Introduction to Algorithms and Data Structures

Lecture 26: Satisfiability and NP-completeness

Mary Cryan

School of Informatics
University of Edinburgh

Reductions between (decision) problems

If I could solve problem Q in polynomial-time, then I would also be able to solve problem R in polynomial-time.

Definition

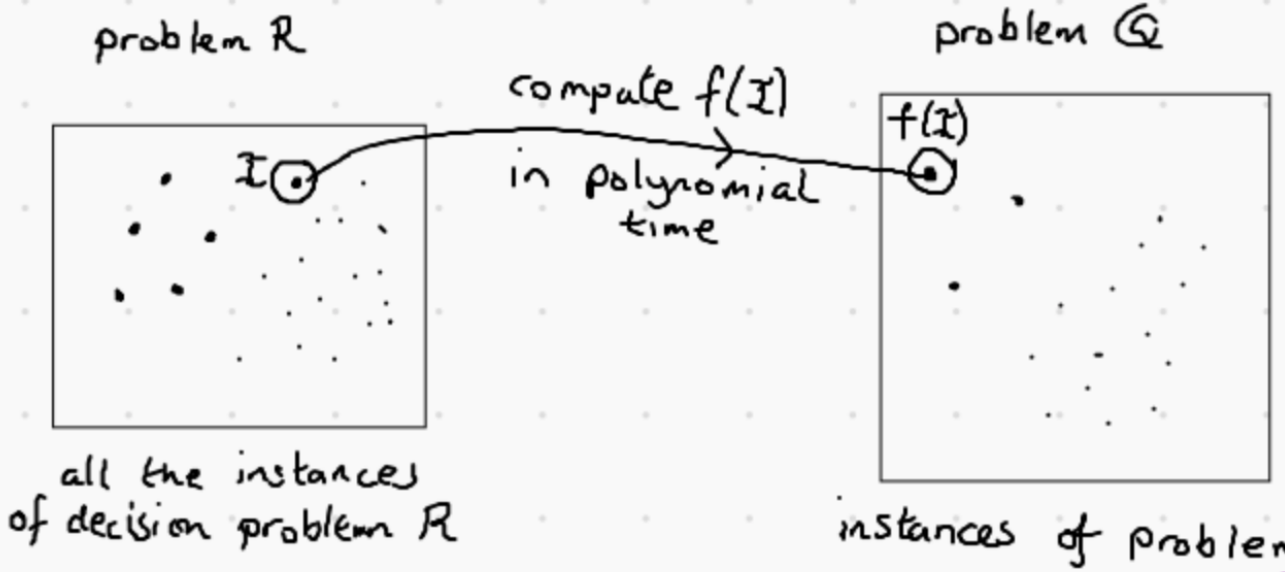
A problem R can be reduced to the problem Q if there is a polynomial-time computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that for all instances $\mathcal{I}$ of R

$$R(\mathcal{I}) = 1 \quad \Leftrightarrow \quad Q(f(\mathcal{I})) = 1$$

- Means that R is no harder (in the sense of polynomial-time computation) than Q. And that Q is "at least as hard" as R.
- We write $R \leq_{\mathrm{P}} Q$.

($\leq_P$ is not like $\leq$, or even $O(\cdot)$. We can ignore polynomial factors)

Reductions between (decision) problems



need to ensure $R(I)=1 \iff Q(f(I))=1$ (need both directions of the $\iff$)

then if we had a polynomial-time algorithm to answer Q, we could incorporate f to also build a polynomial-time algorithm to answer R.

Meaning of $R \leq_P Q$

If I could solve problem Q in polynomial-time, then I would also be able to solve problem R in polynomial-time.

- IF Q happens to be in P (is polynomial-time solvable), then I can also solve R in polynomial-time:
  - Take the input instance $\mathcal{I}$ of R and do the polynomial-time work to compute $f(\mathcal{I})$.
  - Pass $f(\mathcal{I})$ to our polynomial-time algorithm for deciding Q.
  - Return that answer
- IF R happens to be NP-complete (and, we believe, probably not polynomial-time solvable), then Q is also NP-complete.
  - For any problem H in NP, we can reduce it to R with some g function (because R is NP-complete). But if we instead apply $f(g(\cdot))$ to instances of H, this reduces H down to Q.
- Note: $\leq_P$ is not like $\leq$, or even $O(\cdot)$. It allows us to ignore polynomial factors.

NP-completeness

No (NP) problem is any harder than me.

Definition

A decision problem Q is said to be NP-complete if it belongs to the class NP, and it is also the case that for every problem R in NP, $R \leq_{\mathrm{P}} Q$.

The canonical NP-complete problem is Satisfiability.

- This was the first problem to be shown to be NP-complete (late 1960s/early 1970s).
- In the years that followed many other decision problems were shown to be NP complete by reduction to Satisfiability (and the increasing pool of NP-complete problems).

Satisfiability

Definition

We say a propositional logical formula ϕ over the variables $\{x_1, \ldots, x_n\}$ is in Conjunctive Normal (CNF) if it is written in the form

$$\phi = C_1 \wedge C_2 \wedge \ldots C_m$$

where each of the clauses C_i is a "disjunction of literals" over $\{x_1, \ldots, x_n\}$

For example, if $n = 5$, here are two example CNF formulae:

- $\phi_1 = (x_2 \vee \bar{x_4}) \wedge (x_4 \vee \bar{x_2}) \wedge (x_1 \vee x_2 \vee \bar{x_4})$
- $\phi_2 = (x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (\bar{x_3} \vee x_4) \wedge (\bar{x_4} \vee x_5) \wedge (\bar{x_5} \vee \bar{x_1})$

There are 2^n possible assignments to the logical variables of a CNF.

For $n = 5$, consider $x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1$.

- This assignment makes ϕ_1 true (all clauses are satisfied).
- This assignment does not satisfy ϕ_2 (2nd clause is violated).

Cook-Levin theorem

SAT: Given a CNF formula $\phi = C_1 \wedge \ldots \wedge C_m$ over variables $\{x_1, \ldots, x_n\}$, determine whether there is some satisfying assignment for ϕ.

Theorem (Cook-Levin)

SAT *is NP-complete.*

Proved/published by Cook in 1971, and independently (behind the Iron Curtain) by Levin in late 1960s.

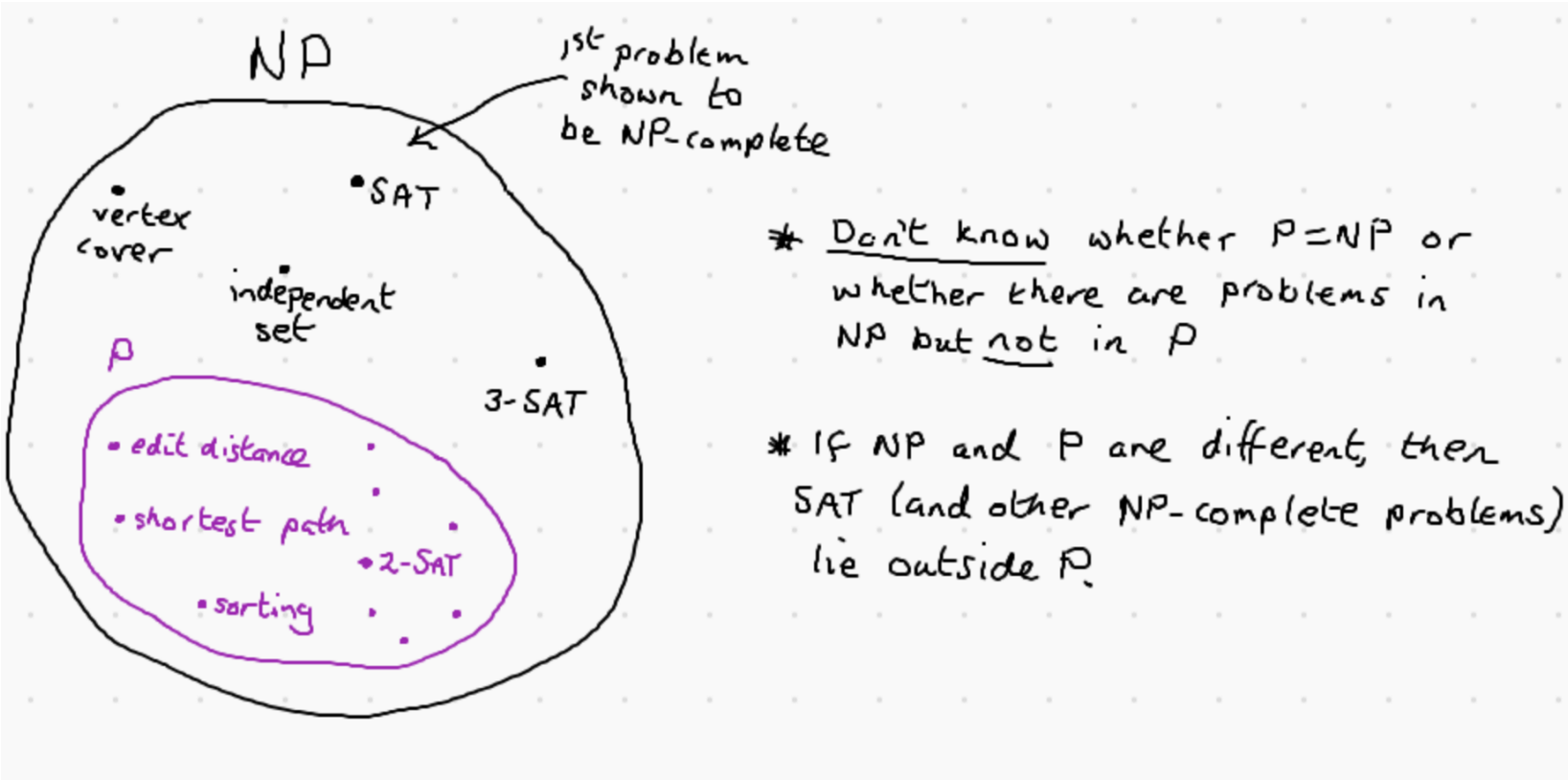
Cook-Levin theorem

For every NP problem R, R "reduces to" $\textsc{Sat}$

Proof works by characterising the behaviour of the polynomial-time verifier for R (on some instance $\mathcal{I}$) as a specially-designed CNF formula.

- Proof works from the "polynomial-time verifier" for the problem R (which must exist), considering the different operations/steps of that verifier.
- All operations/steps of a verifier for the computation of the comparison of a "certificate" against $\mathcal{I}$ can be considered as operations on binary data, can be encoded as Boolean/logical operations.
- Can be used to build a big CNF formula which is true $\Leftrightarrow$ there was some "certificate" for instance $\mathcal{I}$ wrt R.
- The encoding of this algorithm can be shown to be "polynomial-size" in the size of $\mathcal{I}$ (because of the verifier being polynomial-time).
- Full details in *Introduction to Theoretical Computer Science (ITCS)*.

world of NP


NP
1st problem shown to be NP-complete
SAT
vertex cover
independent set
P
3-SAT
• edit distance
• shortest path
• 2-SAT
• sorting
* Don't know whether P=NP or whether there are problems in NP but not in P
* If NP and P are different, then SAT (and other NP-complete problems) lie outside P.

Independent Sets

Our interest is in the INDEPENDENT SET problem.

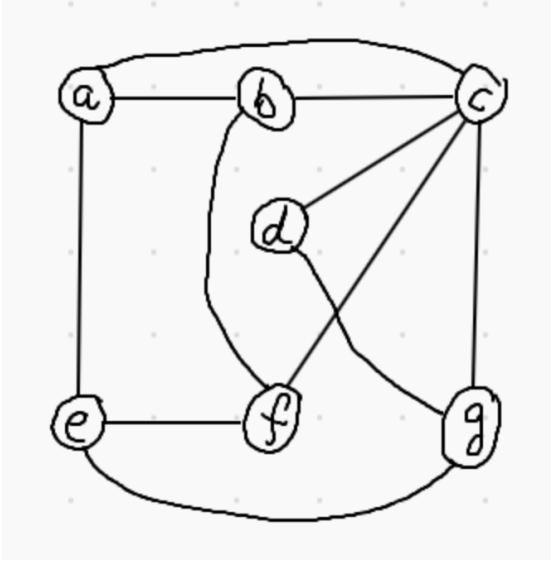
Definition

Given an undirected graph $G = (V, E)$, an Independent Set (IS) is a subset $I \subseteq V$ such that for every pair $u, v \in I$, $(u, v) \notin E$. The *size* of such an independent set is the cardinality $|I|$.

We can consider the following decision problem:

INDEPENDENT SET: Given an undirected graph $G = (V, E)$, and a natural number $k \in \mathbb{N}$, determine whether G has an IS of size $\geq k$.

Independent Set



For this graph, the maximum Independent Set will have size 3.
One solution is $\{b, e, d\}$.

3-CNF and 3-SAT

Definition

The CNF formula ϕ is said to be 3-CNF if each of its clauses C_j is a disjunction of exactly three literals.

3-SAT: Given a 3-CNF formula ϕ over the variables $\{x_1, \ldots, x_n\}$, determine whether there is an assignment of binary values to $\{x_1, \ldots, x_n\}$ that causes all clauses to be satisfied.

Theorem

3-SAT *is NP-complete.*

We will use 3-SAT as our "reducing" problem (R), to show that Independent Set (Q) is also NP-complete.

Reduction: 3-SAT to Independent Set

Our starting point is the 3-SAT problem.

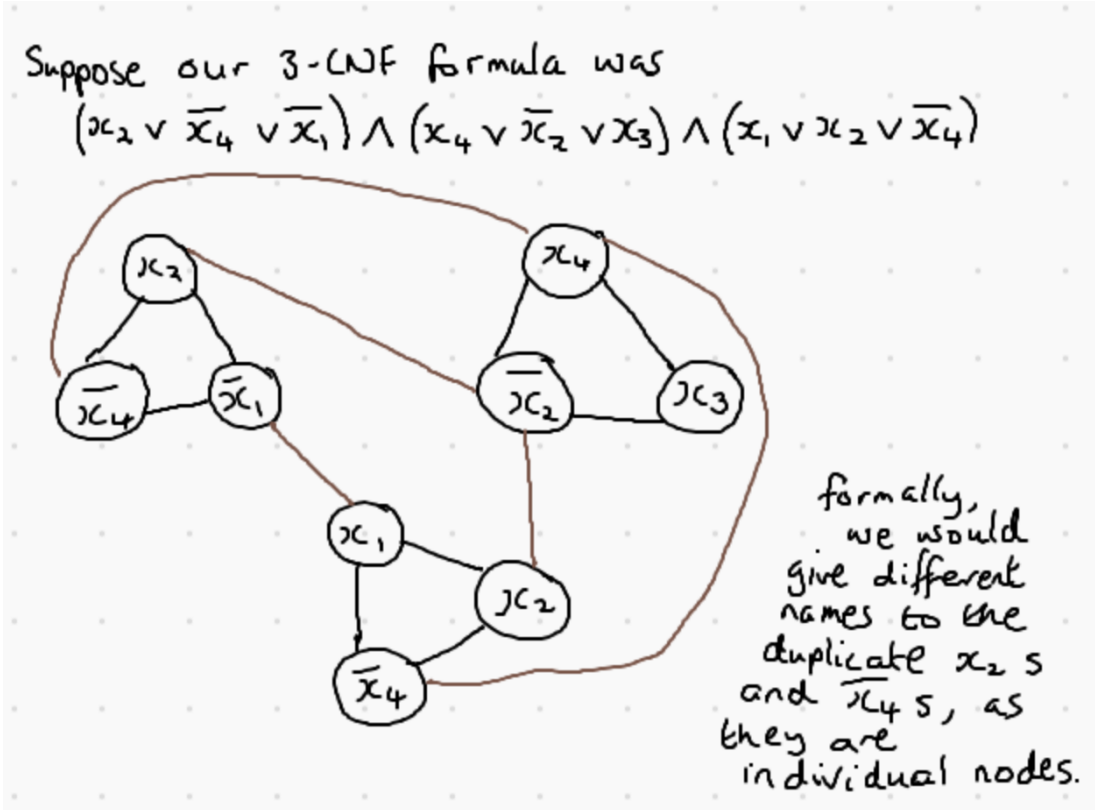
We are given $\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m$, and each of the C_j is $(\ell_{j,1} \vee \ell_{j,2} \vee \ell_{j,3})$ for three *literals* over $\{x_1, \ldots, x_n\}$ (for example, $\ell_{j,1} = x_4, \ell_{j,2} = \bar{x_1}, \ell_{j,3} = \bar{x_9}$).

Interested in an *assignment* to the $\{x_1, \ldots, x_n\}$ which makes *every* clause satisfied.

- C_j is satisfied if *at least one* of its literals are satisfied: $\ell_{j,1}$ or $\ell_{j,2}$, or $\ell_{j,3}$
- For every j, we will add nodes $\ell_{j1,}, \ell_{j,2}, \ell_{j,3}$ to our "Independent Set" graph. We will add edges to connect $\ell_{j,1}, \ell_{j,2}, \ell_{j,3}$ as a triangle ... so *at most one* of $\ell_{j,1}$, $\ell_{j,2}, \ell_{j,3}$ can be chosen to make C_j satisfied.
- For every variable x_i, add an edge between every *positive literal* $\ell_{j,\cdot} = x_i$ and every *negative literal* $\ell_{k,\cdot} = \bar{x_i}$ ... so if we choose x_i to satisfy some clause, the "opposite literal" cannot be used to satisfy any other clause.
- We will have a Independent Set of size $m \Leftrightarrow$ there is some satisfying assignment for ϕ.

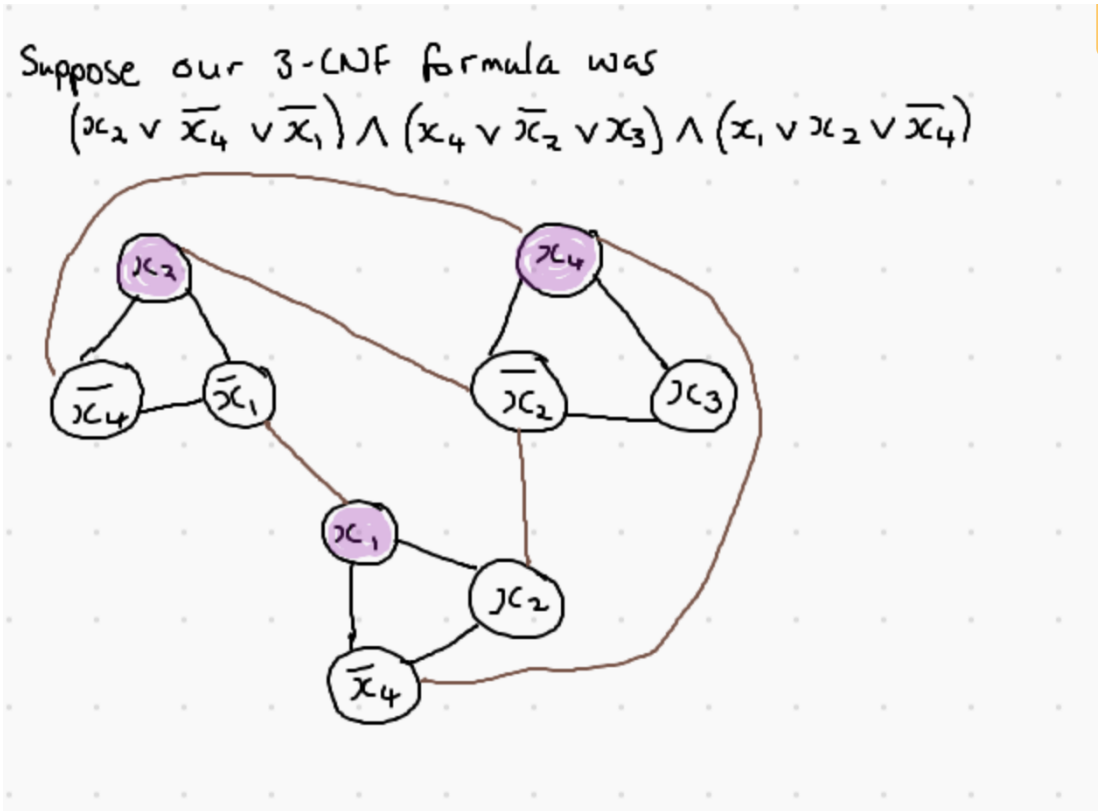
Example: Independent Set reduction

Suppose our 3-CNF formula was $(x_2 \vee \bar{x_4} \vee \bar{x_1}) \wedge (x_4 \vee \bar{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x_4})$.



Example: Independent Set solution

Suppose our 3-CNF formula was $(x_2 \vee \bar{x_4} \vee \bar{x_1}) \wedge (x_4 \vee \bar{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x_4})$.



Example: Independent Set is NP-complete

Size of our "reduction"

- Number of nodes is equal to the number of *literals* of the 3-CNF - $3m$
- Number of edges is at most $3m + 3\frac{m}{2}^2 = O(m^2)$

The construction of the graph (for the INDEPENDENT-SET instance $f(\mathcal{I})$) is methodical; can be done in polynomial-time:

- First build the "triangle" for each clause C_j and its literals.
  (Collectively, all these can be added to G in $O(m)$ time)
- Next iterate through each variable i adding all "clashing pair" edges for x_i.
  (Done naïvely, would take $O(m^2)$ for each x_i, so at most $O(n \cdot m^2)$ overall.)

We have argued why the constructed graph
will have an Independent Set of size $\geq m \Leftrightarrow$ our 3-CNF formula was satisfiable.

$$\text{3-SAT} \quad \leq_P \quad \text{INDEPENDENT SET}$$

Reading and Working

Reading:

Roughgarden 22.1, 22.2, 22.3, 22.4, 22.5. OR

KT (Kleinberg and Tardos) 8.3.Satisfiability

The following short video is worth a look: Short video about Cook-Levin

Working:

- We never showed that 3-SAT was NP-complete. Think about making a reduction from SAT to 3-SAT to show that 3-SAT is also NP-complete.