Algorithms and Data Structures

Matrix Chain Multiplication (cont), Longest Common Subsequence

Dynamic Programming

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Solve the subproblems from the smallest to the largest. When you solve a subproblem, store the solution (e.g., in an array) and use it to solve the larger subproblems.

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For that we will use the *standard* algorithm for matrix multiplication:

```
RECTANGULAR-MATRIX-MULTIPLY(A, B, C, p, q, r)
1  for i = 1 to p
2      for j = 1 to r
3          for k = 1 to q
4              c_ij = c_ij + a_ik · b_kj
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The running time is the number of scalar multiplications, i.e., pqr.

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If instead we do $(A_1 \cdot (A_2 \cdot A_3))$ we have $100 \cdot 5 \cdot 50 = 25000$ scalar multiplications + $10 \cdot 100 \cdot 50 = 50000$ scalar multiplications for a total of 75000.

Full Parenthesisation

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Matrix Chain Multiplication problem: Given a chain $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices where for $i = 1,2,\ldots,n$, matrix A_i has dimension $p_{i-1} \times p_i$, find a full parenthesisation of the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$ that minimises the number of scalar multiplications.

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We can think of the input as a sequence of dimensions $\langle p_0, p_1, p_2, \ldots, p_n \rangle$.

A Dynamic Programming algorithm (bottom-up)

```
MATRIX-CHAIN-ORDER(p, n)
 1  let m[1:n, 1:n] and s[1:n − 1, 2:n] be new tables
 2  for i = 1 to n                          // chain length 1
 3      m[i, i] = 0
 4  for l = 2 to n                          // l is the chain length
 5      for i = 1 to n − l + 1              // chain begins at A_i
 6          j = i + l − 1                   // chain ends at A_j
 7          m[i, j] = ∞
 8          for k = i to j − 1              // try A_{i:k} A_{k+1:j}
 9              q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
10              if q < m[i, j]
11                  m[i, j] = q             // remember this cost
12                  s[i, j] = k             // remember this index
13  return m and s
```

Computing a solution

We have the following relation:

$$M[i,j] = \begin{cases} 0, & \text{if } i = j, \\ \min_{k \in [i,j)} \{M[i,k] + M[k+1,j] + p_{i-1} p_k p_j\} & \text{if } i < j\,. \end{cases}$$

We could now define a recursive algorithm straightforwardly using this.

```
RECURSIVE-MATRIX-CHAIN(p, i, j)
1  if i == j
2      return 0
3  m[i, j] = ∞
4  for k = i to j − 1
5      q = RECURSIVE-MATRIX-CHAIN(p, i, k)
           + RECURSIVE-MATRIX-CHAIN(p, k + 1, j)
           + p_{i−1} p_k p_j
6      if q < m[i, j]
7          m[i, j] = q
8  return m[i, j]
```

Memoization

We write the procedure recursively in a natural manner, but we modify it to save the result of each subproblem in a table.

The procedure first checks if we have computed the solution already in our table.

If we have, it uses it without calling itself recursively.

If we have not, then it recurses to compute it.

A Dynamic Programming algorithm (top-down)

```
MEMOIZED-MATRIX-CHAIN(p, n)
1  let m[1:n, 1:n] be a new table
2  for i = 1 to n
3       for j = i to n
4            m[i, j] = ∞
5  return LOOKUP-CHAIN(m, p, 1, n)
```

```
LOOKUP-CHAIN(m, p, i, j)
1  if m[i, j] < ∞
2       return m[i, j]
3  if i == j
4       m[i, j] = 0
5  else for k = i to j − 1
6            q = LOOKUP-CHAIN(m, p, i, k)
                 + LOOKUP-CHAIN(m, p, k + 1, j) + p_{i−1} p_k p_j
7            if q < m[i, j]
8                 m[i, j] = q
9  return m[i, j]
```

Longest Common Subsequence

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Motivation: A strand of DNA consists of a string of bases, with possible values Adenine, Cytosine, Guanine, and Thymine.

We can express a strand of DNA as a string, e.g.,

$S_1 =$ ACCGGTCGAGTGCGCGGAAGCCGGCCAA

$S_2 =$ GTCGTTCGGAATGCCGTTGCTCTGTAAA

We would like to compare two strands to see how similar they are (in order to see how similar two organisms are).

Various way to do that. Here: Find the longest possible strand S_3 which is a subsequence of both S_1 and S_2.

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$Z = \langle z_1, z_2, \ldots, z_k \rangle$ is a *subsequence* of $X = \langle x_1, x_2, \cdots, x_m \rangle$ if there exists a strictly increasing sequence $\langle i_1, i_2, \ldots, i_k \rangle$ of indices of X such that for all $j = 1,2,\ldots, k$ we have $x_{i_j} = z_j$.

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Example: $Z = \langle B, C, D, B \rangle$ is a subsequence of $X = \langle A,\ B, C, B, D, A, B \rangle$.

Brute Force

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Exponential time.

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Given a sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$, the ith *prefix* of X (for $i = 1,2,\ldots, m$) is defined as $X_i = \langle x_1, x_2, \ldots, x_i \rangle$.

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Example: $X = \langle A, B, C, B, D, A, B \rangle$, $X_4 = \langle A, B, C, B \rangle$ and $X_0 = \langle \rangle$.

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Theorem (Optimal Substructure):
Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be two sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y. Then.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$ and $z_k \neq x_m$, then Z is an LCS of X_{m-1} and Y.
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Assume by contradiction that there exists a longer common subsequence W of X_{m-1} and Y_{n-1} that has length at least k.

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If we append $x_m = y_n$ to W we obtain a common subsequence of X and Y of length $k + 1$ contradicting the fact that Z is a LCS.

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(2) Since $z_k \neq x_m$, Z is a common subsequence of X_{m-1} and Y. Assume by contradiction that it is no a LCS.

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Let W be a longer common subsequence, of length at least $k + 1$.

But W is a common subsequence of X and Y, contradicting the fact that Z is a LCS.

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  - Find a LCS of X and Y_{n-1}.
  - Choose the one that is longer.

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Otherwise we consider the two cases of the previous slide:

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$C[i,0] = C[0,j]$ for all i, j.

Otherwise we consider the two cases of the previous slide:

- If $x_i = x_j$, we have $C[i,j] = C[i-1,j-1] + 1$

Constructing a recursion

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$$C[i, j] = \begin{cases} 0, & \textbf{if } i = 0 \textbf{ or } j = 0, \\ C[i - 1,j - 1] + 1, & \textbf{if } i, j > 0 \textbf{ and } x_i = y_j, \\ \max\{C[i, j - 1] \,.\, C[i - 1,j]\} & \textbf{if } i, j > 0 \textbf{ and } x_i \neq y_j \,. \end{cases}$$

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 8  else PRINT-LCS(b, X, i, j − 1)
```

0	0	0	0
0	★	★	★
0	★	★	★
0			
0			

A Dynamic Programming Algorithm

```
LCS-LENGTH(X, Y, m, n)
 1  let b[1:m, 1:n] and c[0:m, 0:n] be new tables
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 3      c[i, 0] = 0
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 5      c[0, j] = 0
 6  for i = 1 to m          // compute table entries in row-major order
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 8          if x_i == y_j
 9              c[i, j] = c[i − 1, j − 1] + 1
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 3  if b[i, j] == "↖"
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 5      print x_i           // same as y_j
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 7      PRINT-LCS(b, X, i − 1, j)
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0	0	0	0
0	★	★	★
0	★	★	★
0			
0			

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0	0	0	0
0	★	★	★
0	★	★	★
0			
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```

0	0	0	0
0 ←	★	★	★
0	★	★	★
0			
0			

A Dynamic Programming Algorithm

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```

0	0	0	0
0	★	★	★
0	★	★	★
0			
0			

Example

```
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16  return c and b
```

		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0						
B	0						
C	0						
B	0						
D	0						
A	0						
B	0						

Example

```
LCS-LENGTH(X, Y, m, n)
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```

		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0						
B	0						
C	0						
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A	0						
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Example

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		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0						
B	0						
C	0						
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	0	0	0	0	0	0	0
A	0						
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	0	0	0	0	0	0	0
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A	0	0	0				
B	0						
C	0						
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D	0						
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		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0				
B	0						
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```

		B	*D*	*C*	*A*	*B*	*A*
	0	0	0	0	0	0	0
A	0	0	0				
B	0						
C	0						
B	0						
D	0						
A	0						
B	0						

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		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0			
B	0						
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 1  let b[1:m, 1:n] and c[0:m, 0:n] be new tables
 2  for i = 1 to m
 3      c[i, 0] = 0
 4  for j = 0 to n
 5      c[0, j] = 0
 6  for i = 1 to m          // compute table entries in row-major order
 7      for j = 1 to n
 8          if x_i == y_j                Is x_1 = y_4?
 9              c[i, j] = c[i − 1, j − 1] + 1
10              b[i, j] = "↖"
11          elseif c[i − 1, j] ≥ c[i, j − 1]
12              c[i, j] = c[i − 1, j]
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16  return c and b
```

		B	*D*	*C*	*A*	*B*	*A*
	0	0	0	0	0	0	0
A	0	0	0	0			
B	0						
C	0						
B	0						
D	0						
A	0						
B	0						

$C[0,1] = 0$ $C[0,2] = 0$

$C[1,0] = 0$ $C[1,1] = 0$

Example

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LCS-LENGTH(X, Y, m, n)
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```

		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0			
B	0						
C	0						
B	0						
D	0						
A	0						
B	0						

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		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0	1		
B	0						
C	0						
B	0						
D	0						
A	0						
B	0						

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		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0	1		
B	0						
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B	0						
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A	0						
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	0	0	0	0	0	0	0
A	0	0	0	0	1		
B	0						
C	0						
B	0						
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```

		B	*D*	*C*	*A*	*B*	*A*
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	
B	0						
C	0						
B	0						
D	0						
A	0						
B	0						

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```

		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0						
C	0						
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```

		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1				
B	0						
D	0						
A	0						
B	0						

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```

		B	*D*	*C*	*A*	*B*	*A*
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1				
B	0						
D	0						
A	0						
B	0						

Next to compute: $C[3,3]$

Example

```
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```

		B	*D*	*C*	*A*	*B*	*A*
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1				
B	0						
D	0						
A	0						
B	0						

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Example

```
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Is $x_3 = y_3$?

		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1				
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		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1				
B	0						
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		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
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Example

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16  return c and b
```

		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1	2	2	2	2
B	0	1	1	2	2	3	3
D	0	1	2	2	2	3	3
A	0	1	2	2	3	3	4
B	0	1	2	2	3	4	4

Example

```
PRINT-LCS(b, X, i, j)
1  if i == 0 or j == 0
2       return              // the LCS has length 0
3  if b[i, j] == "↖"
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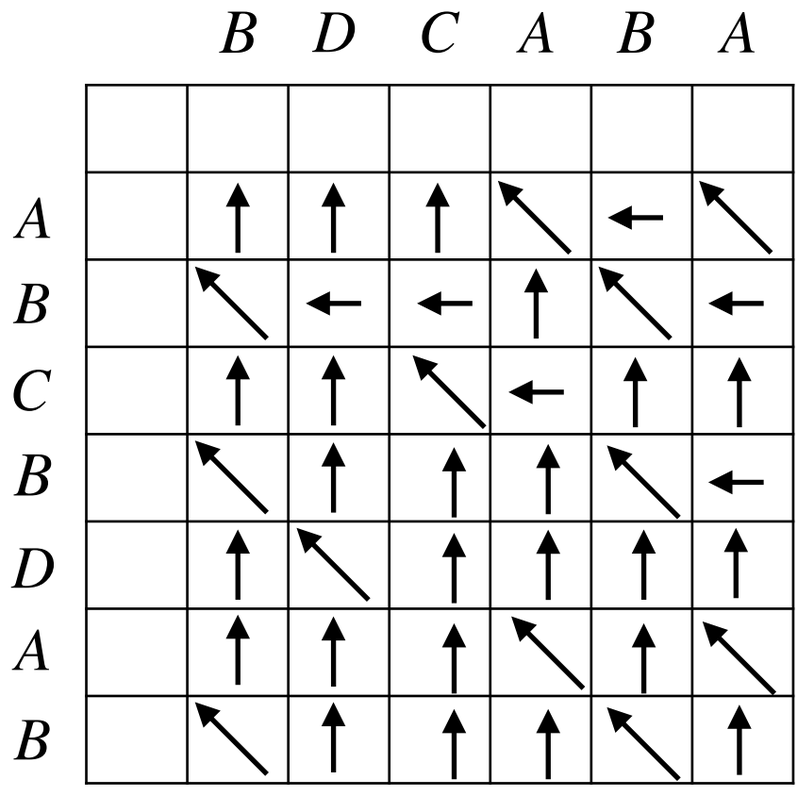
5       print x_i           // same as y_j
6  elseif b[i, j] == "↑"
7       PRINT-LCS(b, X, i − 1, j)
8  else PRINT-LCS(b, X, i, j − 1)
```



Example

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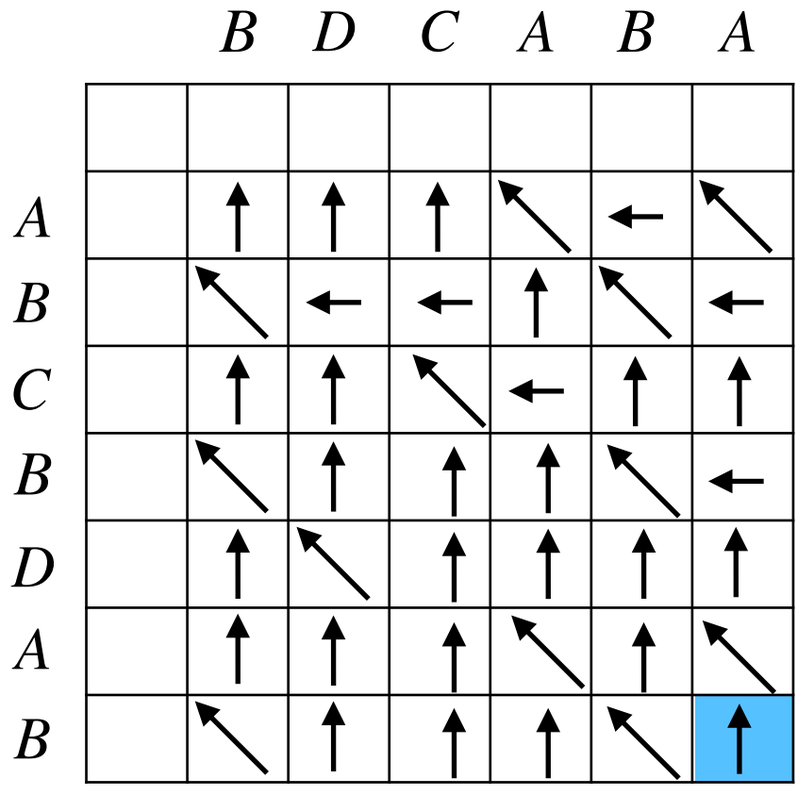

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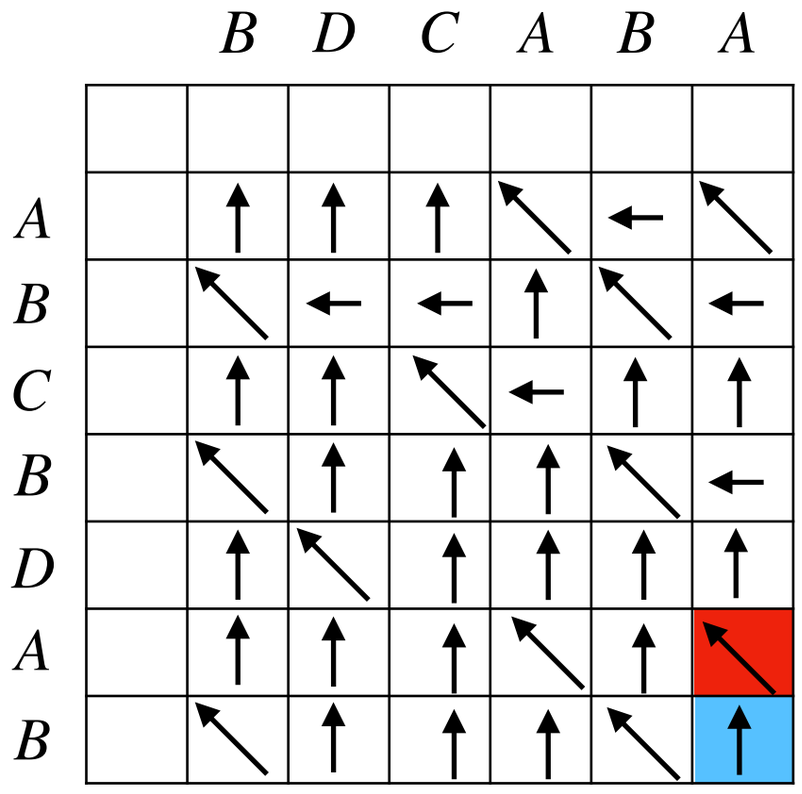

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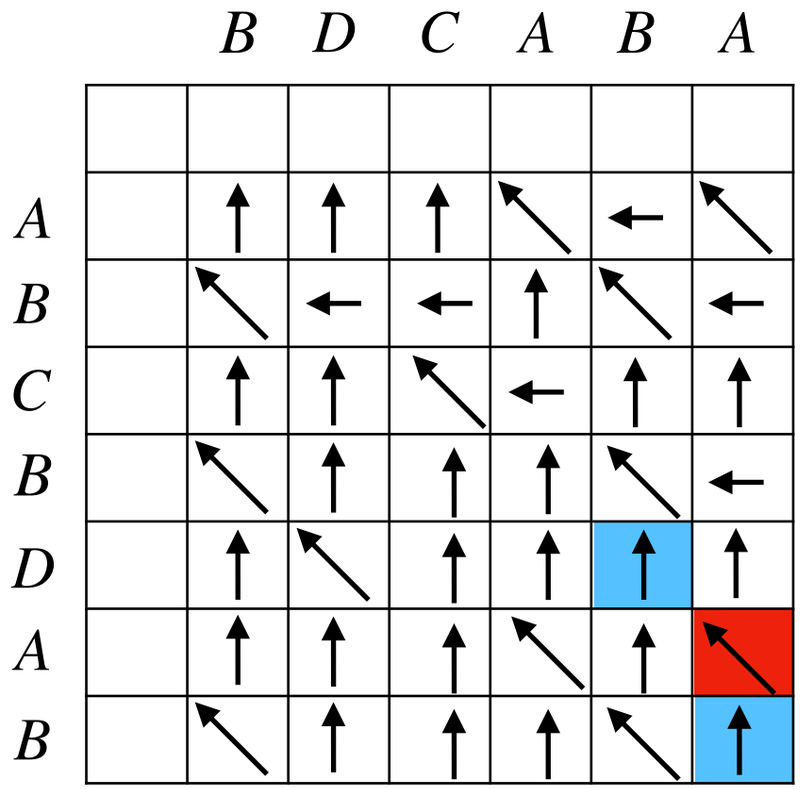

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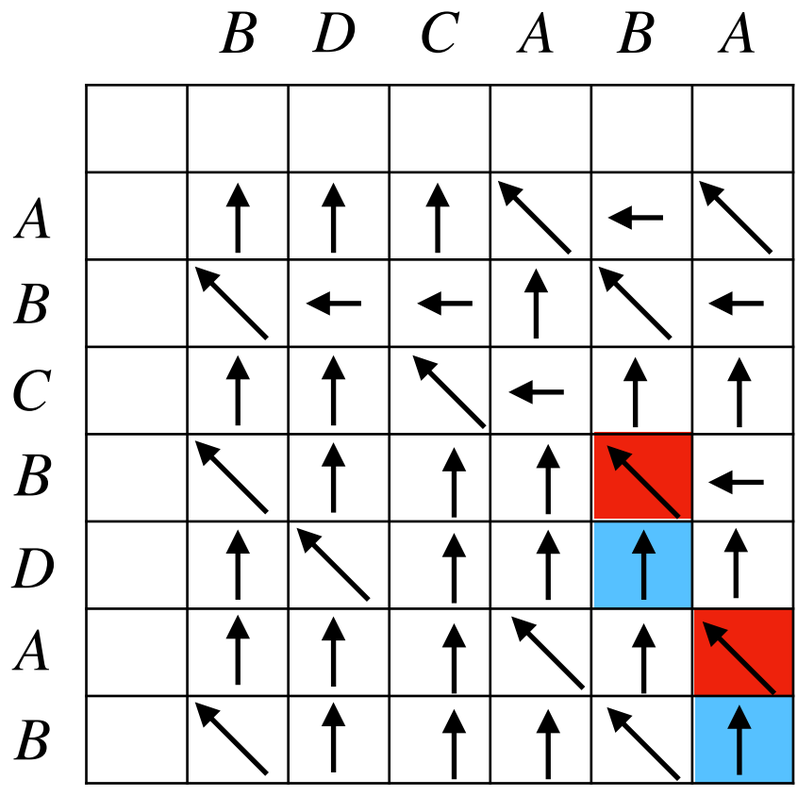

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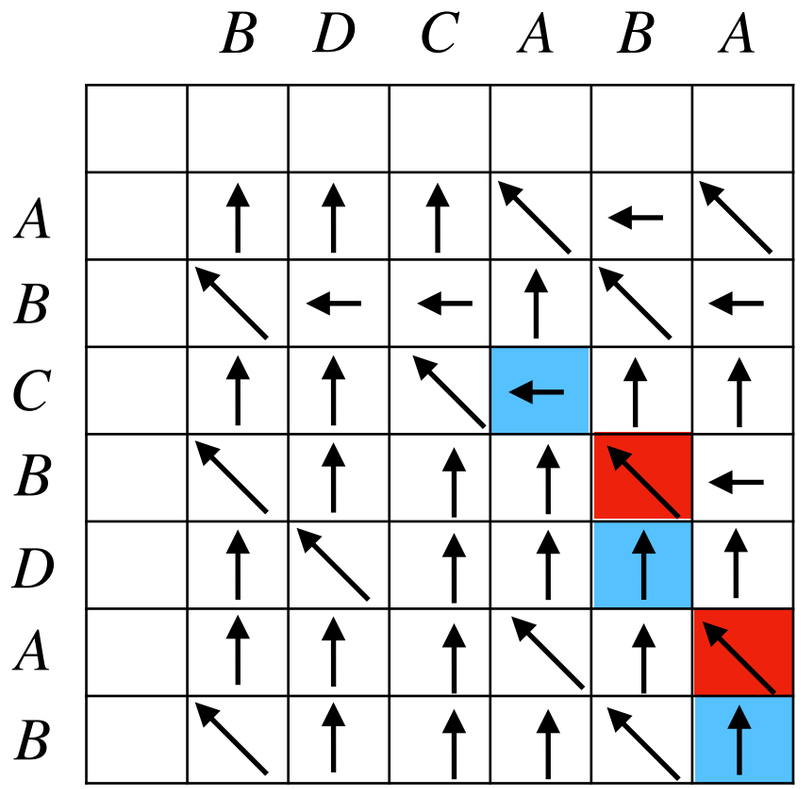

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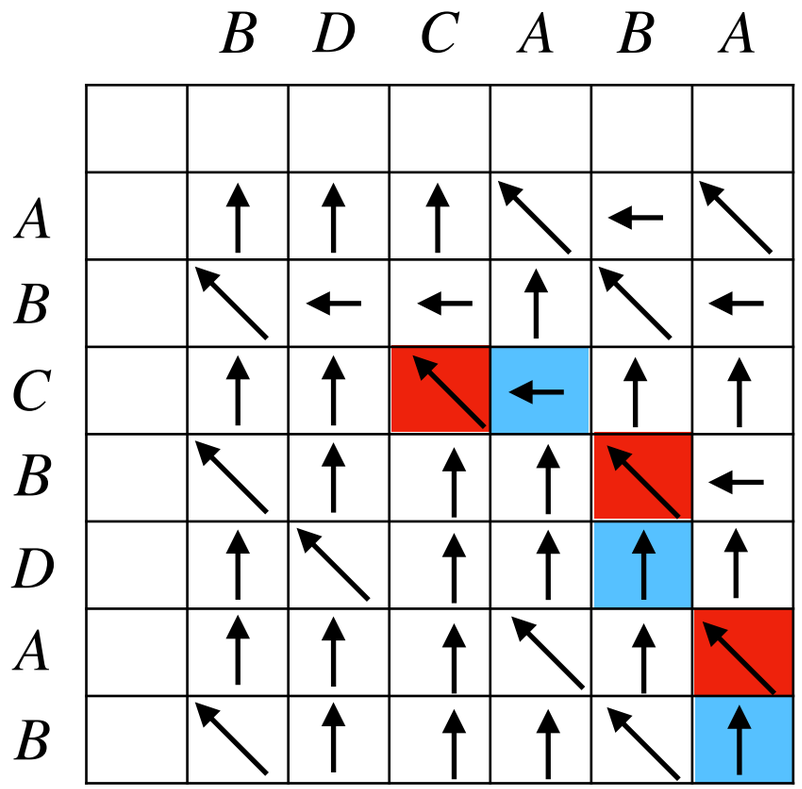

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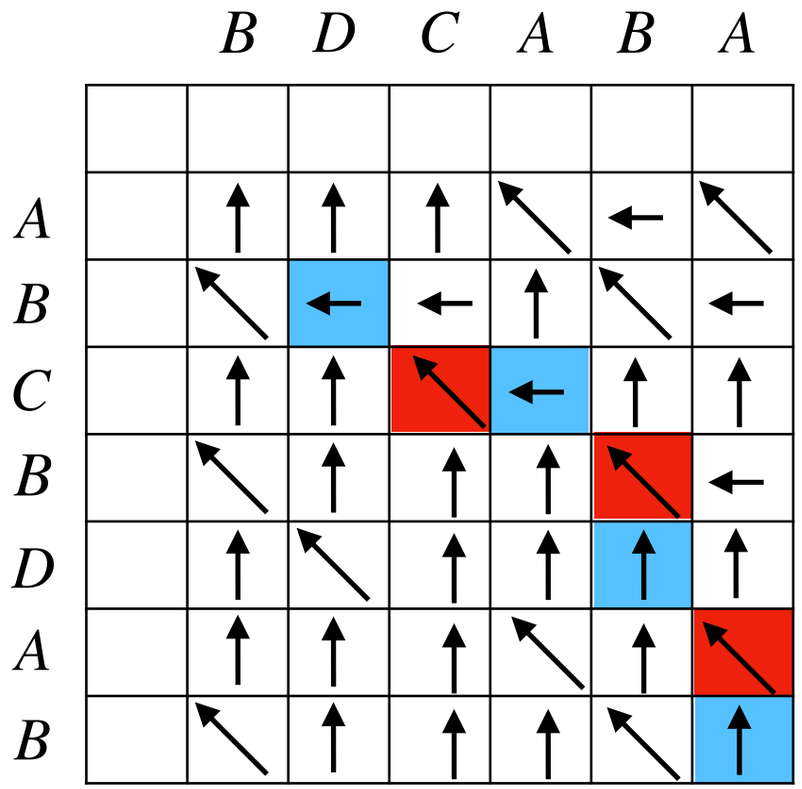

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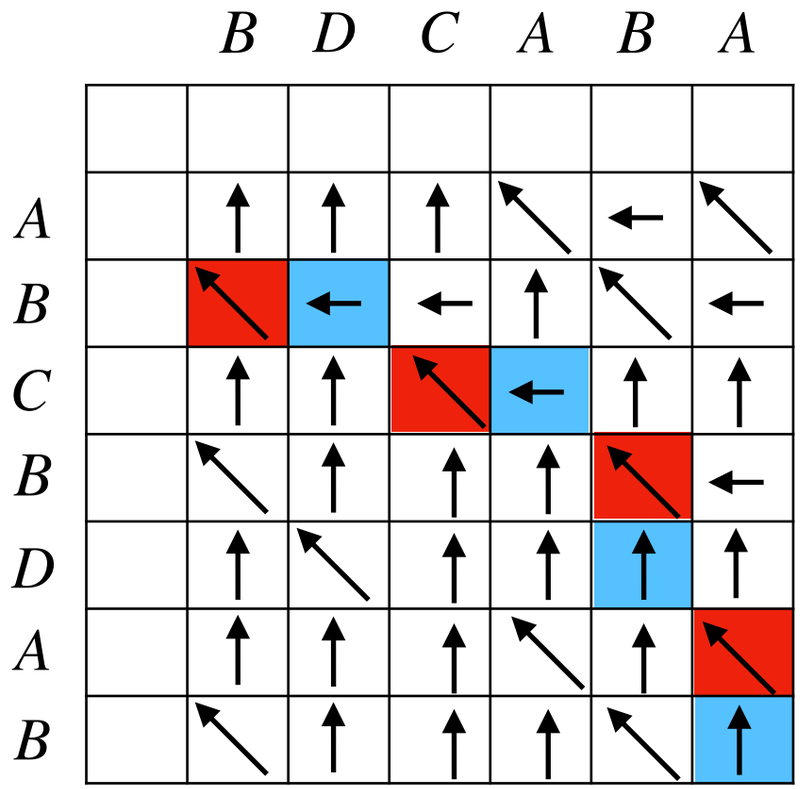

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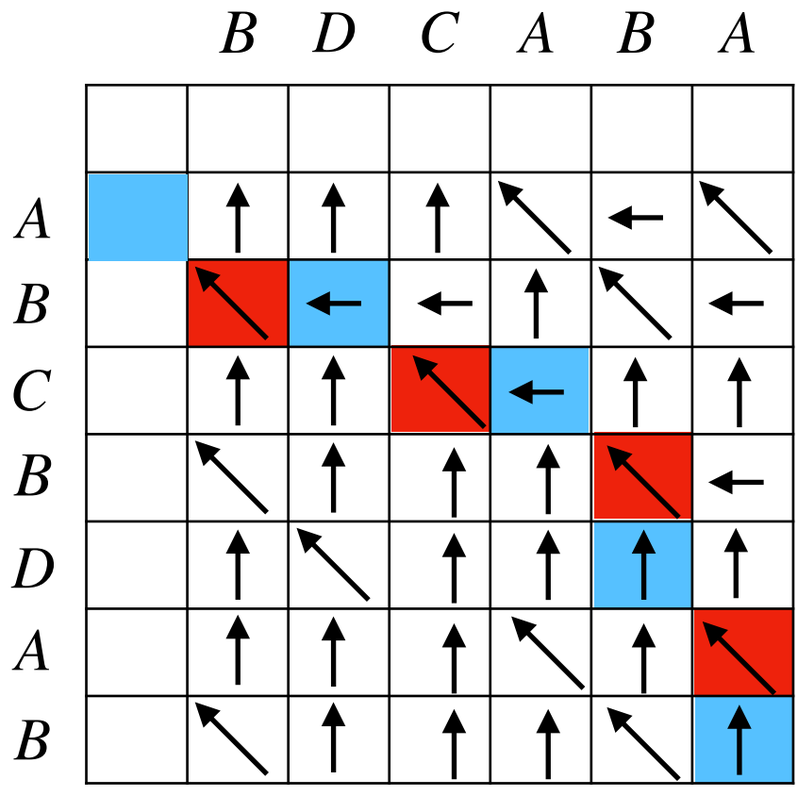

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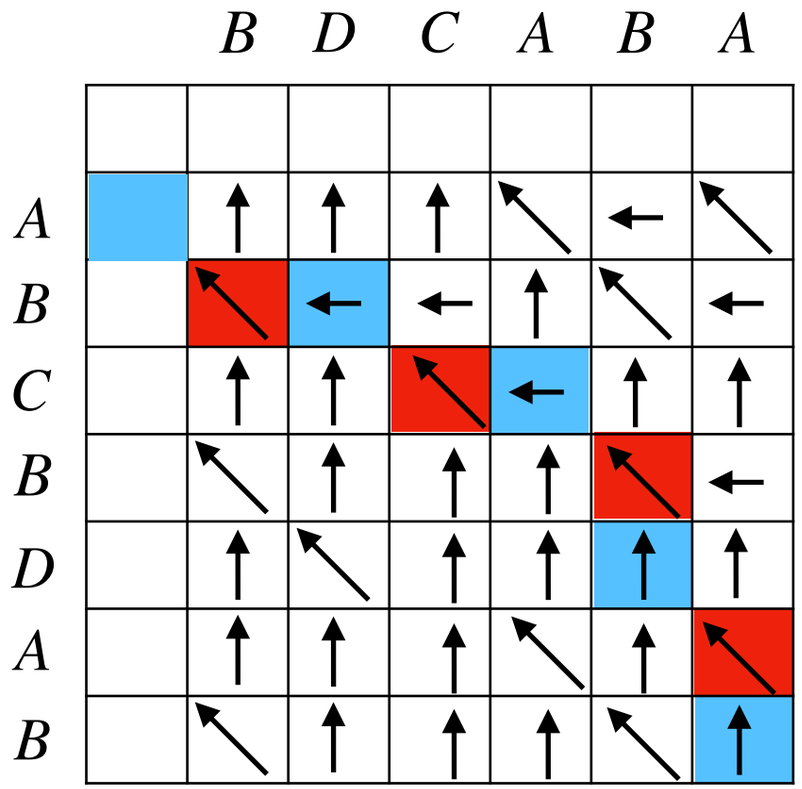

A

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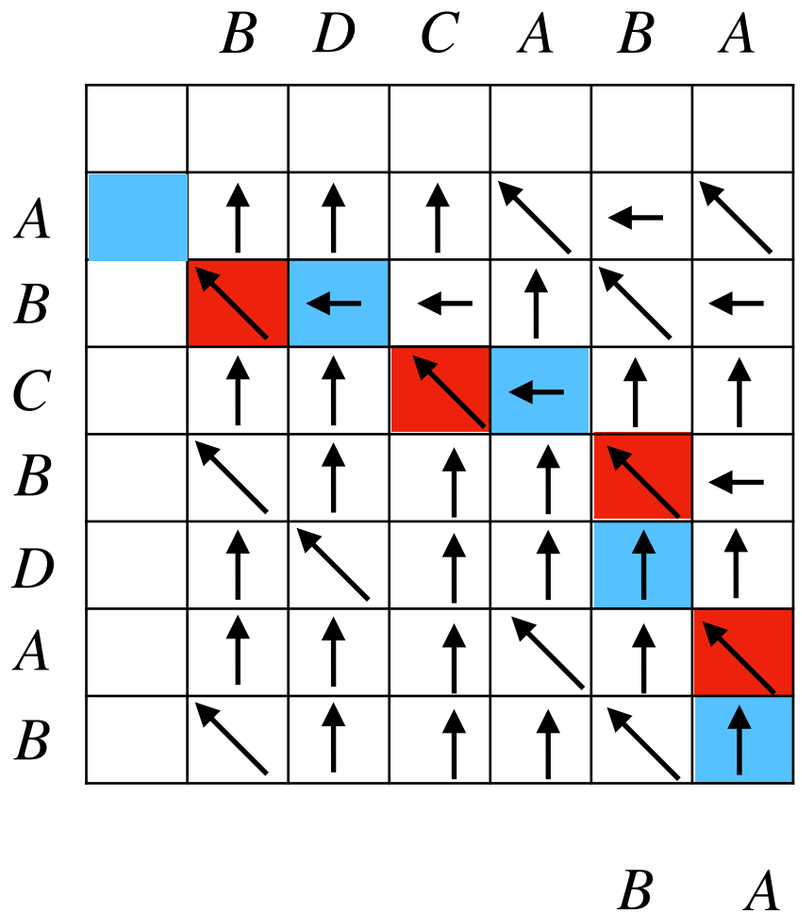

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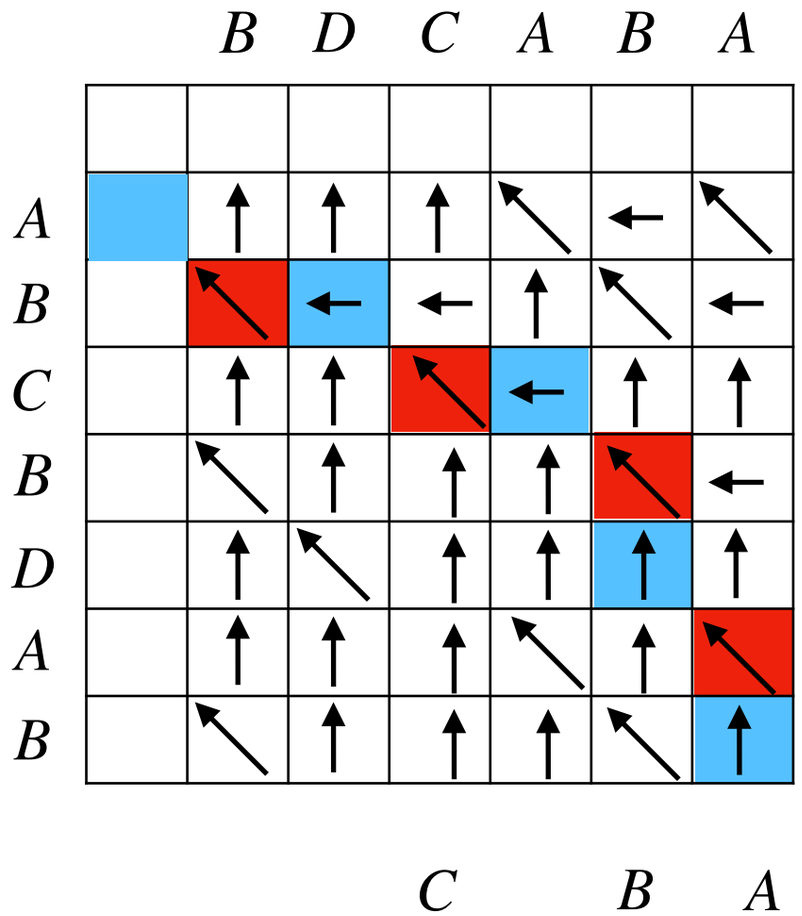

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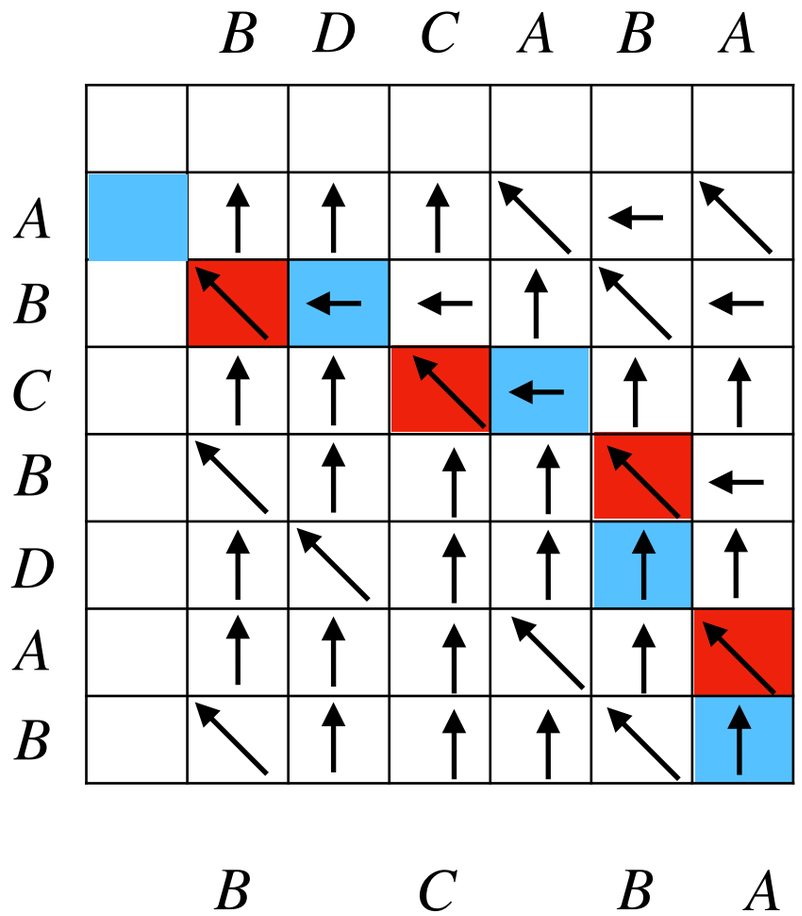

A Dynamic Programming Algorithm

```
LCS-LENGTH(X, Y, m, n)
 1  let b[1:m, 1:n] and c[0:m, 0:n] be new tables
 2  for i = 1 to m
 3      c[i, 0] = 0
 4  for j = 0 to n
 5      c[0, j] = 0
 6  for i = 1 to m          // compute table entries in row-major order
 7      for j = 1 to n
 8          if x_i == y_j
 9              c[i, j] = c[i − 1, j − 1] + 1
10              b[i, j] = "↖"
11          elseif c[i − 1, j] ≥ c[i, j − 1]
12              c[i, j] = c[i − 1, j]
13              b[i, j] = "↑"
14          else c[i, j] = c[i, j − 1]
15              b[i, j] = "←"
16  return c and b

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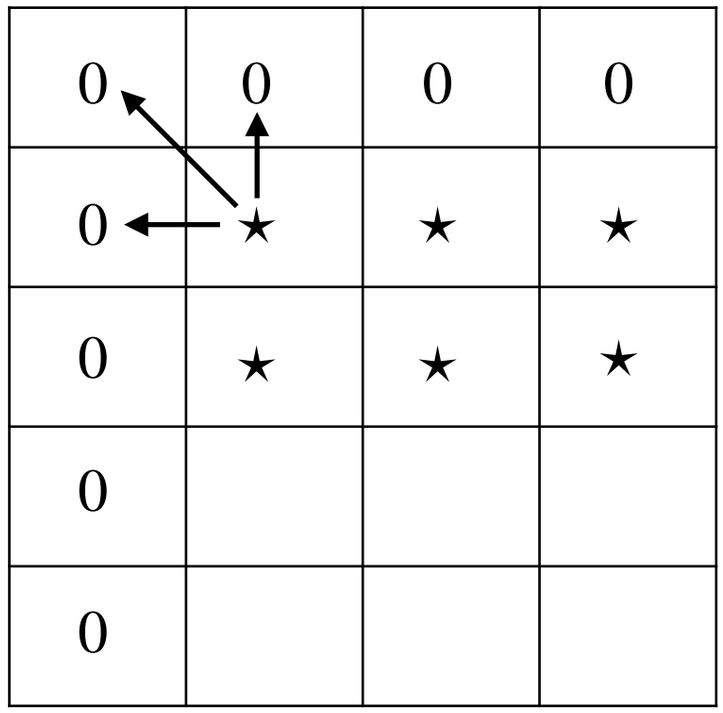

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do we need this?

0	0	0	0
0	★	★	★
0	★	★	★
0			
0			

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can we use less space here?

0	0	0	0
0	★	★	★
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0			
0			

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