ADS Tutorial 7

Instructor: Aris Filos-Ratsikas TA: Kat Molinet

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Problem 1

A cargo plane has three compartments for storing cargo: front, centre, and rear. These comparments have the following limits on both weight and space: the front can fit up to 10 tons and up to 6800 cubic metres of cargo, the centre can fit up to 16 tons and 8700 cubic metres of cargo, and the rear can fit up to 8 tons and 5300 cubic metres of cargo.

Additionally, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the plane. For example, if 8 tons are put in the centre compartment, 5 tons must be put in the front compartment, since $8/16 = 5/10$.

The following four cargoes are available for shippment on the next flight: Any proportion of these cargoes

Cargo	Weight (tons)	Volume (cubic meters/ton)	Profit (GBP/ton)
C_1	18	480	310
C_2	15	650	380
C_3	23	580	350
C_4	12	390	28

can be accepted. The objective is to determine how much (if any) of each cargo C_1, C_2, C_3, and C_4 should be accepted and how to distribute each among the compartments of the plane so that the total profit for the flight is maximised.

Formulate the problem above as a linear program.

Solution

We will use variables for each cargo and each compartment of the plane. In particlar, let x_{ij} denote the amount of cargo C_i, for $i \in \{1, 2, 3, 4\}$, that we allocate to compartment j of the plane, where $j \in \{f, c, r\}$.

We first figure out the objective function. The goal is to maximise the profit. The total amount of units from cargo C_i is $x_{if} + x_{ic} + x_{ir}$, and we multiply that by the corresponding profit of each cargo. Therefore, we obtain the following objective function:

$$\text{Maximise } 310(x_{1f} + x_{1c} + x_{1r}) + 380(x_{2f} + x_{2c} + x_{2r}) + 350(x_{3f} + x_{3c} + x_{3r}) + 28(x_{4f} + x_{4c} + x_{4r}).$$

We now write the constraints. First of all, all the variables have to be non-negative, as we cannot allocate negative amounts of cargo, i.e., $x_{ij} \geq 0$ for all $i \in \{1, 2, 3, 4\}$ and $j \in \{f, c, r\}$. Then we write the capacity constraints. For the weight, we have that the total amount of cargo in each compartment is at most the specified weight for that comparment. For example, for the front we have that

$$18x_{1f} + 15x_{2f} + 23x_{3f} + 12x_{4f} \leq 10.$$

The constraints for the centre and the rear are similar. For the capacity, we have a similar constraint, e.g., for the front:

$$480 \cdot 18 \cdot x_{1f} + 650 \cdot 15 \cdot x_{2f} + 580 \cdot 23 \cdot x_{3f} + 390 \cdot 12 \cdot x_{4f} \leq 6800.$$

The constraints for the centre and the rear are similar. Finally, we have to consider the balance constraint. This can be simply written as:

$$\frac{1}{10}(18x_{1f}+15x_{2f}+23x_{3f}+12x_{4f}) = \frac{1}{16}(18x_{1c}+15x_{2c}+23x_{3c}+12x_{4c}) = \frac{1}{8}(18x_{1r}+15x_{2r}+23x_{3r}+12x_{4r}).$$

Problem 2

Consider a convex polygon P in $\mathbb{R}^2$, given by an unordered list of its corner points $p_1, p_2, \ldots, p_m$, with $p_j = (a_j, b_j)$. Construct a linear program with the following properties:

A. If the origin $(0,0)$ is in the interior of P, then the program is feasible, and the optimal value of its objective function is strictly positive.

B. If the origin $(0,0)$ is not in the interior of P, then the program is either infeasible, or it is feasible and the optimal value of its objective function is at most 0.

Recall that a point is said to be in the interior of P if it is in P but not on the boundary of P, that is it is neither a corner nor on an edge of P. You may also use the fact that a point is in the interior of P if and only if it is a *weighted average* of the corner points of P, with all weights being strictly positive.

Solution

Given the last fact given in the statement of the exercise, we will express a point as a weighted average of the corner points in P. In particular for the origin, we will have that $\sum_{j=1}^{m} w_j a_j = 0$ (for the a coordinate) and $\sum_{j=1}^{m} w_j b_j = 0$ (for the b coordinate). Now we would like our linear program to be feasible and have a strictly positive objective function value if all the weights w_j are strictly positive, and to be infeasible or have zero or negative objective function value otherwise. To express that the weights are positive, we could attempt to use a new variable δ and write $w_j > \delta$. We cannot however have strict inequalities in our constraints. Instead, we will use $w_j \geq \delta$ and add δ to the objective function. Consider the following linear program:

$$\begin{aligned} \text{maximise} \quad & \delta \\ \text{subject to} \quad & \sum_{j=1}^{m} w_j a_j = 0 \\ & \sum_{j=1}^{m} w_j b_j = 0 \text{ for all } j \\ & w_j \geq \delta \end{aligned}$$

Assume that the LP is feasible and the objective function value is strictly positive. That means that $\delta > 0$ and hence $w_j > 0$ for all j, i.e., the origin is in the interior of the polygon. Assume now that the LP is feasible but the objective function value is at most 0. This means that $w_j \leq 0$, and hence the origin is not in the interior. Similarly, if the LP is infeasible, then it is not possible to find values of w_j and δ that satisfy the constraints, which means that the origin cannot be written as a weighted average of the corner points, with strictly positive weights.

Problem 3

Let $G = (V, E)$ be an undirected graph with weights on the nodes (w_v for each node $v \in V$). A *vertex cover* of G is a set of nodes $S \subseteq V$ such that every edge $e \in E$ is incident to some node in S. A *minimum weight vertex cover* is a vertex cover with minimum total weight $\sum_{v \in S} w_v$.

Formulate the problem of finding a minimum weight vertex cover of G as an integer linear program (ILP). Explain the role of your variables and your constraints, and why an optimal solution to the ILP corresponds to a vertex cover of minimum weight. Write the LP-relaxation of your ILP.

Solution

First, we will consider the variables. We will use an indicator variable x_u which will be 1 if node u is in the vertex cover S and 0 otherwise. Our objective function can then be written as

$$\text{Minimise} \sum_{u \in V} x_u w_u$$

Indeed, notice that if a node is included in the vertex cover then it contributes to its weight (since $x_u = 1$), otherwise it does not. Now we have to write the constraints. The easiest one is the integrality constraint $x_u \in \{0, 1\}$, since x_u is an indicator variable. Finally, we need to express the constraint that for every edge $e = (u, v) \in E$, at least one of u and v must be included in S. That we can do via the following constraint;

$$x_u + x_v \geq 1$$

Indeed, if both x_u and x_v are 0 (i.e., none of the two endpoints of the edge is in the vertex cover), then the constraint will be violated. In the end we have the following LP:

$$\begin{aligned} \text{minimise} \quad & \sum_{u \in V} x_u w_u \\ \text{subject to} \quad & x_u + x_v \geq 1, \quad \text{for all } e = (u, v) \in E \\ & x_u \in \{0, 1\} \quad \quad \text{for all } u \in V \end{aligned}$$

Problem 4

Consider the following problem. There are n indivisible items of weights $w_1, \dots, w_n$ to be distributed to m bags. Our goal is to minimise the weight of the heaviest bag. Formulate this problem as an integer linear program. Explain the role of your variables and your constraints, and why an optimal solution to the ILP corresponds to a vertex cover of minimum weight. Write the LP-relaxation of your ILP.

Solution

We start with our choice of variables. We would like to figure out which bag to assign each item to. For that, we will use indicator variables x_{ij} which will be 1 if item i is placed in bag j and 0 otherwise. For our objective function, we want to minimise the weight of the heaviest bag, which can be written as:

$$\text{minimise} \max_{j=1,\dots,m} \sum_{i \in j} x_{ij} w_i$$

Indeed, the weight of bag j is the sum of weights of the items that are assigned to that bag, which are only the items i for which $x_{ij} = 1$. Then we can add the straightforward constraint that each item must be assigned to exactly one bag, i.e.,

$$\sum_{j=1,\dots,m} x_{ij} = 1,$$

as well as the integrailty constraint $x_{ij} = 1$ for $i = 1, \dots, n$ and $j = 1, \dots, m$.

The problem with our formulation is that our objective function is not linear, since it has the max function as a part of it. We will remove the max function using the usual trick: we will introduce a new variable e and we will add the constraint that

$$\sum_{i \in j} x_{ij} w_j \leq e, \text{ for all } j = 1, \dots, m$$

The idea is that e is an upper bound on the value of the objective function: if the maximum of the sums is at most e, then every one of the sums will be at most e. The only thing to be concerned about is whether our LP in an optimal solution will choose some $e' > \max_{j=1,\dots,m} \sum_{i\in j} x_{ij}w_i$ rather than $e = \max_{j=1,\dots,m} \sum_{i\in j} x_{ij}w_i$. However this is not going to happen, because choosing $e = \max_{j=1,\dots,m} \sum_{i\in j} x_{ij}w_i$ is feasible, and results in a smaller objective function value than choosing e'. Our final LP is the following:

$$
\begin{aligned}
\text{minimise} \quad & e \\
\text{subject to} \quad & \sum_{i\in j} x_{ij}w_j \le e, && \text{for all } j = 1,\dots,m \\
& \sum_{j=1}^{m} x_{ij} = 1, && \text{for all } i = 1,\dots,n \\
& x_{ij} \in \{0,1\} \text{ for all } i = 1,\dots,n \text{ and } j = 1,\dots,m
\end{aligned}
$$

Problem 5

A tutor in Algorithms and Data Structures has decided that the tutoring salary is not enough, and has decided to offer private lessons to the students. These will be 1-to-1 lessons, with a duration of 1 hour. The tutor has divided the week into 1-hour slots which are offered to the students. Every student i specifies the following parameters:

- The availability for slot j: this is a parameter a_{ij} which is 1 if the student is availble to take that slot, and 0 otherwise.
- The amount the student is willing to pay for slot j, denoted by p_{ij}.
- The number of lessons q_i that the student wishes to take. The student is willing to either take q_i lessons or none, more or fewer lessons that q_i are not acceptable.

The tutor would like to assign students to slots in a way that maximises their profit. Help the tutor by figure out the optimal assignment by formulating the above problem as an integer linear program. Describe the objective function, the variables, and the constraints.

Solution

We start by identifying appropriate indicator variables: let x_{ij} be 1 if student i is assigned to slot j and 0 otherwise. The profit from this assignment is then $x_{ij}p_{ij}a_{ij}$, as the student will pay p_{ij} for slot j only if the student is available to take the slot (i.e., $a_{ij} = 1$), and if the student is assigned to the slot (i.e., $x_{ij} = 1$). Therefore our objective function is:

$$\text{Maximise} \sum_j \sum_i x_{ij}p_{ij}a_{ij}$$

For our constraints, we have that only one student can be assigned to each slot (since these are 1-to-1 lessons), so we have:

$$\sum_i x_{ij} \le 1, \text{ for all } j$$

We also have the obvious integrality constraints $x_{ij} \in \{0,1\}$ for all i and j. What we are missing are the constraints regarding the number of lessons for each student. We would like a constraint that say that student i can take either q_i lessons or none. Another way to view this is that, if the student is selected for tutoring, then they get q_i lessons, otherwise they are not selected for tutoring. Hence, we can introduce a new indicator variable which is

$$y_i = \begin{cases} 1, & \text{if the student is selected to take lessons,} \\ 0, & \text{otherwise.} \end{cases}$$

Then, we can express the final set of constraints as follows:

$$\sum_j x_{ij}\alpha_{ij} = q_i y_i$$

Indeed, if the student is selected, then will need to take q_i lessons (since in that case $y_i = 1$), otherwise they will take 0 lessons (as in that cases $y_i = 0$). If we add the integrality constraint $y_i \in \{0, 1\}$ for all i, we have our ILP).