Algorithmic Game Theory and Applications

Games and Solution Concepts

The formal definition of a (strategic) game

Definition: A game in normal or strategic form is a tuple $(N, S_1, S_2, \ldots, S_n, u_1, u_2, \ldots, u_n)$ where

1. $N = \{1, \ldots, n\}$ is a set of players (sometimes called "agents").

2. For each player $i \in N$, there is a set S_i of (pure) strategies.

A vector $(s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \ldots \times S_n = S$ is called a strategy profile.

3. For each player $i \in N$, there is a payoff (or utility) function $u_i : S \to \mathbb{R}$ which assigns a numerical value $u_i(s_1, s_2, \ldots, s_n)$ to player i for a given strategy profile $(s_1, s_2, \ldots, s_n)$.

Example 3: Cheating Partners

Classmate You	Take the deal	Don’t take the deal
Take the deal	5 5	-100 20
Don’t take the deal	20 -100	10 10

What would you do?

A First Game: Prisoner's Dilemma

Two criminals have been arrested and are imprisoned in isolation (i.e., they cannot talk to each other!).

The police does not have enough evidence to convict them on the charges, but they do have enough to convict them on other charges (2 years).

The offer each prisoner a bargain: Confess to the crime and you will get a reduced sentence (1 year).

If the other person does not confess, they will be sentenced to 10 years.

But if both prisoners confess, they each receive a sentence of 5 years.

A First Game: Prisoner's Dilemma

Bimatrix Game

Player 1 \ Player 2	Confess	Silent
Confess	-5, -5	-1,-10
Silent	-10, -1	-2, -2

A First Game: Prisoner's Dilemma

Bimatrix Game

How should the prisoners play?

Player 1 \ Player 2	Confess	Silent
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Solution Concept #1:
Dominant Strategy Equilibrium

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(Weakly) Dominant Strategy: A (pure) strategy $s_i \in S_i$ for player $i \in N$ is a (weakly) dominant strategy, if it results in at least as high utility as any other strategy $s_i' \in S_i$, *regardless of the strategies of the others.*

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Mathematically: $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$ and all $s_{-i} \in S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n = S_{-i}$.

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Strictly Dominant Strategy: The inequality above is strict.

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Strictly Dominant Strategy: The inequality above is strict.

(Weakly) Dominant Strategy Equilibrium (DSE): A strategy profile $s = (s_1, \ldots, s_n)$ such that each s_i is a (weakly) dominant strategy.

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Both players confessing is the *logical* outcome of this game.

Both players confessing is the only dominant strategy equilibrium.

	Confess	Silent
Confess	-5	-1
Silent	-10	-2

Player 1 (row player)

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Question: Can a game have multiple strict dominant strategy equilibria?

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Question: Can you propose a game that has more than one weak dominant strategy equilibrium?

Question: Can a game have multiple strict dominant strategy equilibria?

Question: Do all games have weak dominant strategy equilibria?

A Game Without Dominant Strategy Equilibria: Choosing the TV show

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But one of you prefers to watch Peep Show and the other Flight of the Conchords.

	Peep Show	FOTC
Peep Show	10, 7	5, 5
FOTC	1, 1	7, 10

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If Player 2 chooses Peep Show, Player 1 should choose Peep Show.

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If Player 2 chooses Peep Show, Player 1 should choose Peep Show.

If Player 2 chooses FOTC, Player 1 should choose FOTC.

There is no option that is best regardless of the other player - the best choice depends on the choice of the other player!

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(Weakly) Dominant Mixed Strategy: A mixed strategy $x_i \in \Delta(S_i)$ for player $i \in N$ is a (weakly) dominant strategy, if it results in at least as high utility as any other mixed strategy $x_i' \in \Delta(X_i)$, *regardless of the strategies of the others*.

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<u>Proposition:</u> Every pure strategy s_i in the support of a mixed (weakly) dominant strategy x_i is a (weakly) dominant pure strategy.

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Proposition: Every pure strategy s_i in the support of a mixed (weakly) dominant strategy x_i is a (weakly) dominant pure strategy.

This means that if there are no (weakly) dominant pure strategies, then there are also no (weakly) dominant mixed strategies.

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Advantage of DSE: Obviously reasonable outcome - anything else is unreasonable.

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Not all games have DSE.

In fact, even simple games don't have DSE.

Advantage of DSE: Obviously reasonable outcome - anything else is unreasonable.

Drawback of DSE: It is not <u>universal</u> - there are (many) games for which it does not exist.

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Iterated Elimination of Dominated Strategies

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Strictly Dominated Strategy: A (pure) strategy $s_i \in S_i$ for player $i \in N$ is a strictly dominated strategy, if it results in lower utility than some other strategy $s_i' \in S_i$, *regardless of the strategies of the others.*

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Elimination of Dominated Strategies

	L	C	R
U	3, 1	0, 1	0, 0
M	1, 1	1, 1	5, 0
D	0,1	4, 1	0, 0

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	L	C	R
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If Player 2 plays R, they always get a utility of 0.

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	L	C	R
U	3, 1	0, 1	0, 0
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If Player 2 plays C, they always get a utility of 1.

If Player 2 plays R, they always get a utility of 0.

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C is clearly better than R, no matter what Player 1 does.
R is strictly dominated (by C).

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	L	C	R
U	3, 1	0, 1	0, 0
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This means that we can remove R.

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If Player 2 plays L, they always get a utility of 1.

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We usually don't remove weakly dominated strategies.

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Any strictly dominated strategies for Player 1?

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M	1, 1	1, 1
D	0,1	4, 1

Any strictly dominated strategies for Player 1?

We usually don't remove weakly dominated strategies.

Consider M vs (1/2)*U + (1/2)*D

If Player 2 plays L, they always get a utility of 1.

If Player 2 plays C, they always get a utility of 1.

Elimination of Dominated Strategies

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$$\frac{1}{2}u(\text{U,C}) + \frac{1}{2}u(\text{D,C}) = 2$$

Elimination of Dominated Strategies

	L	C
U	3, 1	0, 1
M	1, 1	1, 1
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We usually don't remove weakly dominated strategies.

Any strictly dominated strategies for Player 1?

M is strictly dominated by the mixture of U and L.

Consider M vs (1/2)*U + (1/2)*D

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Elimination of Dominated Strategies

	L	C
U	3, 1	0, 1
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D	0,1	4, 1

Any strictly dominated strategies for Player 1?

M is strictly dominated by the mixture of U and L.

So we can remove it.

If Player 2 plays L, they always get a utility of 1.

If Player 2 plays C, they always get a utility of 1.

We usually don't remove weakly dominated strategies.

Consider M vs (1/2)*U + (1/2)*D

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Elimination of Dominated Strategies

	L	C
U	3, 1	0, 1
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Elimination of Dominated Strategies

	L	C
U	3, 1	0, 1
D	0,1	4, 1

We cannot find any more strictly dominated strategies.

Solution Concept #2: Iterated Elimination of Dominated Strategies

(Weakly) Dominant Strategy Equilibrium (DSE): A strategy profile $s = (s_1, \ldots, s_n)$ such that each s_i is a (weakly) dominant strategy.

Solution Concept #2:
Iterated Elimination of Dominated Strategies

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Iterated Elimination of Dominated Strategies Equilibrium (IEDSE): Any (mixed) strategy profile that assigns zero probability to any strategy that would be removed through iterated elimination of strictly dominated strategies.

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Intuition: DSE means players will only play obviously reasonable strategies, whilst IEDSE says that players will not play obviously unreasonable strategies.

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Intuition: DSE means players will only play obviously reasonable strategies, whilst IEDSE says that players will not play obviously unreasonable strategies.

Clearly: If x is a DSE, it is also an IEDSE.

A Game Without Dominant Strategy Equilibria: Choosing the TV show

	Peep Show	FOTC
Peep Show	10, 7	5, 5
FOTC	1, 1	7, 10

A Game Without Dominant Strategy Equilibria: Choosing the TV show

Are there any strictly dominated strategies here?

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Iterated Elimination of Dominated Strategies Equilibrium

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Drawback of IEDSE: It allows for some (many) potentially unreasonable outcomes.

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Drawback of IEDSE: It allows for some (many) potentially unreasonable outcomes.

Advantage of DSE: It is obviously universal (it always exists).

Conclusion: Not very convincing as a solution concept.

But we will return to it, because it can be useful as a tool for stronger solution concepts.

Back to Choosing the TV show

	Peep Show	FOTC
Peep Show	10, 7	5, 5
FOTC	1, 1	7, 10

Back to Choosing the TV show

If you were Player 1,
would you watch FOTC?

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Peep Show	10, 7	5, 5
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If you were Player 1, would you watch FOTC?

If you were Player 1, would you watch Peep Show?

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Back to Choosing the TV show

If you were Player 1, would you watch FOTC?

If you were Player 1, would you watch Peep Show?

	Peep Show	FOTC
Peep Show	10, 7	5, 5
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Back to Choosing the TV show

If you were Player 1, would you watch FOTC?

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Back to Choosing the TV show

If you were Player 1, would you watch FOTC?

If you were Player 1, would you watch Peep Show?

If you were Player 1, would you watch Peep Show?

If you were Player 1, would you watch FOTC?

	Peep Show	FOTC
Peep Show	10, 7	5, 5
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Back to Choosing the TV show

For either
(Peep Show, Peep Show) or
(FOTC, FOTC),
Player 1 does not want to
deviate to watching the
other show.

	Peep Show	FOTC
Peep Show	10, 7	5, 5
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For either
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Player 1 does not want to
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What about Player 2?

	Peep Show	FOTC
Peep Show	10, 7	5, 5
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Back to Choosing the TV show

For either
(Peep Show, Peep Show) or
(FOTC, FOTC),
Player 1 does not want to *deviate* to watching the other show.

What about Player 2?

Player 2 does not want to deviate either!

	Peep Show	FOTC
Peep Show	10, 7	5, 5
FOTC	1, 1	7, 10

Back to Choosing the TV show

For either
(Peep Show, Peep Show) or
(FOTC, FOTC),
Player 1 does not want to *deviate* to watching the other show.

What about Player 2?

Player 2 does not want to deviate either!

So, assuming that the other player does not change strategy, no player wants to change.

	Peep Show	FOTC
Peep Show	10, 7	5, 5
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Solution Concept #3:
Pure Nash Equilibrium

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Pure Nash Equilibrium (PNE): A pure strategy profile $(s_1, \ldots, s_n)$ such that for any player $i \in N$, *fixing the pure strategies* s_{-i} *of the other players*, player i cannot get higher utility from choosing a different pure strategy.

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Back to Choosing the TV show

Both
(Peep Show, Peep Show)
and
(FOTC, FOTC),
are PNE!

	Peep Show	FOTC
Peep Show	10, 7	5, 5
FOTC	1, 1	7, 10

Before we proceed… DSE vs PNE?

<u>PNE:</u> $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$.

<u>DSE:</u> $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$ and all $s_{-i} \in S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n = S_{-i}$.

Before we proceed… DSE vs PNE?

PNE: $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$.

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What is the difference?

Before we proceed…
DSE vs PNE?

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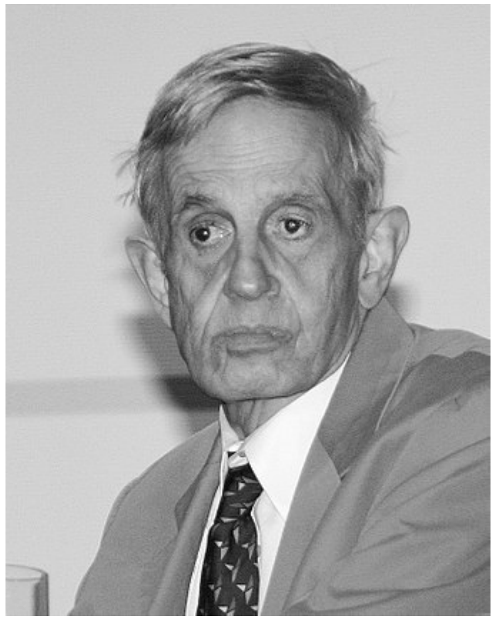
What is the difference?

<u>Which of the following statements is true?</u>
1. Every DSE is a PNE.
2. Every PNE is a DSE.

Solution Concept #3: Pure Nash Equilibrium

Due to Nash (1951).



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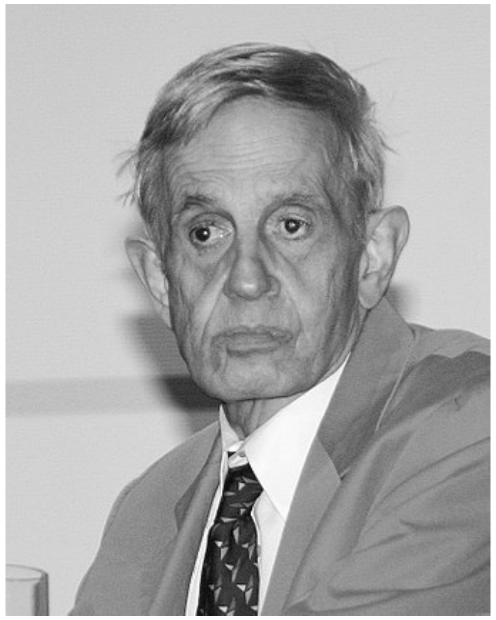

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Advantage of PNE: Much more reasonable outcome - “I won’t change unless the others change”, hence a *stable* outcome.

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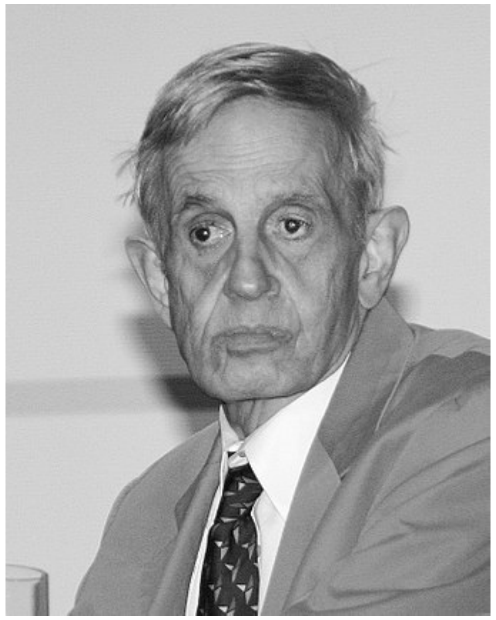

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Is it universal? Do PNE always exist?

Rock-Paper-Scissors

	Rock	Paper	Scissors
Rock (R)	0, 0	-1, 1	1, -1
Paper (P)	1, -1	0, 0	-1, 1
Scissors (S)	-1, 1	1, -1	0, 0

Rock-Paper-Scissors

Is this a PNE?

	Rock	Paper	Scissors
Rock (R)	0, 0	-1, 1	1, -1
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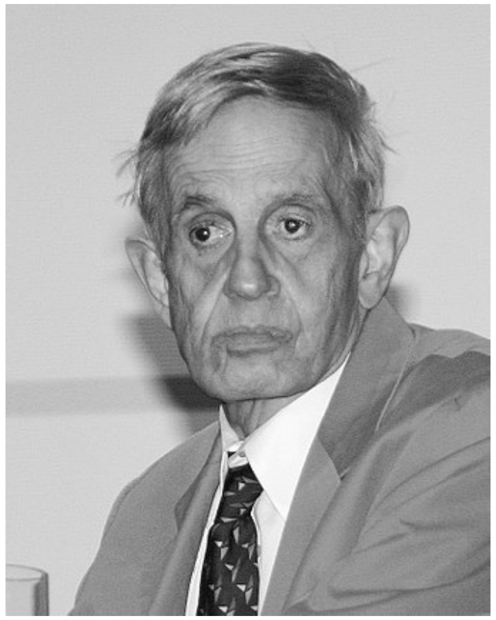
RPS does not have any PNE!

Solution Concept #3:
Pure Nash Equilibrium



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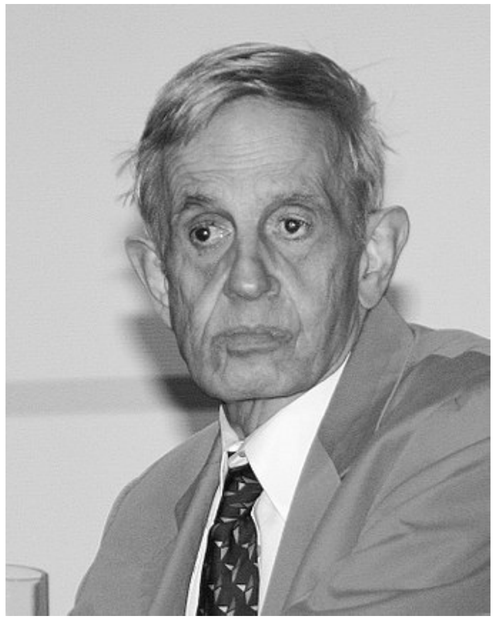
Drawback of PNE: It is not universal - there are games for which it does not exist.



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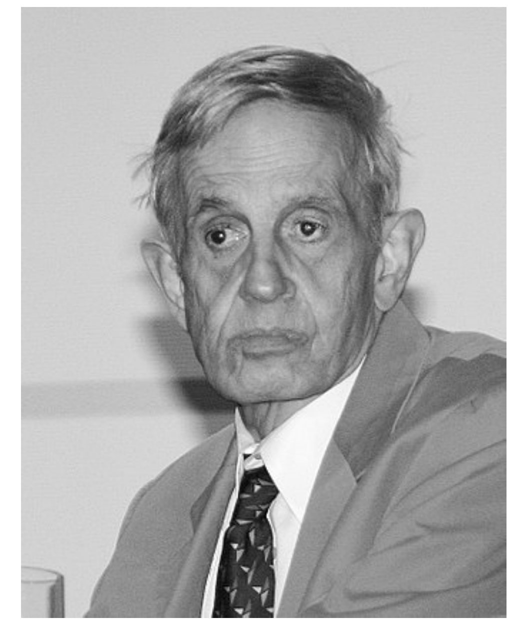

Due to Nash (1951).

Advantage of PNE: Much more reasonable outcome - “I won’t change unless the others change”, hence a *stable* outcome.

Is it universal? Do PNE always exist?

Drawback of PNE: It is not universal - there are games for which it does not exist.

But: There are important classes of games for which it does exist (stay tuned).

Solution Concept #3: Pure Nash Equilibrium

Pure Nash Equilibrium (PNE): A pure strategy profile $(s_1, \ldots, s_n)$ such that for any player $i \in N$, *fixing the pure strategies* s_{-i} *of the other players*, player i cannot get higher utility from choosing a different pure strategy.

Mathematically: $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$.

Equivalently: $s_i \in \arg\max_{\hat{s}_i \in S_i} u_i(\hat{s}_i, s_{-i})$

In words: s_i is a pure strategy that maximises the utility of the player, given the fixed strategies s_{-i} of the other players.

Terminology: s_i is a *pure best response* to s_{-i}.

Terminology: Player i does not have a profitable *unilateral deviation*.

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Equivalently: $s_i \in \arg\max_{\hat{s}_i \in S_i} u_i(\hat{s}_i, s_{-i})$

But what about mixed strategies?

In words: s_i is a pure strategy that maximises the utility of the player, given the fixed strategies s_{-i} of the other players.

Terminology: s_i is a *pure best response* to s_{-i}.

Terminology: Player i does not have a profitable *unilateral deviation*.

Solution Concept #3*: (Mixed) Nash Equilibrium

Pure Nash Equilibrium (MNE): A mixed strategy profile $(x_1, \ldots, x_n)$ such that for any player $i \in N$, *fixing the mixed strategies x_{-i} of the other players*, player i cannot get higher utility from choosing a different mixed strategy.

Mathematically: $u_i(x_i, x_{-i}) \geq u_i(x_i', x_{-i})$ for all $x_i' \in \Delta(S_i)$.

Equivalently: $x_i \in \arg\max_{\hat{x}_i \in \Delta(S_i)} u_i(\hat{x}_i, x_{-i})$

In words: x_i is a mixed strategy that maximises the utility of the player, given the fixed strategies x_{-i} of the other players.

Terminology: x_i is a *(mixed) best response* to x_{-i}.

Terminology: Player i does not have a profitable *unilateral deviation*.

Solution Concept #3*:
(Mixed) Nash Equilibrium

Pure Nash Equilibrium (MNE): A mixed strategy profile $(x_1, \ldots, x_n)$ such that for any player $i \in N$, *fixing the mixed strategies x_{-i} of the other players*, player i cannot get higher utility from choosing a different mixed strategy.

Mathematically: $u_i(x_i, x_{-i}) \geq u_i(x_i', x_{-i})$ for all $x_i' \in \Delta(S_i)$.

Equivalently: $x_i \in \arg\max_{\hat{x}_i \in \Delta(S_i)} u_i(\hat{x}_i, x_{-i})$

Recall:
$u_i(\hat{x}_i, x_{-i}) = \mathbb{E}_{(s_i, s_{-i}) \sim (x_i, x_{-i})}[u_i(s_i, s_{-i})]$

In words: x_i is a mixed strategy that maximises the utility of the player, given the fixed strategies x_{-i} of the other players.

Terminology: x_i is a *(mixed) best response* to x_{-i}.

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Rock-Paper-Scissors

	Rock	Paper	Scissors
Rock (R)	0, 0	-1, 1	1, -1
Paper (P)	1, -1	0, 0	-1, 1
Scissors (S)	-1, 1	1, -1	0, 0

Rock-Paper-Scissors

Consider the symmetric strategy (R, P, S) = (1/3, 1/3, 1/3) for both players. This is a MNE.

	Rock	Paper	Scissors	
Rock (R)	0, 0	-1, 1	1, -1	**1/3**
Paper (P)	1, -1	0, 0	-1, 1	**1/3**
Scissors (S)	-1, 1	1, -1	0, 0	**1/3**
	1/3	**1/3**	**1/3**	

Rock-Paper-Scissors MNE

The symmetric strategy (R, P, S) = (1/3, 1/3, 1/3) for both players is a MNE.

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How can we prove that though?

Rock-Paper-Scissors MNE

The symmetric strategy (R, P, S) = (1/3, 1/3, 1/3) for both players is a MNE.

How can we prove that though?

The definition says: $u_i(x_i, x_{-i}) \geq u_i(x_i', x_{-i})$ for all $x_i' \in \Delta(S_i)$.

Rock-Paper-Scissors MNE

The symmetric strategy (R, P, S) = (1/3, 1/3, 1/3) for both players is a MNE.

How can we prove that though?

The definition says: $u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i})$ for all $x'_i \in \Delta(S_i)$.

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Since $\sum_{s_i \in S_i} x_i'(s_i) = 1$

Rock-Paper-Scissors

Consider the symmetric strategy (R, P, S) = (1/3, 1/3, 1/3) for both players. This is a MNE.

	Rock	Paper	Scissors	
Rock (R)	0, 0	-1, 1	1, -1	**1/3**
Paper (P)	1, -1	0, 0	-1, 1	**1/3**
Scissors (S)	-1, 1	1, -1	0, 0	**1/3**
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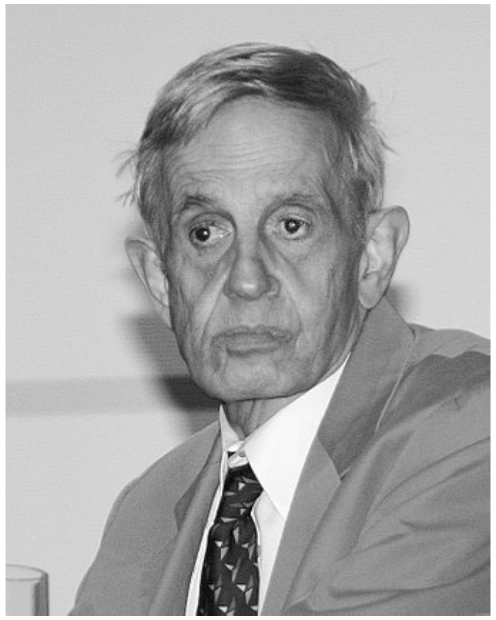
Solution Concept #3*: Mixed Nash Equilibrium



Introduced by Nash in 1951 (in his PhD dissertation).

Advantage of MNE: Much more reasonable outcome - “I won’t change unless the others change”, hence a *stable* outcome.

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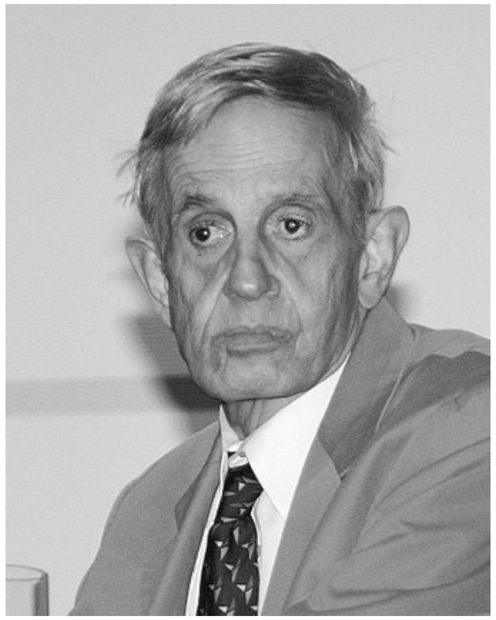

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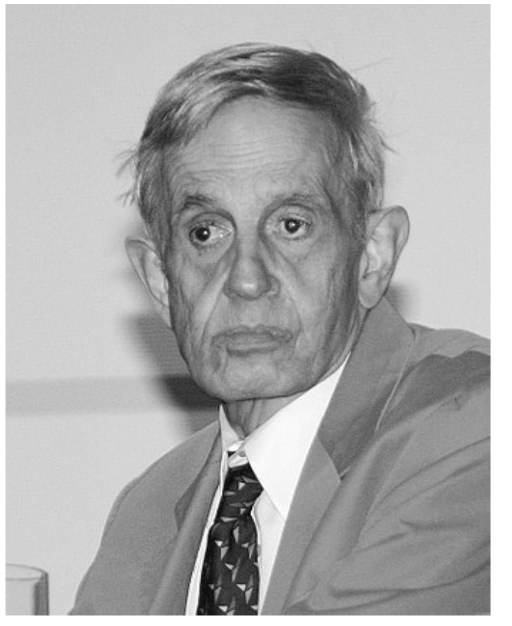

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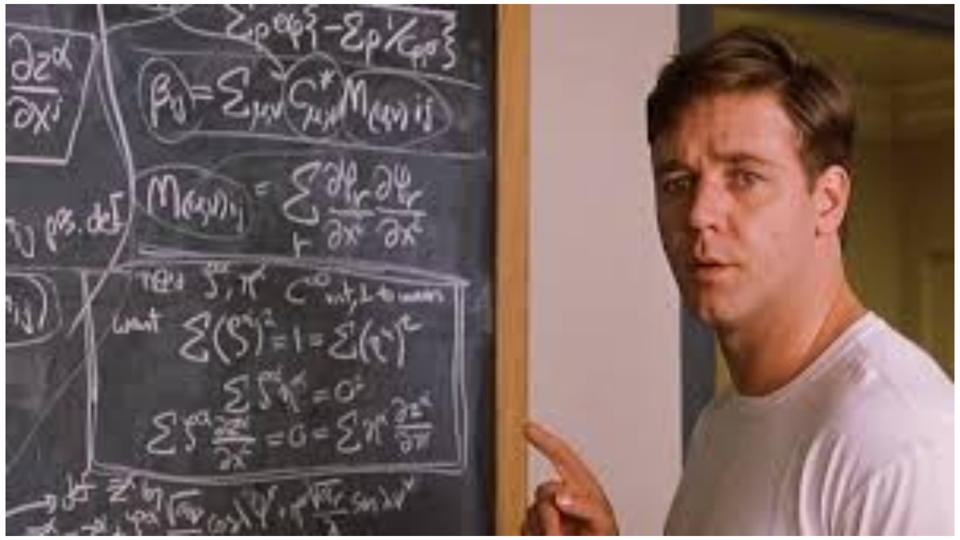
Theorem (Nash 1951): Every (finite normal-form) game has at least one mixed Nash equilibrium.

Homework



“A Beautiful Mind”, 2001 film staring Russell Crowe as John Nash.

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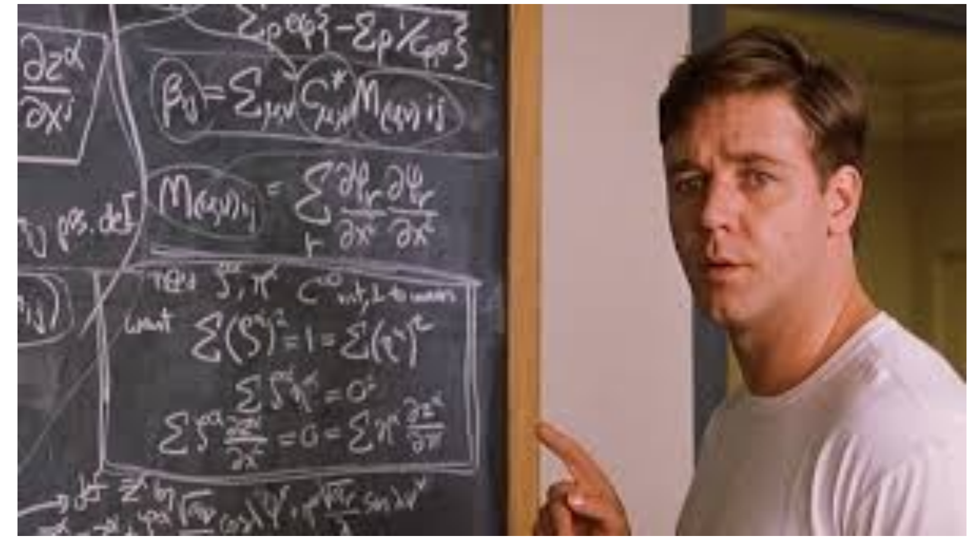

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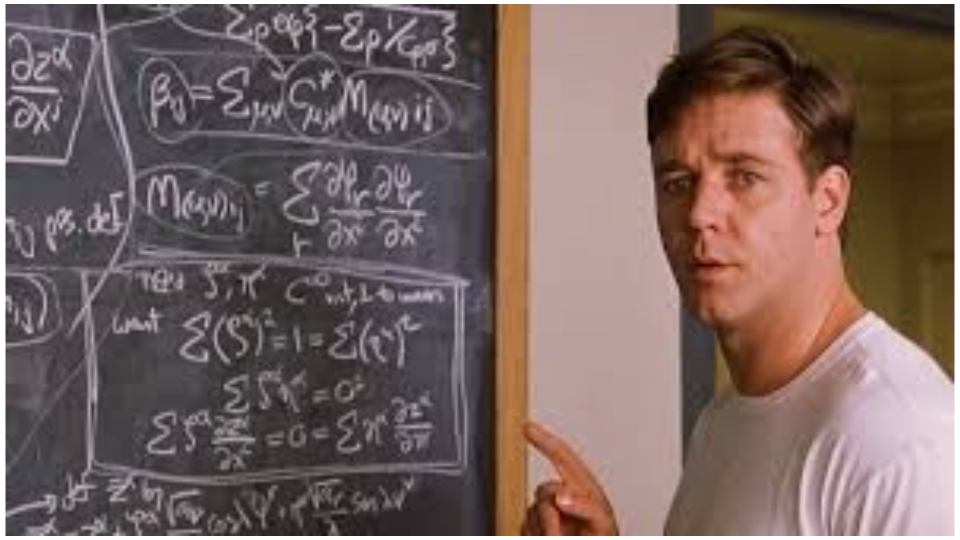

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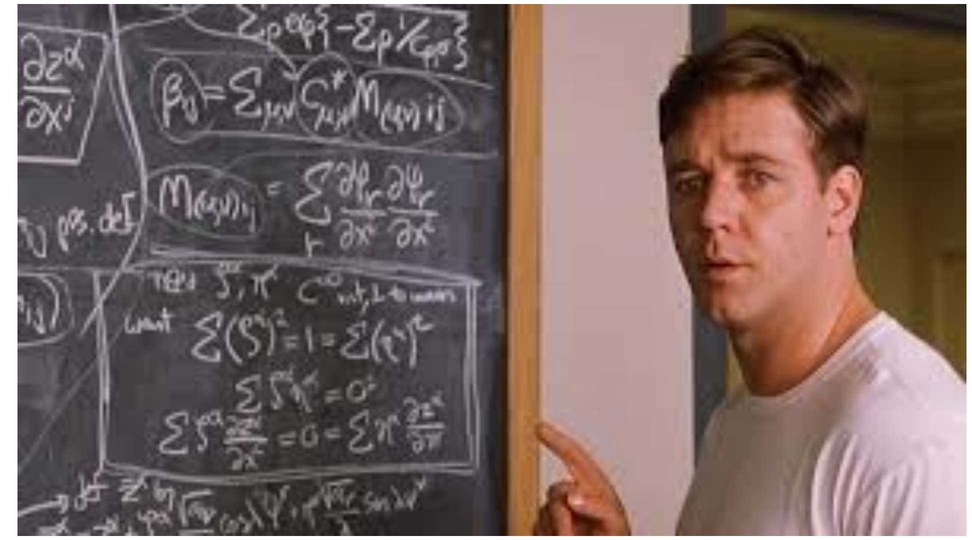

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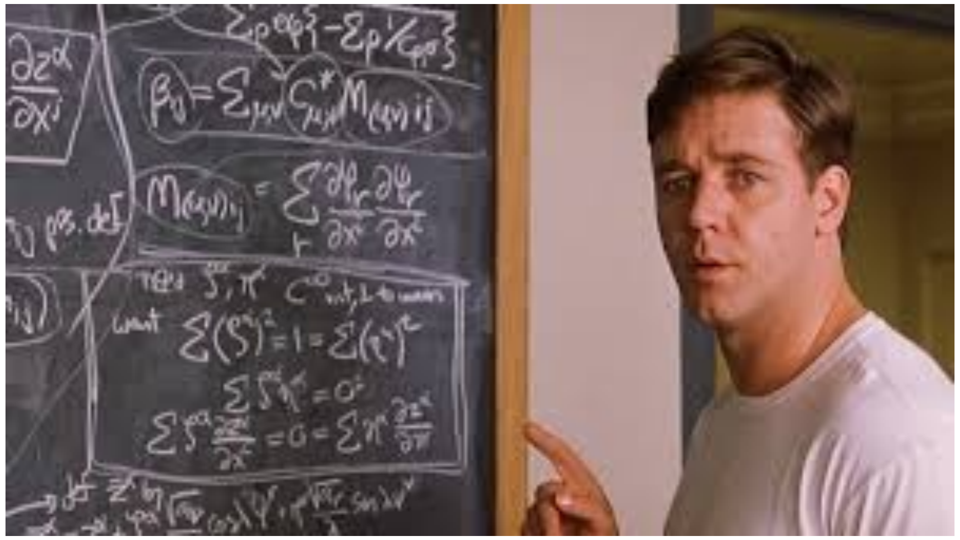

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1. First, think about how to model the situation that you see in the scene as a game.

2. Then, explain why the solution proposed by the fictional John Nash is actually *not* a Nash equilibrium.