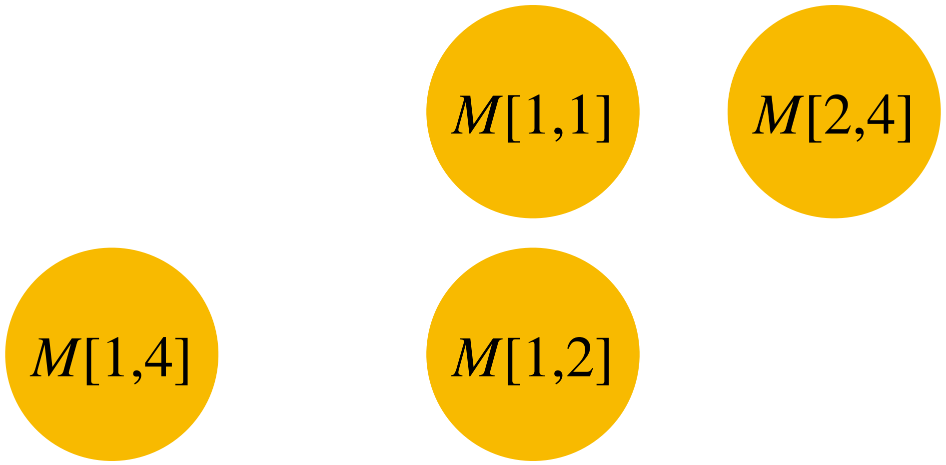

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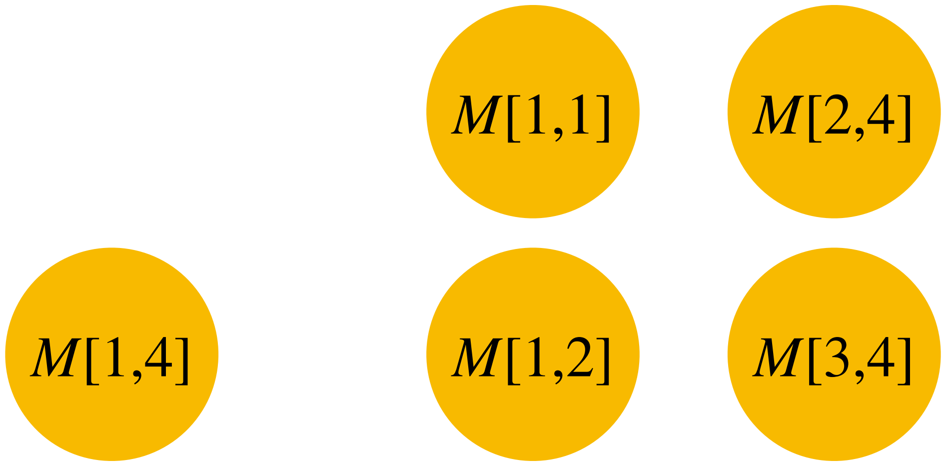

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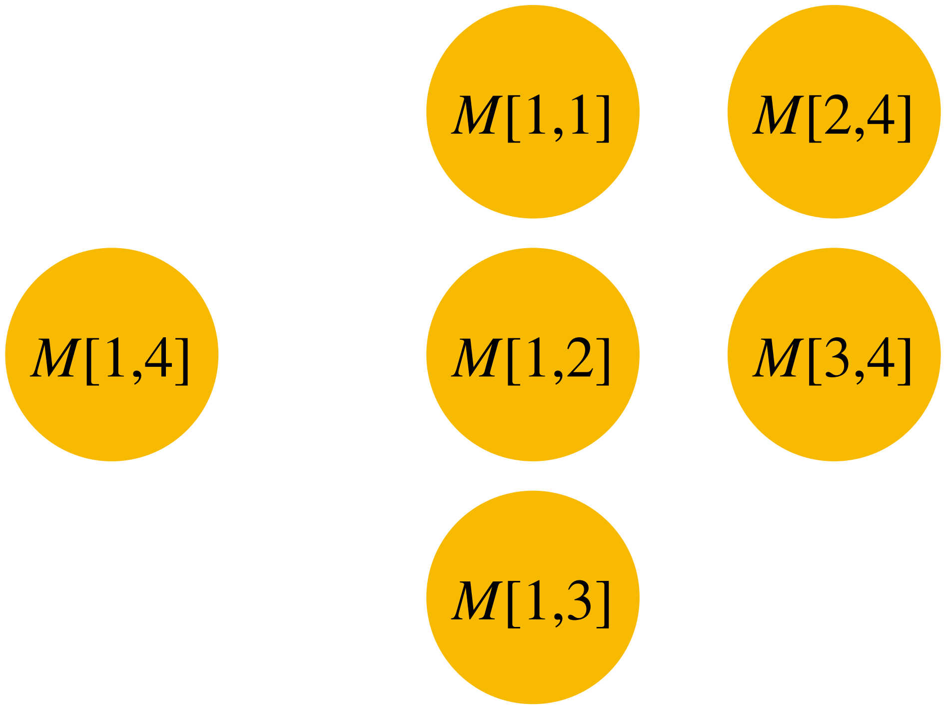

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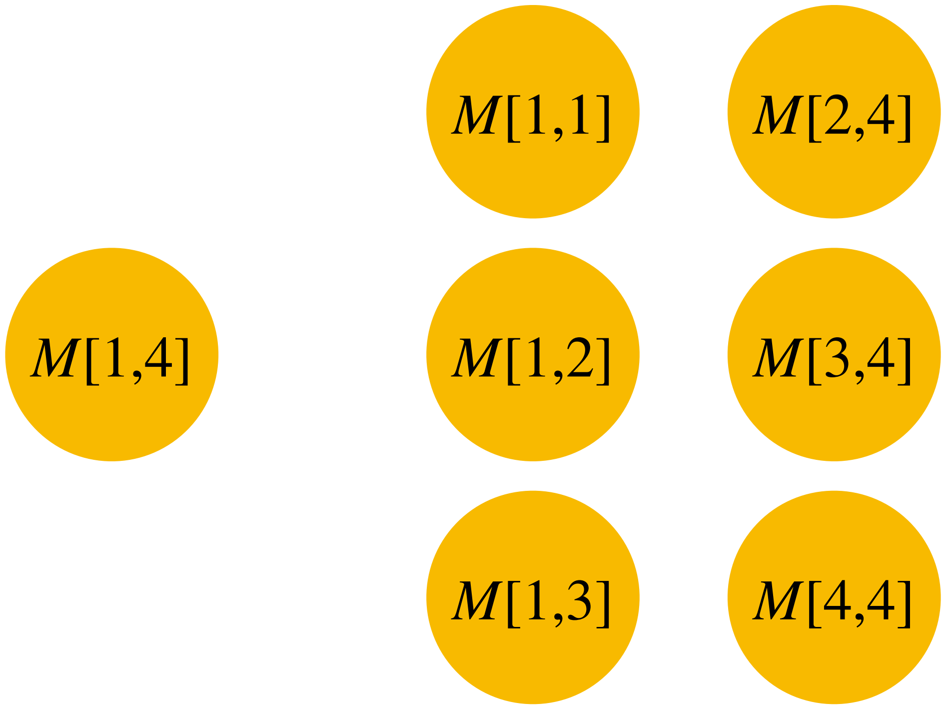

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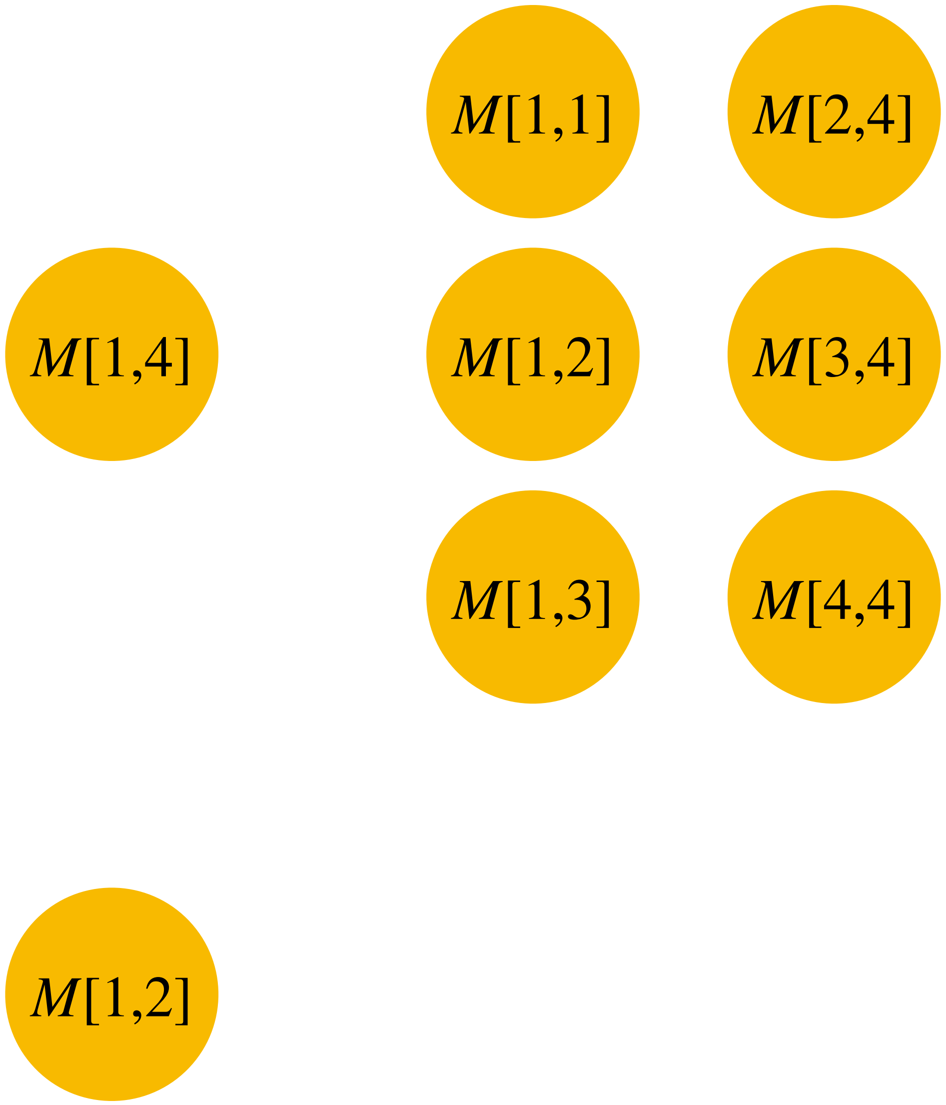

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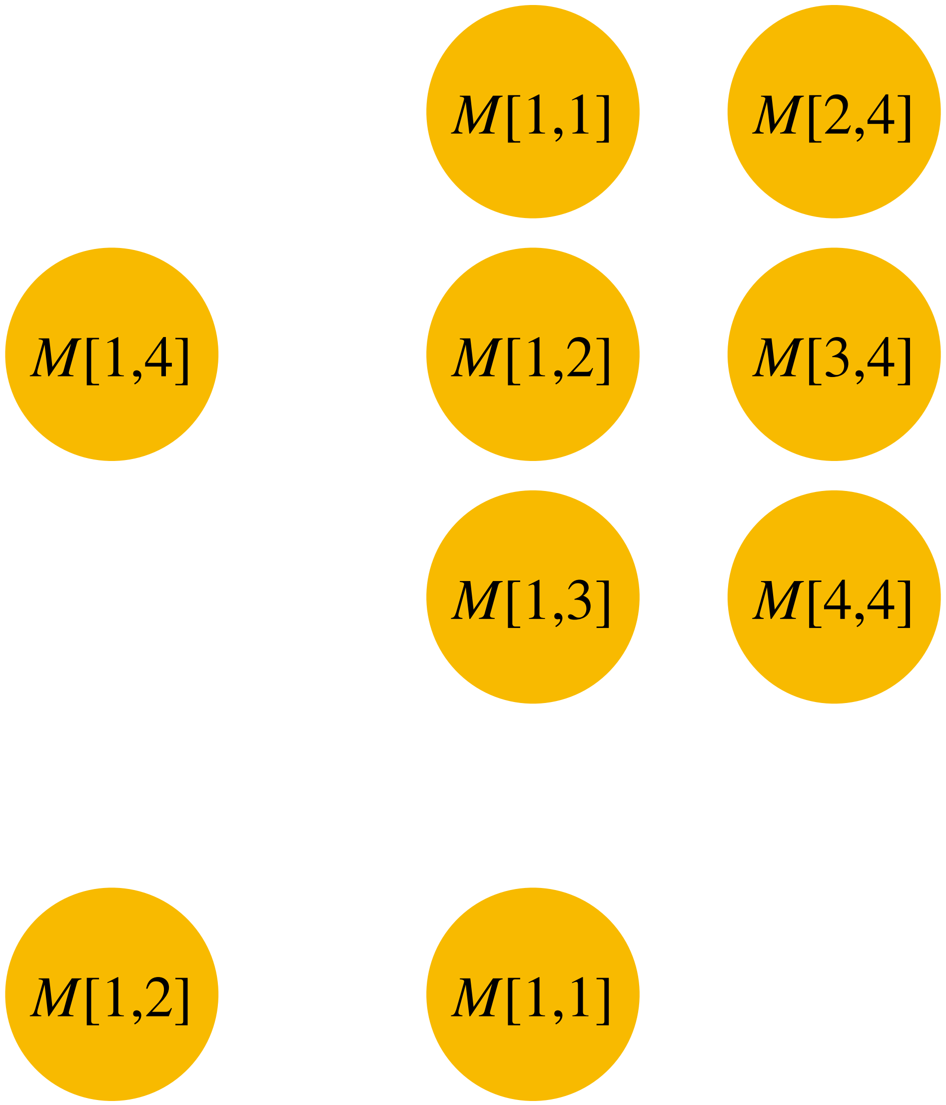

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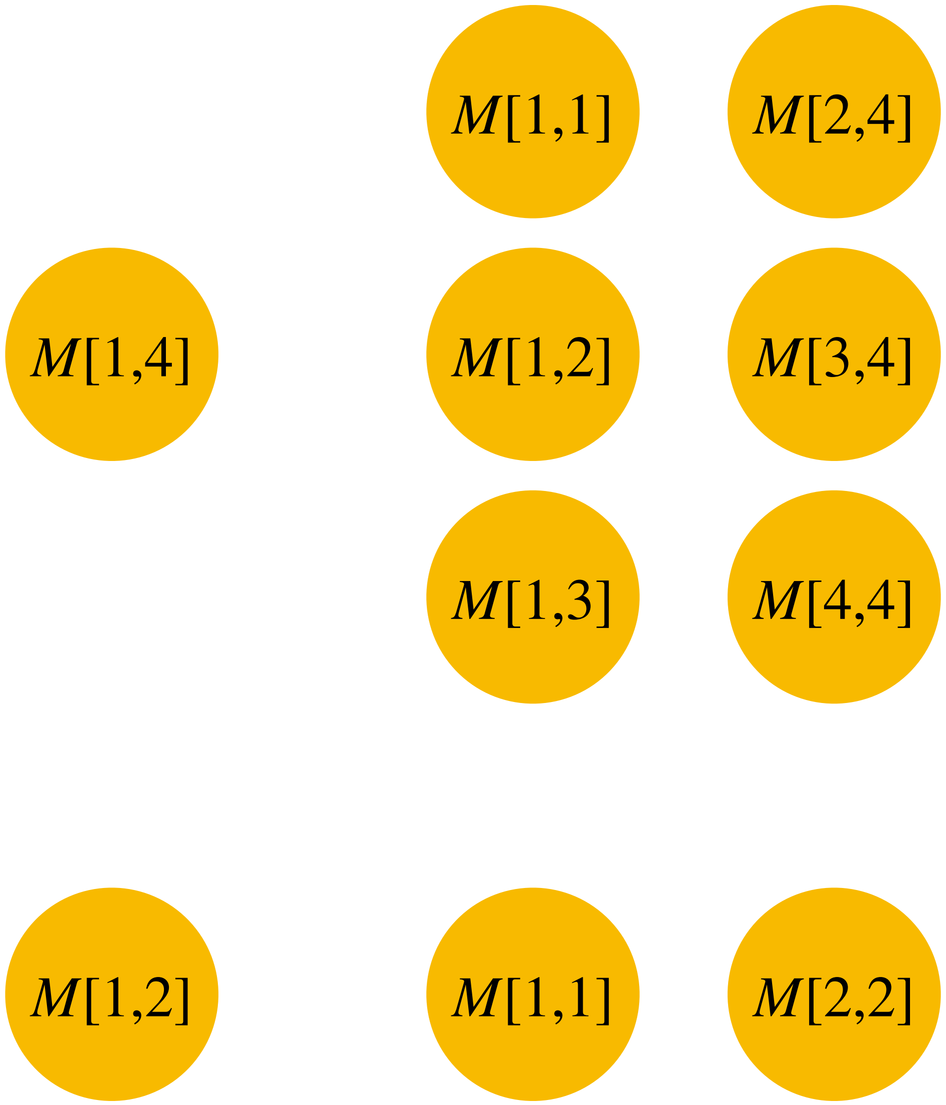

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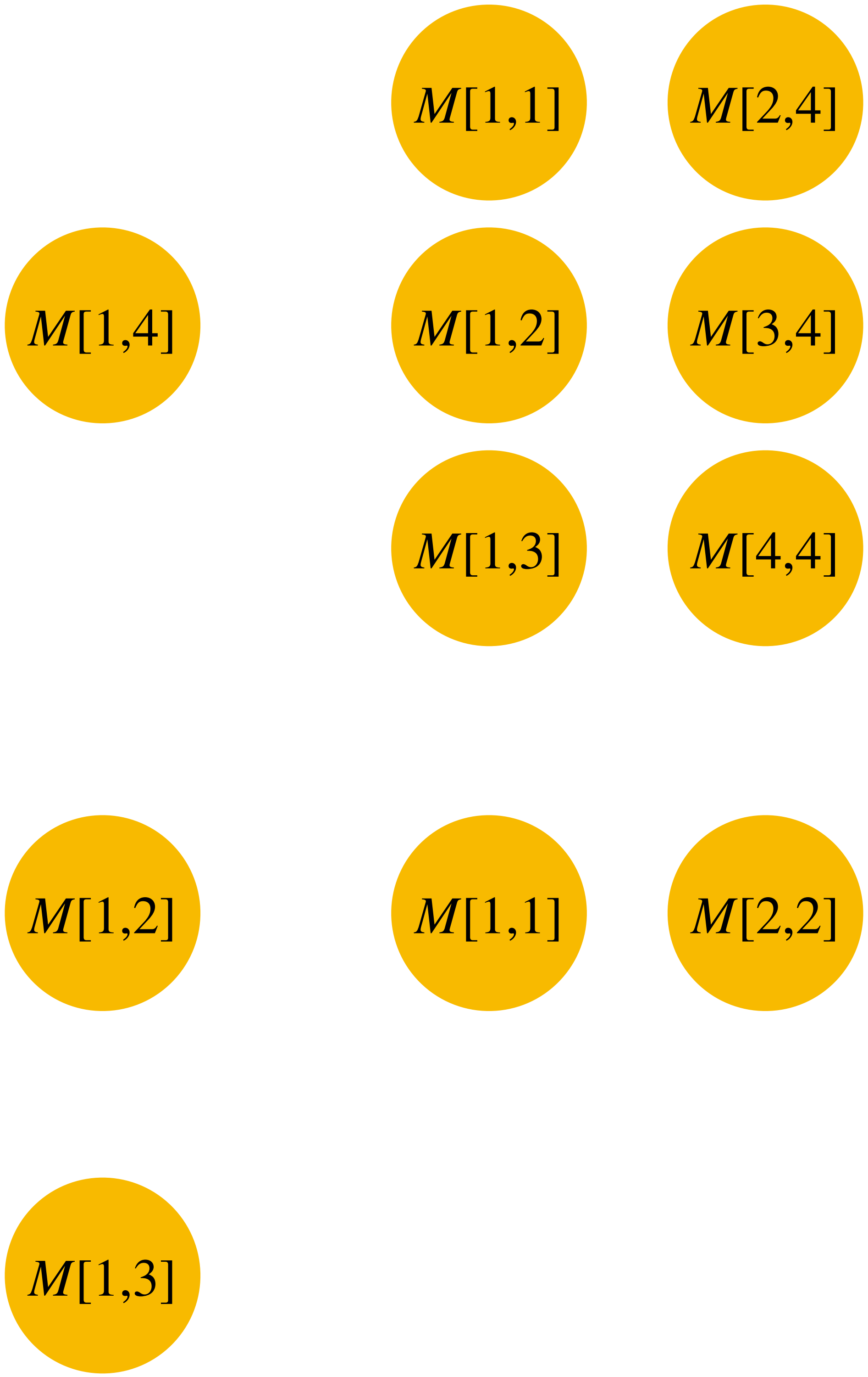

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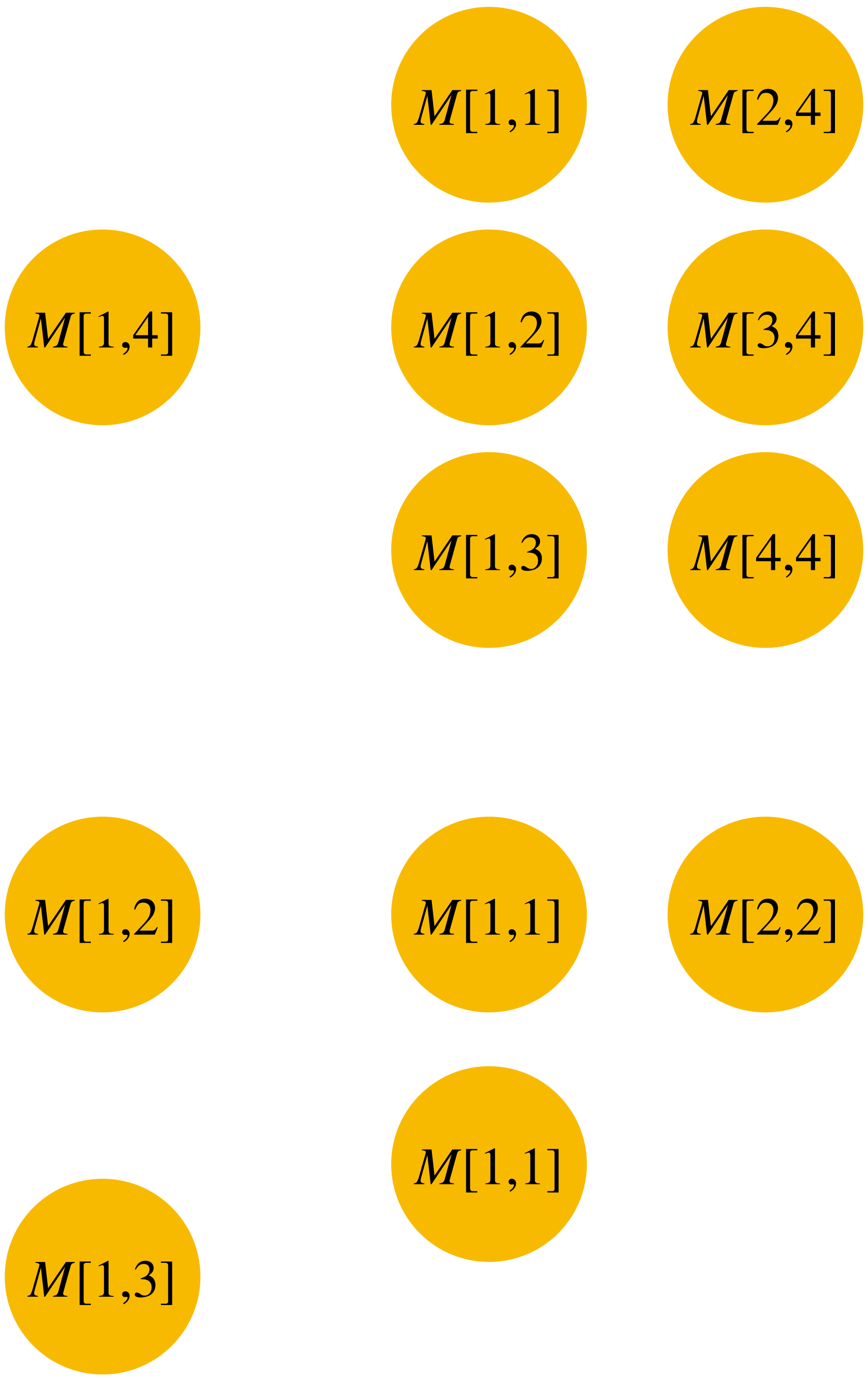

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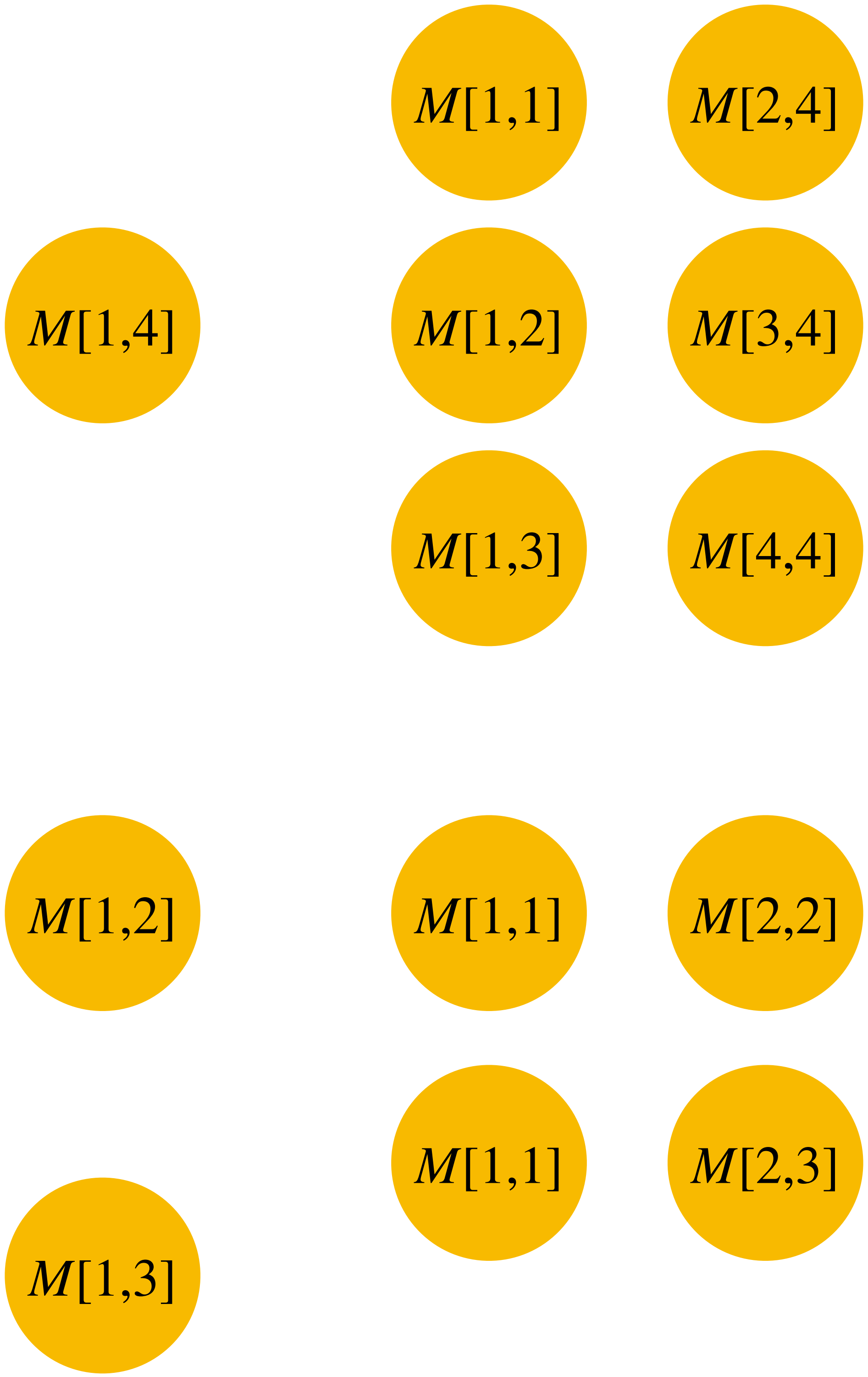

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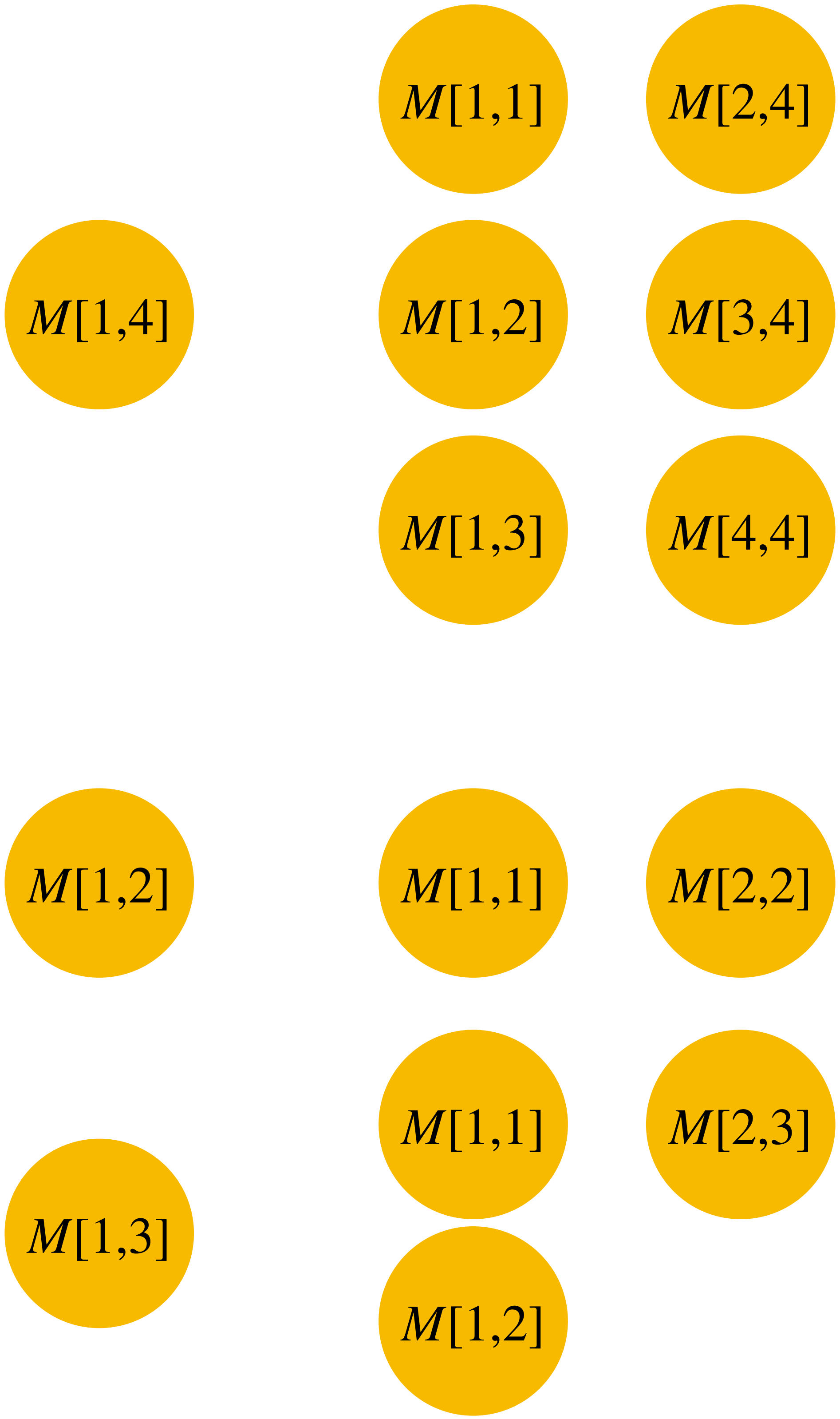

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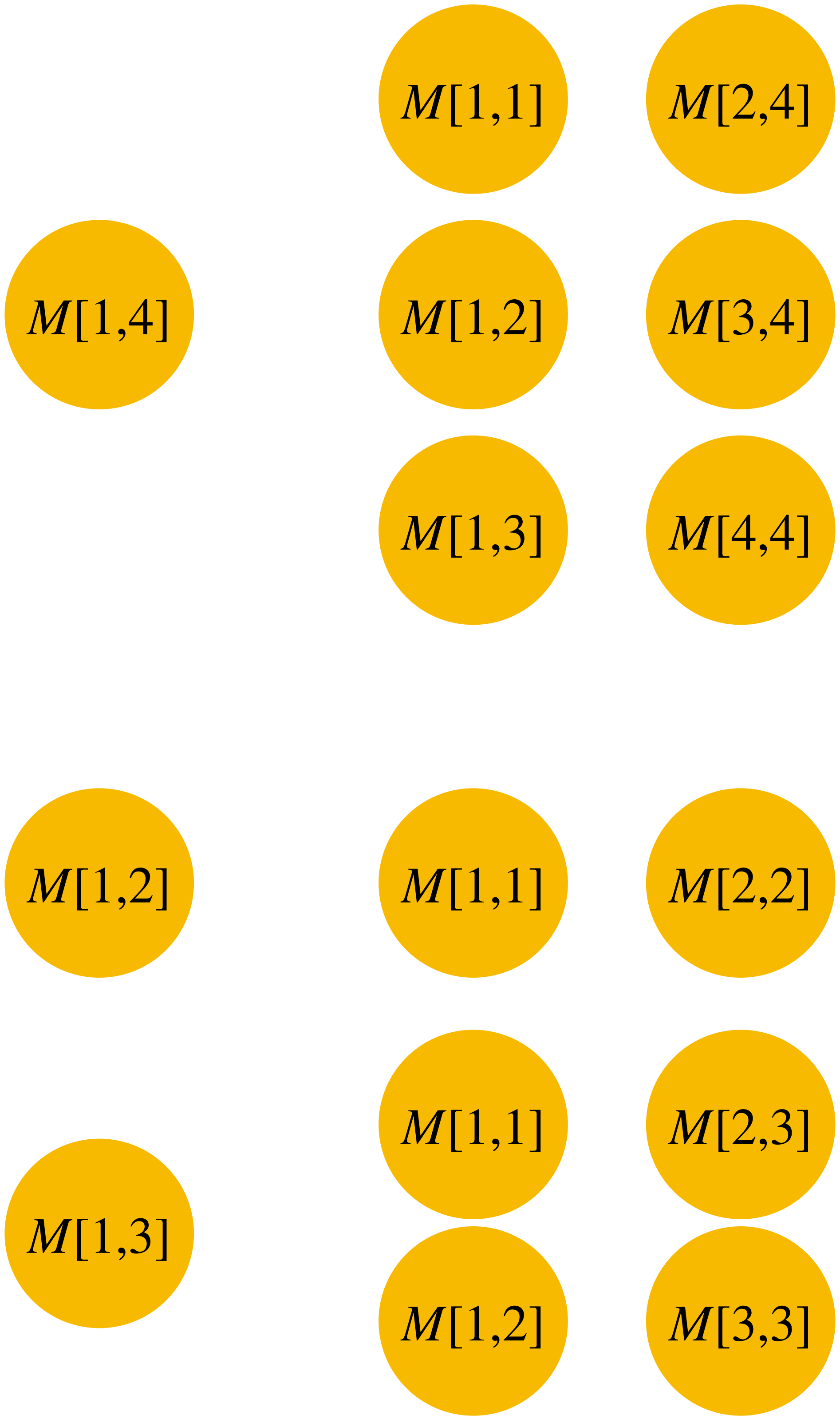

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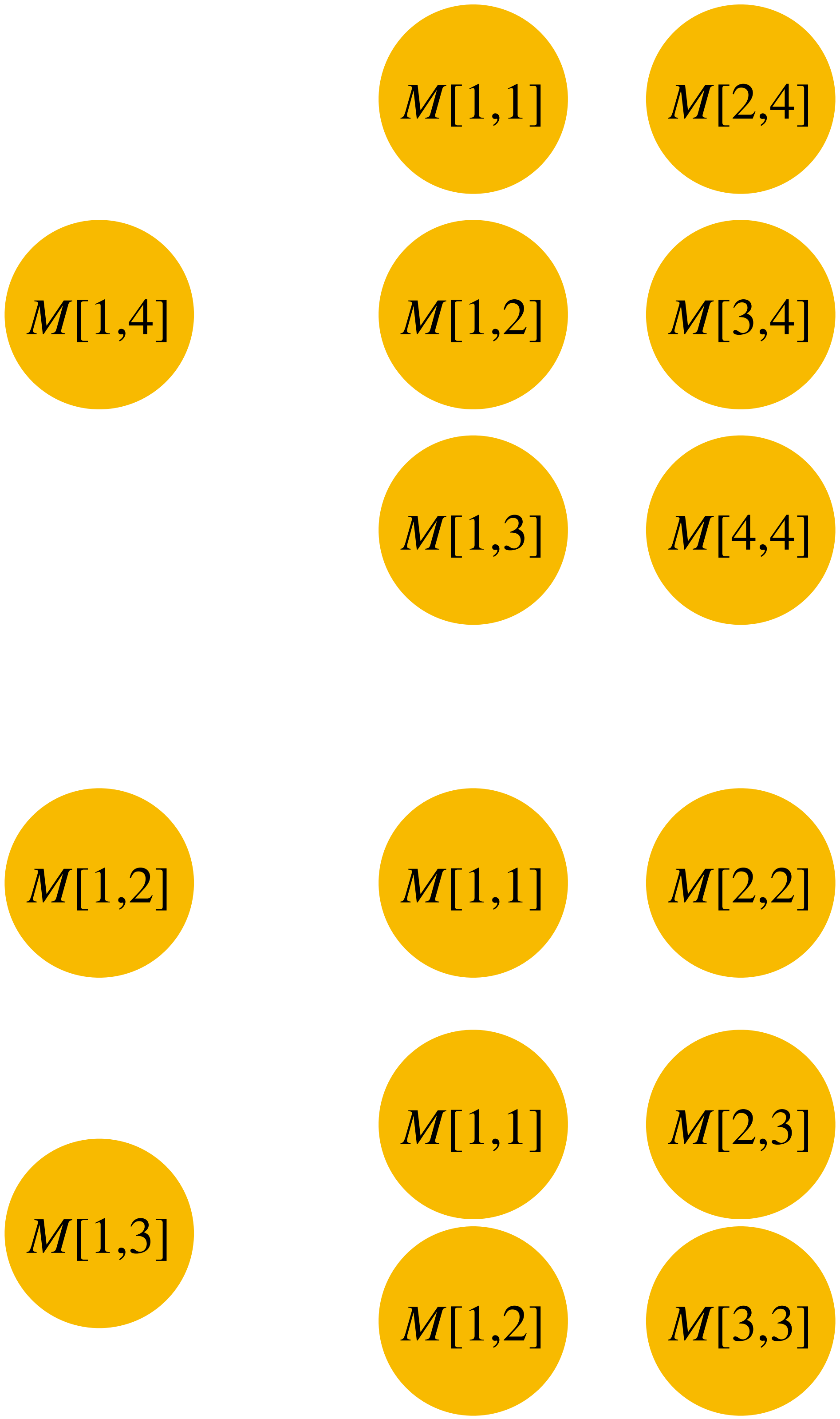

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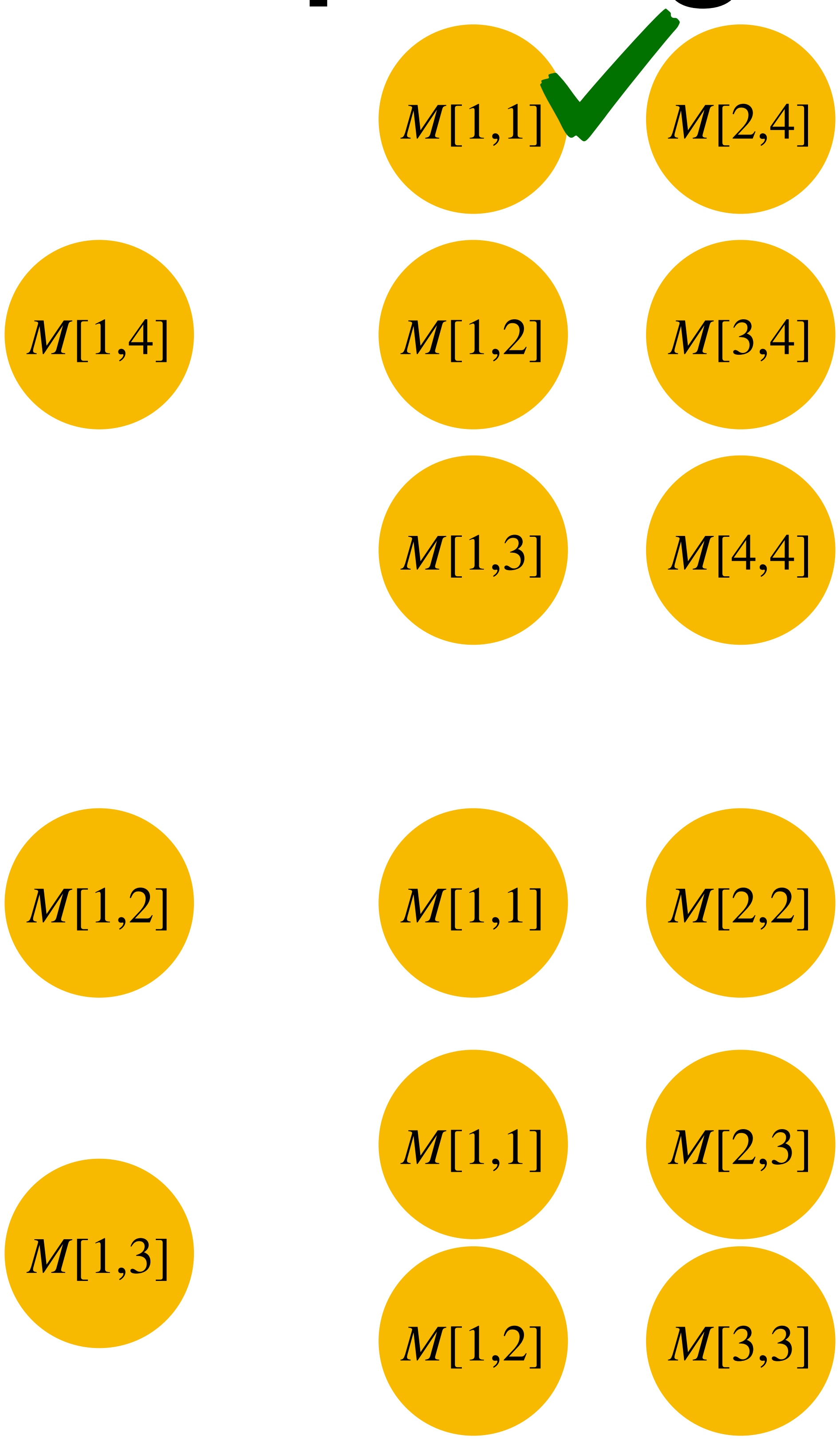
Computing the subproblems



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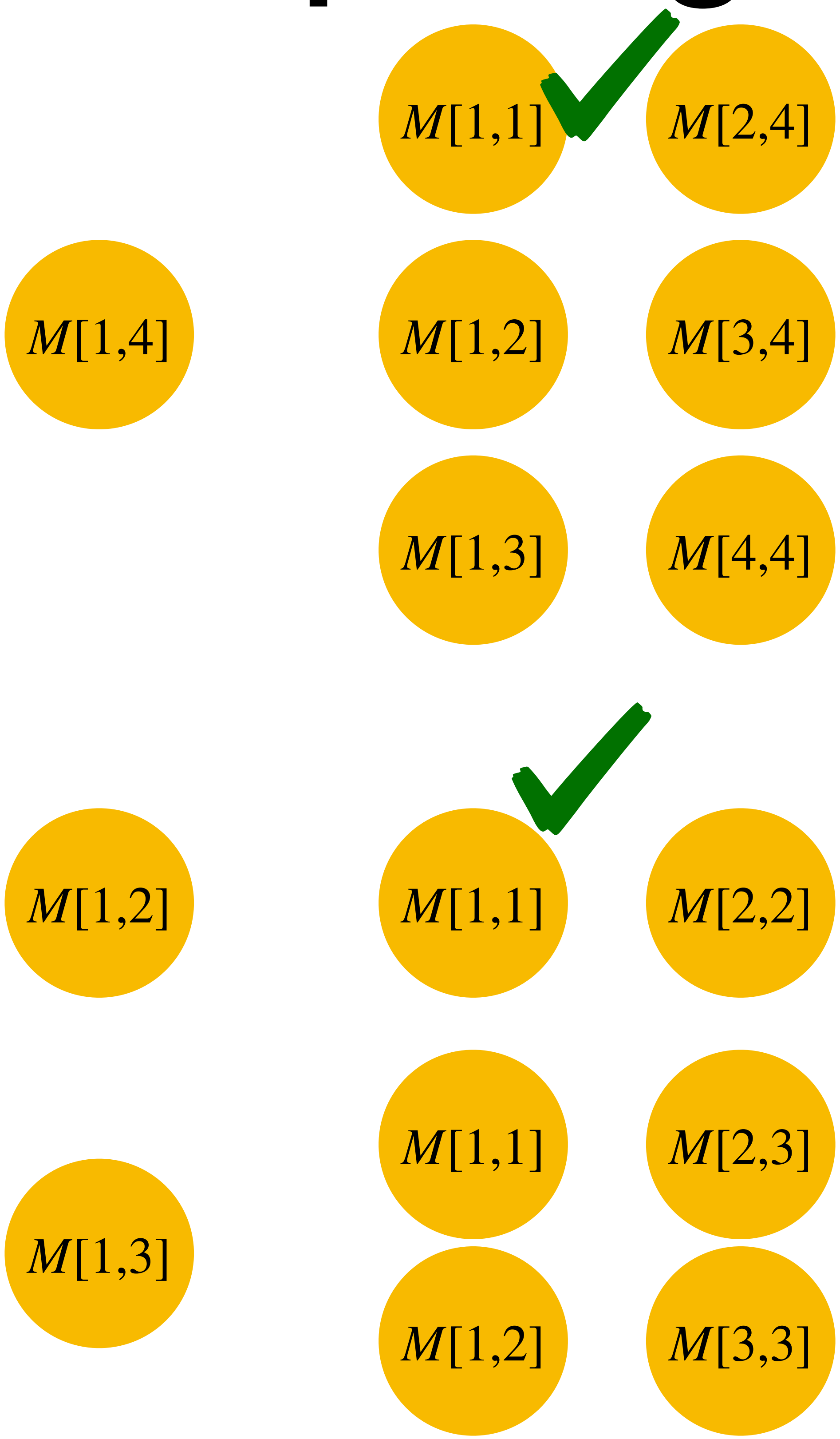

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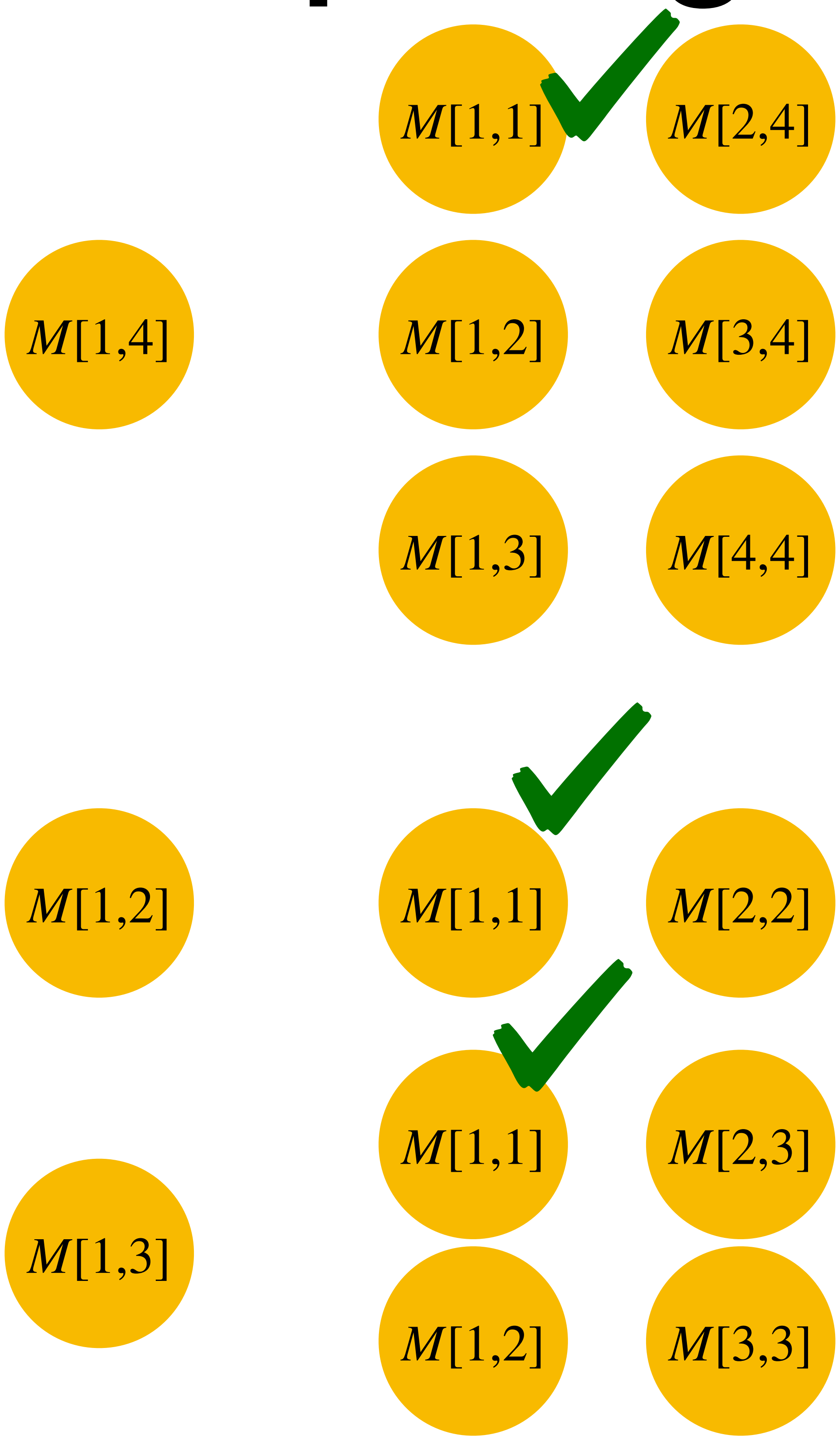

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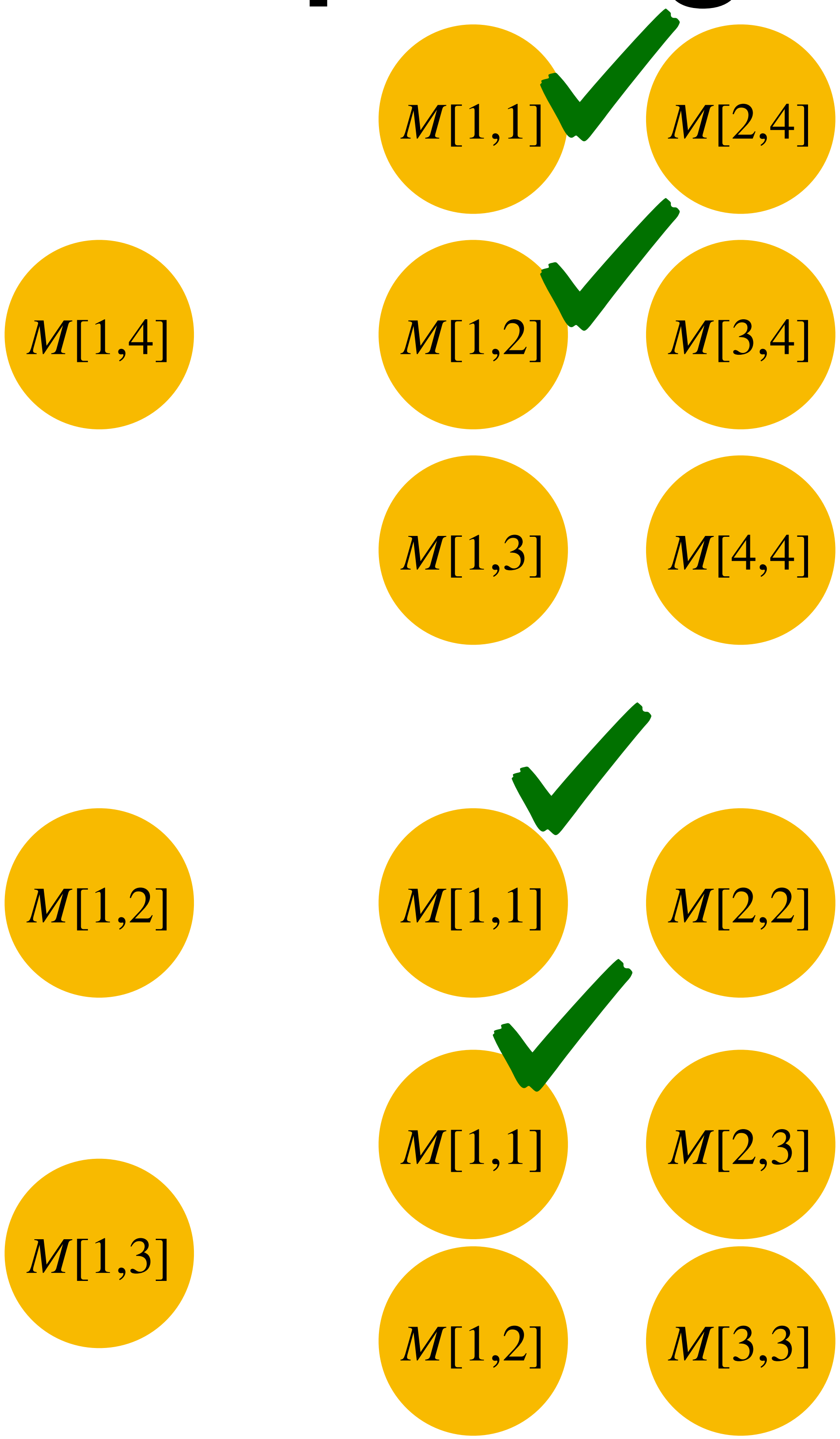

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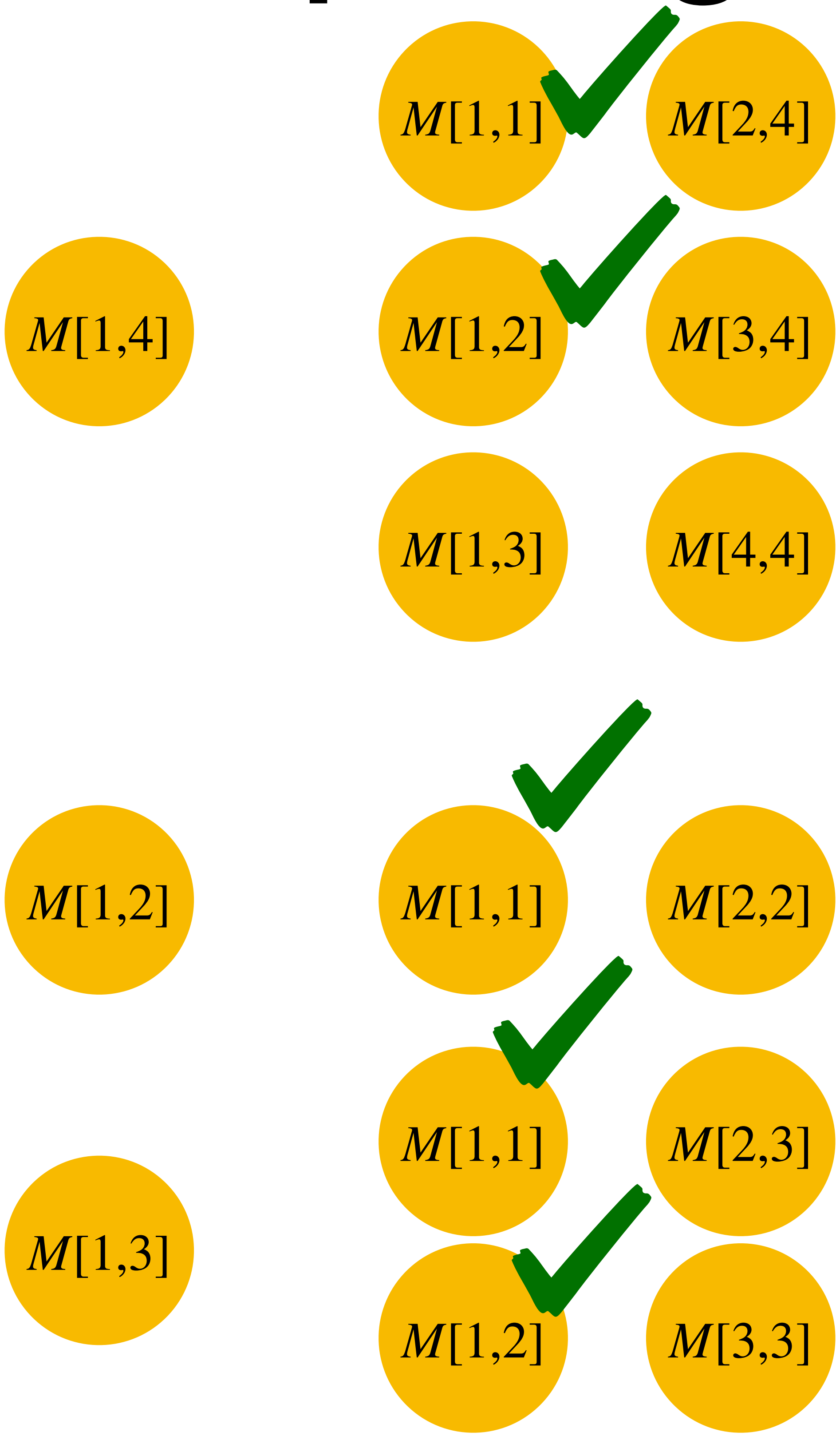

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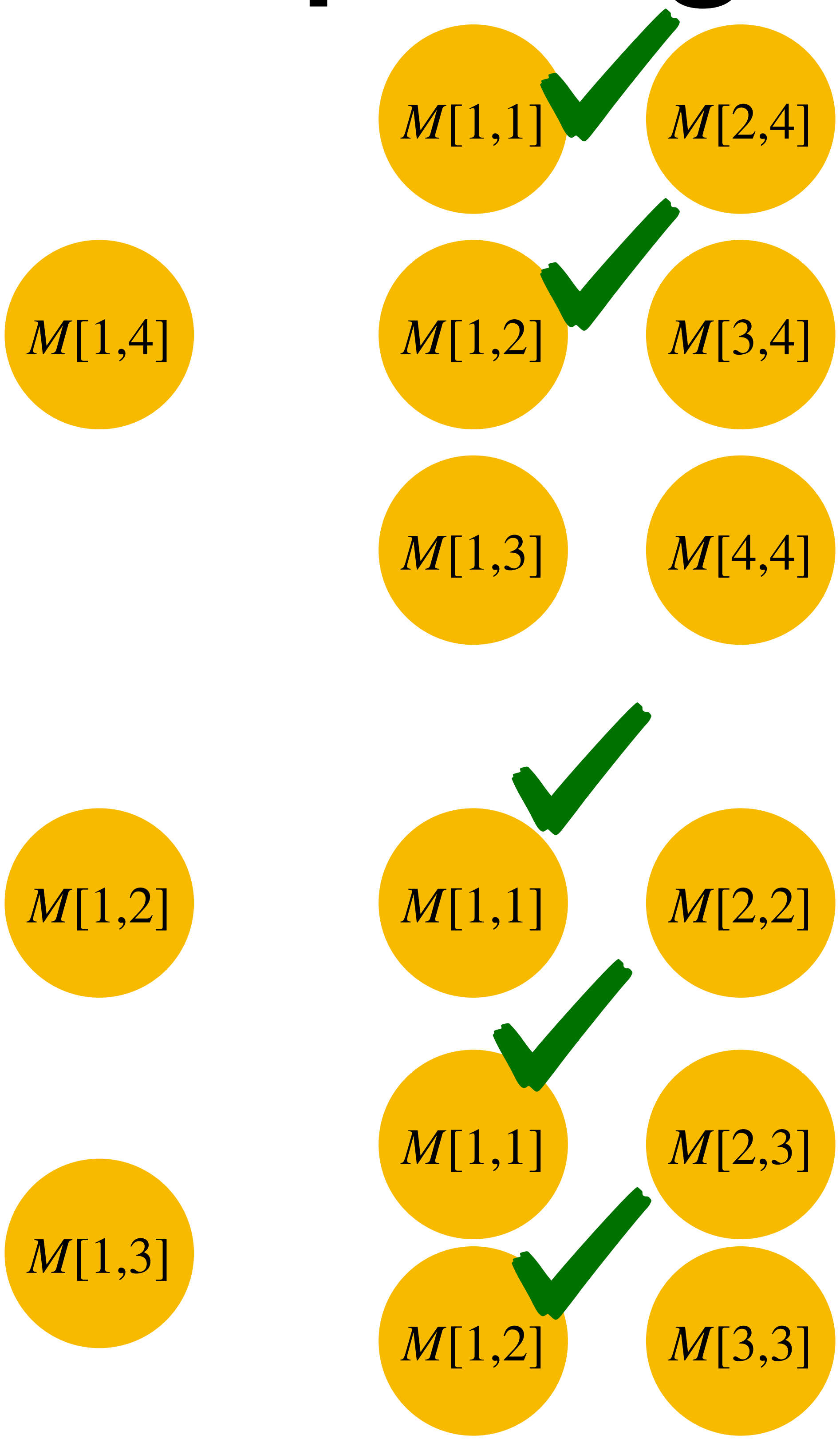

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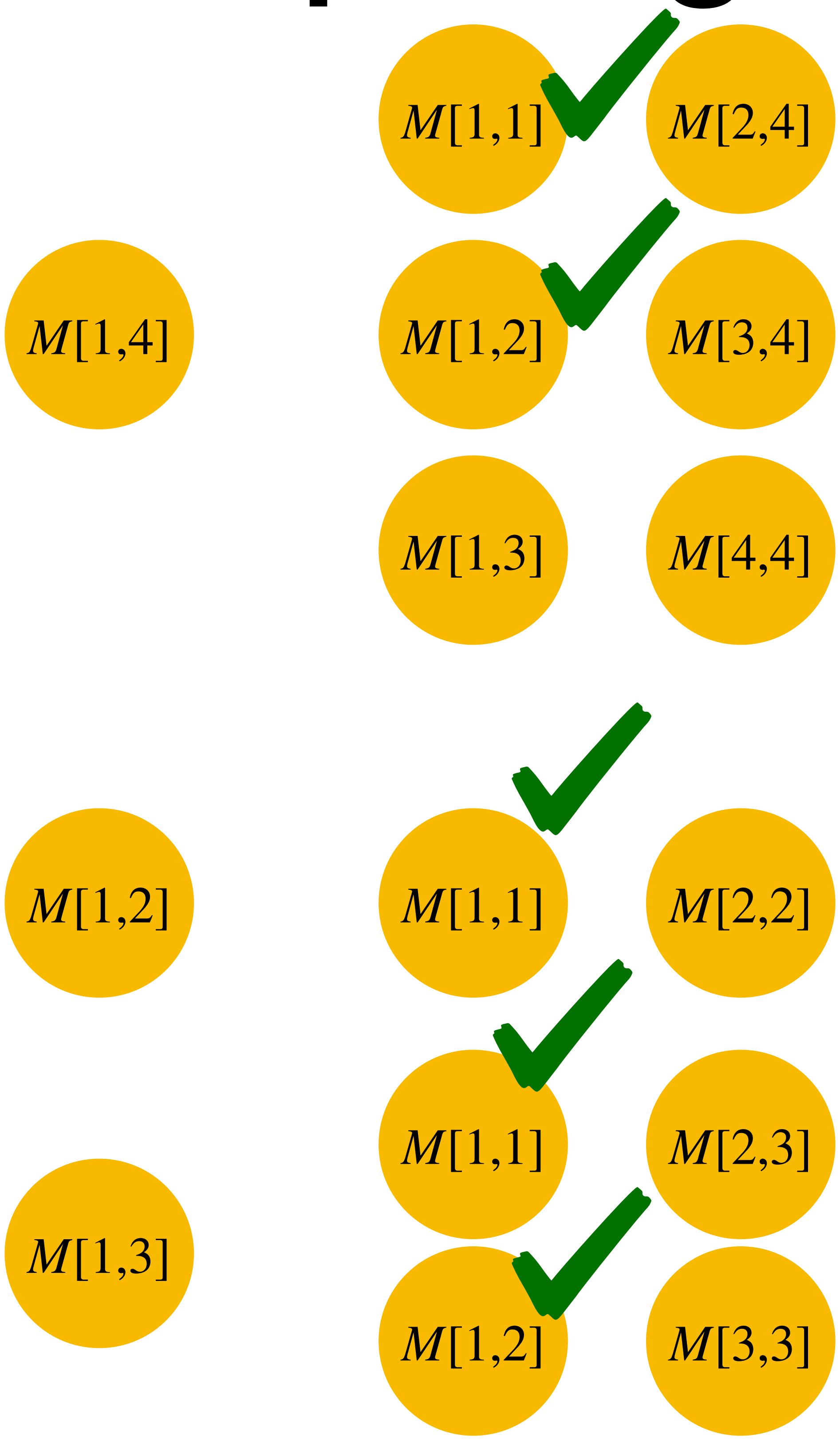
How many distinct subproblems of the problem from i to j are there?

Computing the subproblems



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As many as the choice of i, j that satisfy $i \leq i \leq j \leq n$, i.e., $\binom{n}{2} + n = \Theta(n^2)$

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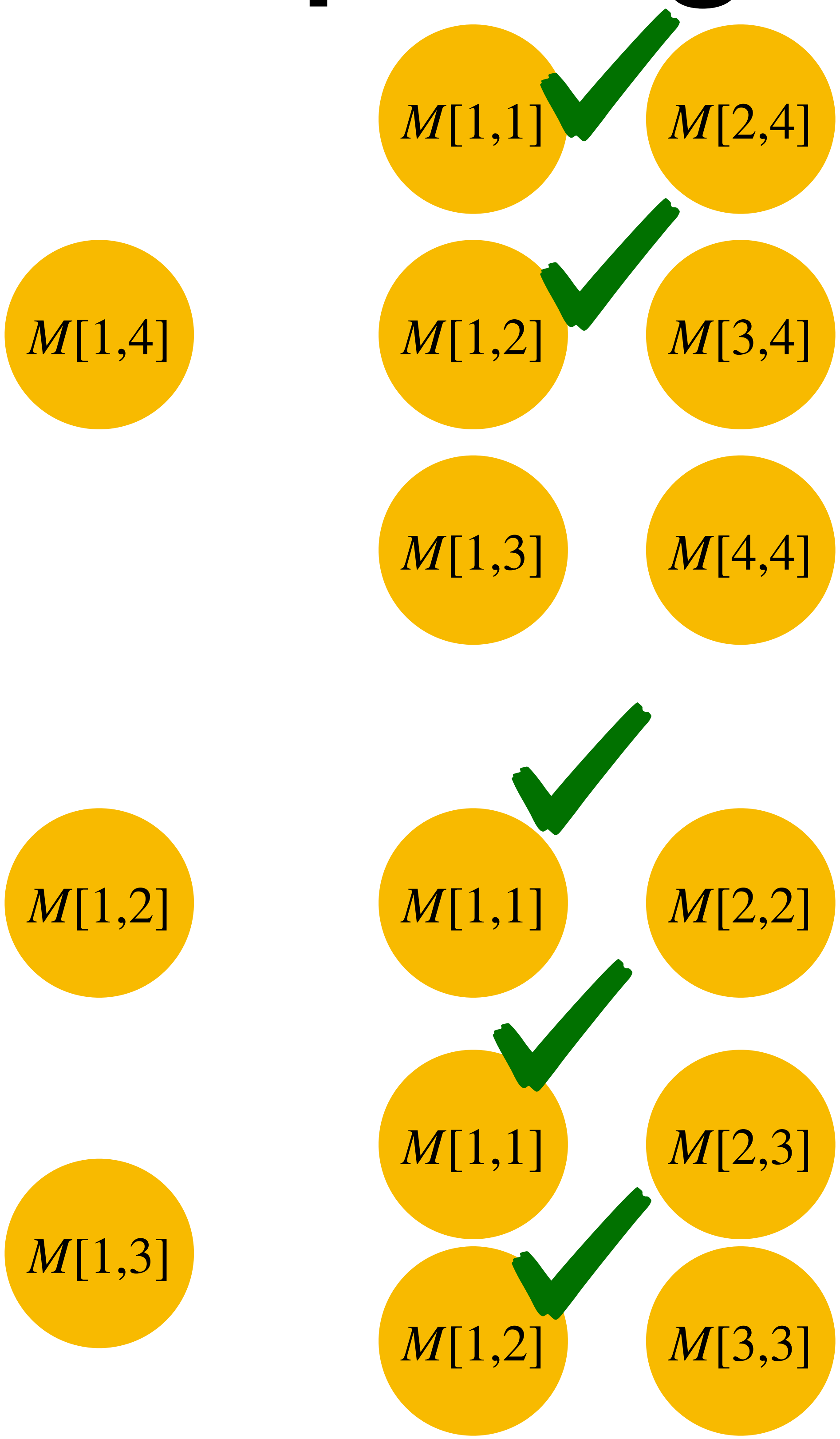

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The key is to *store* the calculation of the subproblems and reuse it.

A Dynamic Programming algorithm

A Dynamic Programming algorithm

To compute $M[i, j]$, we need the values of $M[i, k]$ and $M[k+1, j]$ for all $i \leq k \leq j$.

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So we will have to compute those first.

We work in a *bottom-up manner*.

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1:n, 1:n] and s[1:n-1, 2:n] be new tables
for i = 1 to n                        // chain length 1
    m[i, i] = 0
for l = 2 to n                        // l is the chain length
    for i = 1 to n - l + 1            // chain begins at A_i
        j = i + l - 1                 // chain ends at A_j
        m[i, j] = ∞
        for k = i to j - 1            // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j
            if q < m[i, j]
                m[i, j] = q           // remember this cost
                s[i, j] = k           // remember this index
return m and s
```

Example

$A_1 : 30 \times 35$

$A_2 : 35 \times 15$

$A_3 : 15 \times 5$

$A_4 : 5 \times 10$

$A_5 : 10 \times 20$

$A_6 : 20 \times 25$

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The table M

0					
	0				
		0			
			0		
				0	
					0

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    m[i, i] = 0          } chains of length 1
for l = 2 to n   chains of length 2             // l is the chain length
    for i = 1 to n - l + 1                      // chain begins at A_i
        j = i + l - 1                           // chain ends at A_j
        m[i, j] = ∞
        for k = i to j - 1                      // try A_{i:k} A_{k+1:j}
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        $j = i + l - 1$ // chain ends at A_j $j = 2$
        $m[i,j] = \infty$ We are now computing $M[1,2]$
        for $k = i$ **to** $j - 1$ // try $A_{i:k} A_{k+1:j}$
            $q = m[i,k] + m[k+1,j] + p_{i-1} p_k p_j$
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            if q < m[i, j]
                m[i, j] = q                     // remember this cost
                s[i, j] = k                     // remember this index
return m and s
```

chains of length 1

chains of length 2

$i = 1$

$j = 2$

We are now computing $M[1,2]$

$k = 1$

A Dynamic Programming algorithm

```
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let m[1:n, 1:n] and s[1:n-1, 2:n] be new tables
for i = 1 to n                               // chain length 1
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        j = i + l - 1                        // chain ends at A_j          j = 2
        m[i, j] = ∞                               We are now computing M[1,2]
        for k = i to j - 1                   // try A_{i:k} A_{k+1:j}       k = 1
            q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j      M[1,1] + M[2,2] + p_0 p_1 p_2
            if q < m[i, j]
                m[i, j] = q                  // remember this cost
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MATRIX-CHAIN-ORDER(p, n)

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    for $k = i$ **to** $j - 1$ // try $A_{i:k} A_{k+1:j}$ $k = 1$
      $q = m[i,k] + m[k+1, j] + p_{i-1} p_k p_j$ $M[1,1] + M[2,2] + p_0 p_1 p_2$
      if $q < m[i,j]$ 0
        $m[i,j] = q$ // remember this cost
        $s[i,j] = k$ // remember this index
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A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

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            if $q < m[i,j]$
                $m[i,j] = q$ // remember this cost
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return m and s

Example

$A_1 : 30 \times 35$

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Example

$A_1 : 30 \times 35$

$A_2 : 35 \times 15$

$A_3 : 15 \times 5$

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$A_6 : 20 \times 25$

$p_0 p_1 p_2 = 30 \cdot 35 \cdot 15 = 15750$

The table M

0	15750				
	0				
		0			
			0		
				0	
					0

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
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                m[i, j] = q             // remember this cost
                s[i, j] = k             // remember this index
return m and s
```

chains of length 1

chains of length 2

this goes up to 5

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

let $m[1:n, 1:n]$ and $s[1:n-1, 2:n]$ be new tables   this goes up to 5
for $i = 1$ **to** n // chain length 1
  $m[i, i] = 0$ chains of length 1
for $l = 2$ **to** n chains of length 2   // l is the chain length
  for $i = 1$ **to** $n - l + 1$ // chain begins at A_i $i = 2$
    $j = i + l - 1$ // chain ends at A_j
    $m[i, j] = \infty$
    for $k = i$ **to** $j - 1$ // try $A_{i:k} A_{k+1:j}$
      $q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$
      if $q < m[i, j]$
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MATRIX-CHAIN-ORDER(p, n)

this goes up to 5

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    for i = 1 to n - l + 1              // chain begins at A_i        i = 2
        j = i + l - 1                   // chain ends at A_j          j = 3
        m[i, j] = ∞
        for k = i to j - 1              // try A_{i:k} A_{k+1:j}
            q = m[i,k] + m[k+1, j] + p_{i-1} p_k p_j
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A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

let $m[1:n, 1:n]$ and $s[1:n-1, 2:n]$ be new tables　　　this goes up to 5

for $i = 1$ **to** n　　　// chain length 1

　　$m[i, i] = 0$　　　chains of length 1

for $l = 2$ **to** n　　chains of length 2　　// l is the chain length

　　for $i = 1$ **to** $n - l + 1$　　　// chain begins at A_i　　　$i = 2$

　　　　$j = i + l - 1$　　　// chain ends at A_j　　　$j = 3$

　　　　$m[i, j] = \infty$　　　We are now computing $M[2,3]$

　　　　for $k = i$ **to** $j - 1$　　　// try $A_{i:k} A_{k+1:j}$

　　　　　　$q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$

　　　　　　if $q < m[i, j]$

　　　　　　　　$m[i, j] = q$　　　// remember this cost

　　　　　　　　$s[i, j] = k$　　　// remember this index

return m and s

A Dynamic Programming algorithm

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let $m[1:n, 1:n]$ and $s[1:n-1, 2:n]$ be new tables   this goes up to 5

for $i = 1$ **to** n // chain length 1
    $m[i,i] = 0$ chains of length 1

for $l = 2$ **to** n chains of length 2   // l is the chain length
    for $i = 1$ **to** $n - l + 1$ // chain begins at A_i $i = 2$
        $j = i + l - 1$ // chain ends at A_j $j = 3$
        $m[i,j] = \infty$ We are now computing $M[2,3]$
        for $k = i$ **to** $j - 1$ // try $A_{i:k} A_{k+1:j}$ $k = 2$
            $q = m[i,k] + m[k+1,j] + p_{i-1} p_k p_j$
            if $q < m[i,j]$
                $m[i,j] = q$ // remember this cost
                $s[i,j] = k$ // remember this index

return m and s

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

let $m[1:n, 1:n]$ and $s[1:n-1, 2:n]$ be new tables — this goes up to 5

for $i = 1$ **to** n — // chain length 1

    $m[i, i] = 0$ — chains of length 1

for $l = 2$ **to** n — chains of length 2 — // l is the chain length

    for $i = 1$ **to** $n - l + 1$ — // chain begins at A_i — $i = 2$

        $j = i + l - 1$ — // chain ends at A_j — $j = 3$

        $m[i, j] = \infty$ — We are now computing $M[2,3]$

        for $k = i$ **to** $j - 1$ — // try $A_{i:k} A_{k+1:j}$ — $k = 2$

            $q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$ — $M[2,2] + M[3,3] + p_1 p_2 p_3$

            if $q < m[i, j]$

                $m[i, j] = q$ — // remember this cost

                $s[i, j] = k$ — // remember this index

return m and s

Example

$A_1 : 30 \times 35$

$A_2 : 35 \times 15$

$A_3 : 15 \times 5$

$A_4 : 5 \times 10$

$A_5 : 10 \times 20$

$A_6 : 20 \times 25$

Example

$A_1 : 30 \times 35$

$A_2 : 35 \times 15$

$A_3 : 15 \times 5$

$A_4 : 5 \times 10$

$A_5 : 10 \times 20$

$A_6 : 20 \times 25$

$p_1 p_2 p_3 = 35 \cdot 15 \cdot 5 = 2625$

The table M

0	15750				
	0	2625			
		0			
			0		
				0	
					0

The table M

0	15750				
	0	2625			
		0	750		
			0	1000	
				0	5000
					0

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1:n, 1:n] and s[1:n−1, 2:n] be new tables
for i = 1 to n                                 // chain length 1
    m[i, i] = 0          } chains of length 1
for l = 2 to n                                 // l is the chain length
    for i = 1 to n − l + 1                     // chain begins at A_i
        j = i + l − 1                          // chain ends at A_j
        m[i, j] = ∞
        for k = i to j − 1                     // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k+1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                    // remember this cost
                s[i, j] = k                    // remember this index
return m and s
```

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1:n, 1:n] and s[1:n-1, 2:n] be new tables
for i = 1 to n                                   // chain length 1
    m[i, i] = 0          chains of length 1
for l = 2 to n  chains of length 3               // l is the chain length
    for i = 1 to n - l + 1                       // chain begins at A_i
        j = i + l - 1                            // chain ends at A_j
        m[i, j] = ∞
        for k = i to j - 1                       // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                      // remember this cost
                s[i, j] = k                      // remember this index
return m and s
```

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                                   // chain length 1
    m[i, i] = 0                  chains of length 1
for l = 2 to n    chains of length 3             // l is the chain length
    for i = 1 to n − l + 1                       // chain begins at A_i          i = 1
        j = i + l − 1                            // chain ends at A_j
        m[i, j] = ∞
        for k = i to j − 1                       // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                      // remember this cost
                s[i, j] = k                      // remember this index
return m and s
```

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

let $m[1:n, 1:n]$ and $s[1:n-1, 2:n]$ be new tables
for $i = 1$ **to** n // chain length 1
  $m[i, i] = 0$ chains of length 1
for $l = 2$ **to** n chains of length 3 // l is the chain length
  for $i = 1$ **to** $n - l + 1$ // chain begins at A_i $i = 1$
    $j = i + l - 1$ // chain ends at A_j $j = 3$
    $m[i, j] = \infty$
    for $k = i$ **to** $j - 1$ // try $A_{i:k} A_{k+1:j}$
      $q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$
      if $q < m[i, j]$
        $m[i, j] = q$ // remember this cost
        $s[i, j] = k$ // remember this index
return m and s

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                                  // chain length 1
    m[i, i] = 0          chains of length 1
for l = 2 to n   chains of length 3             // l is the chain length
    for i = 1 to n − l + 1                      // chain begins at A_i          i = 1
        j = i + l − 1                           // chain ends at A_j            j = 3
        m[i, j] = ∞                                   We are now computing M[1,3]
        for k = i to j − 1                      // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                     // remember this cost
                s[i, j] = k                     // remember this index
return m and s
```

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                                  // chain length 1
    m[i, i] = 0                 chains of length 1
for l = 2 to n    chains of length 3            // l is the chain length
    for i = 1 to n − l + 1                      // chain begins at A_i          i = 1
        j = i + l − 1                           // chain ends at A_j            j = 3
        m[i, j] = ∞                                  We are now computing M[1,3]
        for k = i to j − 1                      // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                     // remember this cost
                s[i, j] = k                     // remember this index
return m and s
```

$k = 1$

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                                // chain length 1
    m[i, i] = 0
for l = 2 to n                                // l is the chain length
    for i = 1 to n − l + 1                    // chain begins at A_i
        j = i + l − 1                         // chain ends at A_j
        m[i, j] = ∞
        for k = i to j − 1                    // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                   // remember this cost
                s[i, j] = k                   // remember this index
return m and s
```

$\}$ chains of length 1

chains of length 3

$i = 1$

$j = 3$

We are now computing $M[1,3]$

$k = 1$ $\quad M[1,1] + M[2,3] + p_0 p_1 p_3 = 0 + 2625 + 5250 = 7875$

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                               // chain length 1
    m[i, i] = 0
for l = 2 to n                               // l is the chain length
    for i = 1 to n − l + 1                   // chain begins at A_i
        j = i + l − 1                        // chain ends at A_j
        m[i, j] = ∞
        for k = i to j − 1                   // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                  // remember this cost
                s[i, j] = k                  // remember this index
return m and s
```

chains of length 1

chains of length 3

$i = 1$

$j = 3$

We are now computing $M[1,3]$

$k = 1$ $\quad M[1,1] + M[2,3] + p_0 p_1 p_3 = 0 + 2625 + 5250 = 7875$

$k = 2$

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

let $m[1:n, 1:n]$ and $s[1:n-1, 2:n]$ be new tables
for $i = 1$ **to** n // chain length 1
    $m[i,i] = 0$ chains of length 1
for $l = 2$ **to** n chains of length 3 // l is the chain length
    for $i = 1$ **to** $n - l + 1$ // chain begins at A_i $i = 1$
        $j = i + l - 1$ // chain ends at A_j $j = 3$
        $m[i,j] = \infty$ We are now computing $M[1,3]$
        for $k = i$ **to** $j - 1$ // try $A_{i:k} A_{k+1:j}$
            $q = m[i,k] + m[k+1, j] + p_{i-1} p_k p_j$
            if $q < m[i,j]$
                $m[i,j] = q$ // remember this cost
                $s[i,j] = k$ // remember this index
return m and s

$k = 1$ $M[1,1] + M[2,3] + p_0 p_1 p_3 = 0 + 2625 + 5250 = 7875$

$k = 2$ $M[1,2] + M[3,3] + p_0 p_2 p_3 = 15750 + 0 + 2250 = 18000$

The table M

0	15750	7875			
	0	2625			
		0	750		
			0	1000	
				0	5000
					0

The table M

0	15750	7875	9375		
	0	2625	4375		
		0	750	2500	
			0	1000	3500
				0	5000
					0

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1:n, 1:n] and s[1:n-1, 2:n] be new tables
for i = 1 to n                              // chain length 1
    m[i, i] = 0                 chains of length 1
for l = 2 to n                              // l is the chain length
    for i = 1 to n - l + 1                  // chain begins at A_i
        j = i + l - 1                       // chain ends at A_j
        m[i, j] = ∞
        for k = i to j - 1                  // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                 // remember this cost
                s[i, j] = k                 // remember this index
return m and s
```

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                                   // chain length 1
    m[i, i] = 0            chains of length 1
for l = 2 to n   chains of length 4              // l is the chain length
    for i = 1 to n − l + 1                       // chain begins at A_i
        j = i + l − 1                            // chain ends at A_j
        m[i, j] = ∞
        for k = i to j − 1                       // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                      // remember this cost
                s[i, j] = k                      // remember this index
return m and s
```

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

let $m[1:n, 1:n]$ and $s[1:n-1, 2:n]$ be new tables
for $i = 1$ **to** n // chain length 1
  $m[i, i] = 0$ chains of length 1
for $l = 2$ **to** n chains of length 4 // l is the chain length
  for $i = 1$ **to** $n - l + 1$ // chain begins at A_i $i = 2$
    $j = i + l - 1$ // chain ends at A_j
    $m[i, j] = \infty$
    for $k = i$ **to** $j - 1$ // try $A_{i:k} A_{k+1:j}$
      $q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$
      if $q < m[i, j]$
        $m[i, j] = q$ // remember this cost
        $s[i, j] = k$ // remember this index
return m and s

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

let $m[1:n, 1:n]$ and $s[1:n-1, 2:n]$ be new tables
for $i = 1$ **to** n // chain length 1
    $m[i, i] = 0$
chains of length 1
for $l = 2$ **to** n chains of length 4 // l is the chain length
    for $i = 1$ **to** $n - l + 1$ // chain begins at A_i $i = 2$
        $j = i + l - 1$ // chain ends at A_j $j = 5$
        $m[i, j] = \infty$
        for $k = i$ **to** $j - 1$ // try $A_{i:k} A_{k+1:j}$
            $q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$
            if $q < m[i, j]$
                $m[i, j] = q$ // remember this cost
                $s[i, j] = k$ // remember this index
return m and s

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                       // chain length 1
    m[i, i] = 0          } chains of length 1
for l = 2 to n   chains of length 4  // l is the chain length
    for i = 1 to n − l + 1           // chain begins at A_i        i = 2
        j = i + l − 1                // chain ends at A_j          j = 5
        m[i, j] = ∞                       We are now computing M[2,5]
        for k = i to j − 1           // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q          // remember this cost
                s[i, j] = k          // remember this index
return m and s
```

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                          // chain length 1
    m[i, i] = 0          } chains of length 1
for l = 2 to n    chains of length 4    // l is the chain length
    for i = 1 to n − l + 1              // chain begins at A_i         i = 2
        j = i + l − 1                   // chain ends at A_j           j = 5
        m[i, j] = ∞                         We are now computing M[2,5]
        for k = i to j − 1              // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q             // remember this cost
                s[i, j] = k             // remember this index
return m and s
```

$k = 1$

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                              // chain length 1
    m[i, i] = 0            } chains of length 1
for l = 2 to n    chains of length 4        // l is the chain length
    for i = 1 to n − l + 1                  // chain begins at A_i          i = 2
        j = i + l − 1                       // chain ends at A_j            j = 5
        m[i, j] = ∞                              We are now computing M[2,5]
        for k = i to j − 1                  // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                 // remember this cost
                s[i, j] = k                 // remember this index
return m and s
```

$k = 1$ $\quad M[2,2] + M[3,5] + p_1 p_2 p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1:n, 1:n] and s[1:n-1, 2:n] be new tables
for i = 1 to n                                 // chain length 1
    m[i, i] = 0            chains of length 1
for l = 2 to n     chains of length 4          // l is the chain length
    for i = 1 to n - l + 1                     // chain begins at A_i        i = 2
        j = i + l - 1                          // chain ends at A_j          j = 5
        m[i, j] = ∞                              We are now computing M[2,5]
        for k = i to j - 1                     // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                    // remember this cost
                s[i, j] = k                    // remember this index
return m and s
```

$k = 1$ $\quad M[2,2] + M[3,5] + p_1 p_2 p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$

$k = 2$

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1:n, 1:n] and s[1:n-1, 2:n] be new tables
for i = 1 to n                              // chain length 1
    m[i,i] = 0
for l = 2 to n                              // l is the chain length
    for i = 1 to n - l + 1                  // chain begins at A_i
        j = i + l - 1                       // chain ends at A_j
        m[i,j] = ∞
        for k = i to j - 1                  // try A_{i:k} A_{k+1:j}
            q = m[i,k] + m[k+1,j] + p_{i-1} p_k p_j
            if q < m[i,j]
                m[i,j] = q                  // remember this cost
                s[i,j] = k                  // remember this index
return m and s
```

$\left.\begin{array}{l}\textbf{for } i = 1 \textbf{ to } n \\ \quad m[i,i] = 0\end{array}\right\}$ chains of length 1

chains of length 4

$i = 2$

$j = 5$

We are now computing $M[2,5]$

$k = 1$ $\quad M[2,2] + M[3,5] + p_1 p_2 p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$

$k = 2$ $\quad M[2,3] + M[4,5] + p_1 p_3 p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                         // chain length 1
    m[i, i] = 0
for l = 2 to n                         // l is the chain length
    for i = 1 to n − l + 1             // chain begins at A_i
        j = i + l − 1                  // chain ends at A_j
        m[i, j] = ∞
        for k = i to j − 1             // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q            // remember this cost
                s[i, j] = k            // remember this index
return m and s
```

chains of length 1

chains of length 4

$i = 2$

$j = 5$

We are now computing $M[2,5]$

$k = 1$ $\quad M[2,2] + M[3,5] + p_1 p_2 p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$

$k = 2$ $\quad M[2,3] + M[4,5] + p_1 p_3 p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$

$k = 3$

A Dynamic Programming algorithm

```
MATRIX-CHAIN-ORDER(p, n)
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                                // chain length 1
    m[i, i] = 0            } chains of length 1
for l = 2 to n    chains of length 4          // l is the chain length
    for i = 1 to n − l + 1                    // chain begins at A_i          i = 2
        j = i + l − 1                         // chain ends at A_j            j = 5
        m[i, j] = ∞                                  We are now computing M[2,5]
        for k = i to j − 1                    // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                   // remember this cost
                s[i, j] = k                   // remember this index
return m and s
```

$k = 1 \qquad M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$

$k = 2 \qquad M[2,3] + M[4,5] + p_1p_3p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$

$k = 3 \qquad M[2,4] + M[5,5] + p_1p_4p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375$

The table M

0	15750	7875	9375		
	0	2625	4375	7125	
		0	750	2500	
			0	1000	3500
				0	5000
					0

The table M

0	15750	7875	9375	11875	15125
	0	2625	4375	7125	10500
		0	750	2500	5375
			0	1000	3500
				0	5000
					0

The table M

0	15750	7875	9375	11875	15125
	0	2625	4375	7125	10500
		0	750	2500	5375
			0	1000	3500
				0	5000
					0

optimal cost

Computing the optimal solution

```
MATRIX-CHAIN-ORDER(p, n)
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                                    // chain length 1
    m[i, i] = 0          } chains of length 1
for l = 2 to n   chains of length 4               // l is the chain length
    for i = 1 to n − l + 1                        // chain begins at A_i        i = 2
        j = i + l − 1                             // chain ends at A_j          j = 5
        m[i, j] = ∞                                    We are now computing M[2,5]
        for k = i to j − 1                        // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                       // remember this cost
                s[i, j] = k                       // remember this index
return m and s
```

$k = 1$ $\quad M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$

$k = 2$ $\quad M[2,3] + M[4,5] + p_1p_3p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$

$k = 3$ $\quad M[2,4] + M[5,5] + p_1p_4p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375$

Computing the optimal solution

MATRIX-CHAIN-ORDER(p, n)

let $m[1:n, 1:n]$ and $s[1:n-1, 2:n]$ be new tables
  for $i = 1$ **to** n // chain length 1
    $m[i,i] = 0$
chains of length 1
for $l = 2$ **to** n chains of length 4 // l is the chain length
  for $i = 1$ **to** $n - l + 1$ // chain begins at A_i $i = 2$
    $j = i + l - 1$ // chain ends at A_j $j = 5$
    $m[i,j] = \infty$ We are now computing $M[2,5]$
    for $k = i$ **to** $j - 1$ // try $A_{i:k} A_{k+1:j}$
      $q = m[i,k] + m[k+1,j] + p_{i-1} p_k p_j$
      if $q < m[i,j]$
        $m[i,j] = q$ // remember this cost
        $s[i,j] = k$ // remember this index
return m and s

$k = 1$ $M[2,2] + M[3,5] + p_1 p_2 p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$

$k = 2$ $M[2,3] + M[4,5] + p_1 p_3 p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$ optimal split

$k = 3$ $M[2,4] + M[5,5] + p_1 p_4 p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375$

Computing the optimal solution

MATRIX-CHAIN-ORDER(p, n)

```
let m[1 : n, 1 : n] and s[1 : n − 1, 2 : n] be new tables
for i = 1 to n                                  // chain length 1
    m[i, i] = 0
for l = 2 to n                                  // l is the chain length
    for i = 1 to n − l + 1                      // chain begins at A_i
        j = i + l − 1                           // chain ends at A_j
        m[i, j] = ∞
        for k = i to j − 1                      // try A_{i:k} A_{k+1:j}
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q                     // remember this cost
                s[i, j] = k                     // remember this index
return m and s
```

chains of length 1

chains of length 4

$i = 2$

$j = 5$

We are now computing $M[2,5]$

$k = 1$ $\quad M[2,2] + M[3,5] + p_1 p_2 p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$

$k = 2$ $\quad M[2,3] + M[4,5] + p_1 p_3 p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$ optimal split

$k = 3$ $\quad M[2,4] + M[5,5] + p_1 p_4 p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375$

Computing the optimal solution

MATRIX-CHAIN-ORDER(p, n)

```
let m[1:n, 1:n] and s[1:n-1, 2:n] be new tables
for i = 1 to n                              // chain length 1
    m[i,i] = 0                              chains of length 1
for l = 2 to n        chains of length 4    // l is the chain length
    for i = 1 to n - l + 1                  // chain begins at A_i          i = 2
        j = i + l - 1                       // chain ends at A_j            j = 5
        m[i,j] = ∞                                  We are now computing M[2,5]
        for k = i to j - 1                  // try A_{i:k} A_{k+1:j}
            q = m[i,k] + m[k+1,j] + p_{i-1} p_k p_j
            if q < m[i,j]
                m[i,j] = q                  // remember this cost
                s[i,j] = k                  // remember this index    S[2,5] = 2
return m and s
```

$k = 1$ $\quad M[2,2] + M[3,5] + p_1 p_2 p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$

$k = 2$ $\quad M[2,3] + M[4,5] + p_1 p_3 p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$ optimal split

$k = 3$ $\quad M[2,4] + M[5,5] + p_1 p_4 p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375$

The table S

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

The table S

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$

The table S

$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$

The table S

$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$

$A_{4:6} = (A_{4:5}) \cdot (A_{6:6})$

The table S

$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$

$A_{1:6} = (A_{1:1} \cdot (A_{2:3})) \cdot ((A_{4:5}) \cdot A_{6:6})$

$A_{4:6} = (A_{4:5}) \cdot (A_{6:6})$

The table S

$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$

$A_{2:3} = (A_{2:2}) \cdot (A_{3:3})$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$

$A_{1:6} = (A_{1:1} \cdot (A_{2:3})) \cdot ((A_{4:5}) \cdot A_{6:6})$

$A_{4:6} = (A_{4:5}) \cdot (A_{6:6})$

The table S

$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$

$A_{2:3} = (A_{2:2}) \cdot (A_{3:3})$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$

$A_{1:6} = (A_{1:1} \cdot (A_{2:3})) \cdot ((A_{4:5}) \cdot A_{6:6})$

$A_{4:6} = (A_{4:5}) \cdot (A_{6:6})$

$A_{4:5} = (A_{4:4}) \cdot (A_{5:5})$

The table S

$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$

$A_{2:3} = (A_{2:2}) \cdot (A_{3:3})$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$

$A_{1:6} = (A_{1:1} \cdot (A_{2:3})) \cdot ((A_{4:5}) \cdot A_{6:6})$

$A_{4:6} = (A_{4:5}) \cdot (A_{6:6})$

$A_{4:5} = (A_{4:4}) \cdot (A_{5:5})$

$$A_{1:6} = (A_{1:1} \cdot (A_{2:2} \cdot A_{3:3})) \cdot ((A_{4:4} \cdot A_{5:5}) \cdot A_{6:6}) = (A_1 \cdot (A_2 \cdot A_3)) \cdot ((A_4 \cdot A_5) \cdot A_6)$$

Dynamic Programming vs Divide and Conquer

DP is an optimisation technique and is only applicable to problems with optimal substructure.

DP splits the problem into parts, finds solutions to the parts and joins them.

The parts are not significantly smaller and are overlapping.

In DP, the subproblem dependency can be represented by a DAG.

DQ is not normally used for optimisation problems.

DQ splits the problem into parts, finds solutions to the parts and joins them.

The parts are significantly smaller and do not normally overlap.

In DQ, the subproblem dependency can be represented by a tree.