ADS Tutorial 6 Solutions

Instructor: Aris Filos-Ratsikas TA: Kat Molinet

November 11, 2024

Problem 1

Solve the following linear program using the simplex method.

$$\begin{array}{ll} \text{maximise} & 6x_1 + 6x_2 + 5x_3 + 9x_4 \\ \\ \text{subject to} & 2x_1 + x_2 + x_3 + 3x_4 \leq 5 \\ & x_1 + 3x_2 + x_3 + 2x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Show each dictionary and each basic feasible solution produced during the execution of the algorithm. Explain which variable is the entering variable and which one is the leaving variable and why.

Solution

First, we reformulate the linear program by defining slack variables w_1 and w_2 as below and constraining their values to be non-negative:

$$\begin{aligned} w_1 &= 5 - 2x_1 - x_2 - x_3 - 3x_4 \\ w_2 &= 3 - x_1 - 3x_2 - x_3 - 2x_4, \end{aligned}$$

giving us the following dictionary:

$$\begin{array}{rcccccc} \zeta = & & +6x_1 & +6x_2 & +5x_3 & +9x_4 \\ \hline w_1 = & 5 & -2x_1 & -x_2 & -x_3 & -3x_4 \\ w_2 = & 3 & -x_1 & -3x_2 & -x_3 & -2x_4 \end{array}$$

$$x_1,\, x_2,\, x_3,\, x_4,\, w_1,\, w_2 \geq 0$$

We select the origin as our initial feasible solution, giving us

$$x_1 = 0, \;\; x_2 = 0, \;\; x_3 = 0, \;\; x_4 = 0, \;\; w_1 = 5, \;\; w_2 = 3, \;\; \zeta = 0.$$

Our first observation is that, since the coefficients of ζ are all positive, we can increase our objective function by increasing any of x_1, x_2, x_3, x_4. Suppose we choose to increase x_4; i.e., we make x_4 our *entering variable*. By how much should we increase x_4? As much as we can, without violating any of our constraints. From our dictionary, we see that only constraints which might apply are w_1 and w_2. In particular, increasing x_4 by $5/3$ would make constraint w_1 go tight; but even before we hit this constraint, we would first hit constraint w_2 at $x_4 = 3/2$. So we set $x_4 = 3/2$ and define w_2 as the *leaving variable* of this iteration. Our new, improved solution is thus

$$x_1 = 0, \;\; x_2 = 0, \;\; x_3 = 0, \;\; x_4 = \frac{3}{2}, \;\; w_1 = \frac{1}{2}, \;\; w_2 = 0, \;\; \zeta = \frac{27}{2}.$$

Now we wish to reformulate our LP so that ζ and the constraints are written in terms of x_1, x_2, x_3 and w_2, rather than x_1, x_2, x_3, x_4. For this, we note that the equation for w_2 above implies that

$$x_4 = \frac{1}{2}(3 - w_2 - x_1 - 3x_2 - x_3).$$

Making this substitution into our original LP dictionary above gives us the following updated dictionary:

$$\begin{array}{rrrrrr}
\zeta = & \frac{27}{2} & - \frac{9}{2}w_2 & + \frac{3}{2}x_1 & - \frac{15}{2}x_2 & + \frac{1}{2}x_3 \\
\hline
w_1 = & \frac{1}{2} & + \frac{3}{2}w_2 & - \frac{1}{2}x_1 & + \frac{7}{2}x_2 & + \frac{1}{2}x_3 \\
x_4 = & \frac{3}{2} & - \frac{1}{2}w_2 & - \frac{1}{2}x_1 & - \frac{3}{2}x_2 & - \frac{1}{2}x_3 \\
\end{array}$$
$$x_1,\, x_2,\, x_3,\, x_4,\, w_1,\, w_2 \geq 0$$

We continue on in a similar fashion. From our revised LP, we see that we can increase the value of our objective function ζ by increasing x_1 or x_3. Suppose we choose x_1 as our entering variable. Looking at constraints w_1 and x_4, we see that increasing x_1 will hit constraint w_1 (our new leaving variable) first at $x_1 = 1$, since $\min\{1, 3\} = 1$. This gives the following improved solution:

$$x_1 = 1,\;\; x_2 = 0,\;\; x_3 = 0,\;\; x_4 = 1,\;\; w_1 = 0,\;\; w_2 = 0,\;\; \zeta = 15.$$

Rewriting ζ and the constraints in terms of w_1, w_2, x_2, x_3 using the fact that $x_1 = 1 - 2w_1 + 3w_2 + 7x_2 + x_3$ (from w_1 in the previous dictionary) gives the following updated dictionary:

$$\begin{array}{rrrrrr}
\zeta = & 15 & - 3w_1 & & + 3x_2 & + 2x_3 \\
\hline
x_1 = & 1 & - 2w_1 & + 3w_2 & + 7x_2 & + x_3 \\
x_4 = & 1 & + w_1 & - 2w_2 & - 5x_2 & - x_3 \\
\end{array}$$
$$x_1,\, x_2,\, x_3,\, x_4,\, w_1,\, w_2 \geq 0$$

From here, we see that we have two possible entering variables: x_2 and x_3. Suppose we choose x_3 and increase it until we hit constraint x_1 or x_4. Since we see from the equation for x_1 that this constraint is infeasible (i.e., there is no positive value of x_3 which makes $x_1 = 0$), we know that the leaving vairable must be x_4, with the constraint going tight at $x_3 = 1$. Using the fact that $x_3 = 1 + w_1 - 2w_2 - 5x_2 - x_4$, we rewrite the LP dictionary as follows:

$$\begin{array}{rrrrrr}
\zeta = & 17 & - w_1 & - 4w_2 & - 7x_2 & - 2x_4 \\
\hline
x_1 = & 2 & - w_1 & + w_2 & + 2x_2 & - x_4 \\
x_3 = & 1 & + w_1 & - 2w_2 & - 5x_2 & - x_4 \\
\end{array}$$
$$x_1,\, x_2,\, x_3,\, x_4,\, w_1,\, w_2 \geq 0$$

Since our objective function ζ has all-negative coefficients, increasing any of w_1, w_2, x_2, x_4 can only decrease its value. Thus, we have maximized our solution at $(x_1, x_2, x_3, x_4) = (2, 0, 1, 0)$ and $(w_1, w_2) = (0, 0)$, with objective value $\zeta = 17$.

Problem 2

Consider the following linear program.

$$\begin{array}{ll}
\text{maximise} & 2x_1 + x_2 \\
 & \\
\text{subject to} & 2x_1 + x_2 \leq 4 \\
 & 2x_1 + 3x_2 \leq 3 \\
 & 4x_1 + x_2 \leq 5 \\
 & x_1 + 5x_2 \leq 1 \\
 & x_1, x_2 \geq 0
\end{array}$$

A. Solve the LP above using the simplex method. Show each dictionary and each basic feasible solution produced during the execution of the algorithm. Explain which variable is the entering variable and which one is the leaving variable and why.

B. Solve the LP above by drawing the feasible region in two dimensions and checking the objective function value on each of its corners.

Solution

A. Original dictionary:

$$\begin{array}{rrrr}
\zeta = & & +2x_1 & +x_2 \\
\hline
w_1 = & 4 & -2x_1 & -x_2 \\
w_2 = & 3 & -2x_1 & -3x_2 \\
w_3 = & 5 & -4x_1 & -x_2 \\
w_4 = & 1 & -x_1 & -5x_2
\end{array}$$

$$x_1,\, x_2,\, w_1,\, w_2,\, w_3,\, w_4, \geq 0$$

Initial solution:

$$x_1 = 0,\;\; x_2 = 0,\;\; w_1 = 4,\;\; w_2 = 3,\;\; w_3 = 5,\;\; w_4 = 1,\;\; \zeta = 0.$$

Possible entering variables: x_1, x_2. Suppose we select x_1. Then the corresponding leaving variable would be w_4, since $\min\{2, 3/2, 5/4, 1\} = 1$. Updated solution:

$$x_1 = 1,\;\; x_2 = 0,\;\; w_1 = 4,\;\; w_2 = 3,\;\; w_3 = 5,\;\; w_4 = 0,\;\; \zeta = 2.$$

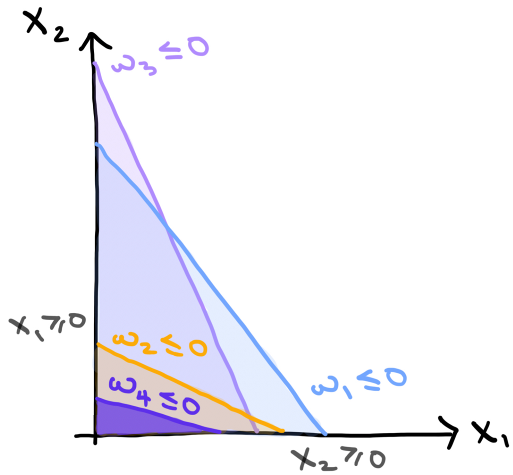
Rewriting LP dictionary using the fact that $x_1 = 1 - w_4 - 5x_2$:

$$\begin{array}{rrrr}
\zeta = & 2 & -2w_4 & -9x_2 \\
\hline
w_1 = & 2 & +2w_4 & +9x_2 \\
w_2 = & 1 & +2w_4 & +7x_2 \\
w_3 = & 1 & +4w_4 & +19x_2 \\
x_1 = & 1 & -w_4 & -5x_2
\end{array}$$

$$x_1,\, x_2,\, w_1,\, w_2,\, w_3,\, w_4, \geq 0$$

Since both coefficients of ζ are negative, we know that our solution, $(x_1, x_2) = (1, 0)$ and $\zeta = 2$, must be maximal.

B. Since this optimization problem is a linear program, we know that its solution lies at one of the vertices of the feasible region (assuming the feasible region is bounded). Graphing each of the constraints, we find that the feasible region lies at the intersection of the constraints $x_1 \geq 0$, $x_2 \geq 0$, and $x_1 + 5x_2 \leq 1$:



The feasible region is the dark purple triangle in the bottom left. The vertices of this feasible region are $(0,0)$, $(0,1/5)$ and $(1,0)$; and value of our objective function ζ at each of these vertices is as follows:

$$\zeta(0,0) = 0, \ \ \zeta(0,1/5) = 1/5, \ \ \zeta(1,0) = 2.$$

Thus, we confirm what we showed in part A: that the solution $x_1 = 1$, $x_2 = 0$ maximizes ζ over the feasible region, and that $\zeta(1,0) = 2$.

Problem 3

Solve the following linear program using the simplex method.

$$\begin{array}{ll} \text{maximise} & 2x_1 - 6x_2 \\ \\ \text{subject to} & -x_1 - x_2 - x_3 \leq -2 \\ & 2x_1 - x_2 + x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Show each dictionary and each basic feasible solution produced during the execution of the algorithm. Explain which variable is the entering variable and which one is the leaving variable and why.

Solution

First, we note that, unlike in the previous exercises, the right-hand sides of our constraints are *not* all non-negative, and so initializing our dictionary as we've done previously gives us a dictionary which is infeasible. So instead, we follow the process outlined in section 2.3 of Vanderbei to initialize our dictionary.

We start by defining the following auxiliary problem:

$$\begin{array}{ll} \text{maximise} & -x_0 \\ \\ \text{subject to} & -x_0 - x_1 - x_2 - x_3 \leq -2 \\ & -x_0 + 2x_1 - x_2 + x_3 \leq 1 \\ & x_0, x_1, x_2, x_3 \geq 0 \end{array}$$

The initialized dictionary of this auxiliary problem is as follows:

$$\begin{array}{rrrrrr} \eta = & & -\,x_0 & & & \\ \hline w_1 = & -\,2 & +\,x_0 & +\,x_1 & +\,x_2 & +\,x_3 \\ w_2 = & 1 & +\,x_0 & -\,2x_1 & +\,x_2 & -\,x_3 \end{array}$$

$$x_0, x_1, x_2, x_3, w_1, w_2 \geq 0$$

and has initial solution

$$x_0 = 0, \;\; x_1 = 0, \;\; x_2 = 0, \;\; x_3 = 0, \;\; w_1 = -2, \;\; w_2 = 1, \;\; \eta = 0.$$

To convert this into a feasible dictionary, we pivot with x_0 as the entering variable and the "most infeasible variable" w_1 as the leaving variable to get the updated dictionary

$$\begin{array}{rrrrrr} \eta = & -2 & -w_1 & +x_1 & +x_2 & +x_3 \\ \hline w_2 = & 3 & +w_1 & -3x_1 & & -2x_3 \\ x_0 = & 2 & +w_1 & -x_1 & -x_2 & -x_3 \end{array}$$
$$x_0, x_1, x_2, x_3, w_1, w_2 \geq 0$$

and solution

$$x_0 = 2, \;\; x_1 = 0, \;\; x_2 = 0, \;\; x_3 = 0, \;\; w_1 = 0, \;\; w_2 = 3, \;\; \eta = -2.$$

Now we proceed as usual, applying the simplex method. Our possible entering variables are x_1, x_2, x_3. Suppose we choose x_2, which hits its first constraint, x_0 at $x_2 = 2$. So we let x_0 be the leaving variable and set $x_2 = 2$. Note that under this assignment, $\eta = 0$; i.e., η is optimal. Thus, we now drop x_0 from the LP and reintroduce our original objective function ζ in terms of w_1, x_1, x_3, using the equation for x_2 from $\eta = 0$ in the previous dictionary:

$$\zeta = 2x_1 - 6x_2 = 2x_1 - 6(2 + w_1 - x_1 - x_3) = -12 - 6w_1 + 8x_1 + 6x_3,$$

giving us the following updated dictionary:

$$\begin{array}{rrrrr} \zeta = & -12 & -6w_1 & +8x_1 & +6x_3 \\ \hline w_2 = & 3 & +w_1 & -3x_1 & -2x_3 \\ x_2 = & 2 & +w_1 & -x_1 & -x_3 \end{array}$$
$$x_0, x_1, x_2, x_3, w_1, w_2 \geq 0$$

and solution

$$x_0 = 0, \;\; x_1 = 0, \;\; x_2 = 2, \;\; x_3 = 0, \;\; w_1 = 0, \;\; w_2 = 3, \;\; \zeta = -12.$$

From here, we see that ζ is not yet maximal (since it still has terms with positive coefficients), and that x_1 and x_3 are our possible entering variables. Suppose we choose x_3. Then w_2 would be our corresponding leaving variable, since $\min\{3/2.2\} = 3/2$ (meaning that we hit constraint w_2 first when increasing x_3), and we let $x_3 = 3/2$. Our updated dictionary is below:

$$\begin{array}{rrrrr} \zeta = & -3 & -3w_1 & -3w_2 & -x_1 \\ \hline x_2 = & \frac{1}{2} & +\frac{1}{2}w_1 & +\frac{1}{2}w_2 & +\frac{1}{2}x_1 \\ x_3 = & \frac{3}{2} & +\frac{1}{2}w_1 & -\frac{1}{2}w_2 & -\frac{3}{2}x_1 \end{array}$$
$$x_0, x_1, x_2, x_3, w_1, w_2 \geq 0$$

Since all the coefficients of ζ are negative, we know we cannot do any better than our current solution

$$x_0 = 0, \;\; x_1 = 0, \;\; x_2 = \frac{1}{2}, \;\; x_3 = \frac{3}{2}, \;\; w_1 = 0, \;\; w_2 = 0, \;\; \zeta = -3.$$