Introduction to Algorithms and Data Structures

Lecture 28: Dealing with NP-completeness (exhaustive search)

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Implications of NP-complete status

When prove a problem is NP-complete*, we no longer expect to be able to design polynomial-time algorithms to generate exact solutions (to the Decision problem or to an Optimization version)*

What are our options?

- Heuristic methods ("rules of thumb") that might not *guarantee* good results, but behave well in practice.
- Might there be a polynomial-time algorithm to search for an *approximate* solution rather than an exact one? (L25)
- Brute-force methods that run in exponential-time (today)
- Recursive backtracking (today)

Today we mostly dealing with the search version of the NP-complete problem - though we will resolve the decision question too.

We will demonstrate our methods wrt SAT.

“Brute force”

Might not have studied it formally but probably we know what it means:
“try all possibilities”

If our propositional formula is in CNF over n logical variables, then there are 2^n different assignments to iterate through.

We have $\Phi = C_1 \wedge \ldots \wedge C_m$

Consider a specific test assignment $\mathbf{x} = x_1 \ldots x_n \in \{0, 1\}^n$

- Each clause C_j can be checked for satisfiability wrt $\mathbf{x}$ in time $O(|C_j|)$.
- The formula Φ can be checked for satisfiability in time $\sum_{j=1}^{m} O(|C_j|) \; = \; O(|\Phi|)$.

So the full “brute force” algorithm could be carried out in $O(2^n \cdot |\phi|)$ time.

For a 3-CNF formula this would be $O(2^n \cdot m)$ time

Recursive backtracking (basic algorithm)

We set-up the exploration of the search space to exploit shared properties of the collection of the assignments $\mathbf{x} \in \{0,1\}^n$.

Work with respect to a "current partial assignment" $\mathbf{b} = b_i \mid i \in \mathcal{I}$.

Algorithm SAT-backtrack($\Phi = C_1 \wedge \ldots \wedge C_m, \mathcal{I}, \mathbf{b}$)

1. **if** $(m = 0)$ **then return** T
2. **else if** Φ contains an empty clause **then return** F
3. **else** choose an unassigned variable x_i $(i \in [n] \setminus \mathcal{I})$ how?
4. $\Phi' \leftarrow \Phi(x_i \leftarrow 0)$
   (simplifying Φ' based on this new assignment)
5. **if** SAT-backtrack($\Phi', \mathcal{I} \cup \{i\}, \mathbf{b} \cdot 0$)
6. **then return** T
7. **else** $\Phi' \leftarrow \Phi(x_i \leftarrow 1)$
   (simplifying Φ' based on this new assignment)
8. **return** SAT-backtrack($\Phi', \mathcal{I} \cup \{i\}, \mathbf{b} \cdot 1$)

Example: recursive backtracking

We take Φ over 4 variables $\{x_1, x_2, x_3, x_4\}$ with the 10 clauses

$$\begin{aligned}\Phi \quad = \quad & (x_1 \vee x_2) \wedge (x_1 \vee \bar{x_3}) \wedge (\bar{x_1} \vee \bar{x_2} \vee x_3) \wedge (\bar{x_2} \vee \bar{x_3}) \wedge (x_2 \vee x_3) \\ & \wedge (x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (\bar{x_2} \vee x_3 \vee x_4) \wedge (\bar{x_1} \vee \bar{x_4}) \wedge (\bar{x_3} \vee x_4) \wedge (x_1 \vee \bar{x_4})\end{aligned}$$

We work through this example in the following slides.

Example: recursive backtracking

$(x_1 \vee x_2)$

$(x_1 \vee \overline{x_3})$

$(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

$(\overline{x_2} \vee \overline{x_3})$

$(x_2 \vee x_3)$

$(x_1 \vee x_2 \vee x_3 \vee x_4)$

$(\overline{x_2} \vee x_3 \vee x_4)$

$(\overline{x_1} \vee \overline{x_4})$

$(\overline{x_3} \vee \overline{x_4})$

$(x_1 \vee \overline{x_4})$

Example: recursive backtracking

$(\cancel{x_1} \vee x_2)$

$(\cancel{x_1} \vee \bar{x}_3)$

✓ $(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

$(\bar{x}_2 \vee \bar{x}_3)$

$(x_2 \vee x_3)$

$(\cancel{x_1} \vee x_2 \vee x_3 \vee x_4)$

$(\bar{x}_2 \vee x_3 \vee x_4)$

✓ $(\bar{x}_1 \vee \bar{x}_4)$

$(\bar{x}_3 \vee \bar{x}_4)$

$(\cancel{x_1} \vee \bar{x}_4)$

Example: recursive backtracking

C_1 empty $(\cancel{x_1} \vee \cancel{x_2})$

$(\cancel{x_1} \vee \bar{x}_3)$

✓ $(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

✓ $(\bar{x}_2 \vee \bar{x}_3)$

$(\cancel{x_2} \vee x_3)$

$(\cancel{x_1} \vee \cancel{x_2} \vee x_3 \vee x_4)$

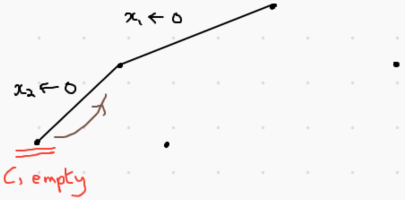
✓ $(\bar{x}_2 \vee x_3 \vee x_4)$

✓ $(\bar{x}_1 \vee \bar{x}_4)$

$(\bar{x}_3 \vee \bar{x}_4)$

$(\cancel{x_1} \vee \bar{x}_4)$

Example: recursive backtracking



$(\cancel{x_1} \vee x_2)$

$(\cancel{x_1} \vee \overline{x_3})$

✓ $(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

$(\overline{x_2} \vee \overline{x_3})$

$(x_2 \vee x_3)$

$(\cancel{x_1} \vee x_2 \vee x_3 \vee x_4)$

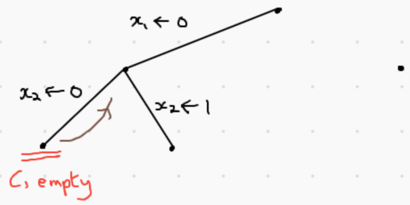
$(\overline{x_2} \vee x_3 \vee x_4)$

✓ $(\overline{x_1} \vee \overline{x_4})$

$(\overline{x_3} \vee \overline{x_4})$

$(\cancel{x_1} \vee \overline{x_4})$

Example: recursive backtracking



✓ $(\cancel{x_1} \vee x_2)$

$(\cancel{x_1} \vee \bar{x}_3)$

✓ $(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

$(\cancel{\bar{x}_2} \vee \bar{x}_3)$

✓ $(x_2 \vee x_3)$

✓ $(\cancel{x_1} \vee x_2 \vee x_3 \vee x_4)$

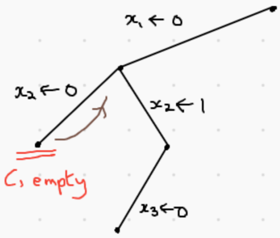
$(\cancel{\bar{x}_2} \vee x_3 \vee x_4)$

✓ $(\bar{x}_1 \vee \bar{x}_4)$

$(\bar{x}_3 \vee \bar{x}_4)$

$(\cancel{x_1} \vee \bar{x}_4)$

Example: recursive backtracking



✓ $(\cancel{x_1} \vee x_2)$

✓ $(\cancel{x_1} \vee \bar{x}_3)$

✓ $(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

✓ $(\cancel{\bar{x}_2} \vee \bar{x}_3)$

✓ $(x_2 \vee x_3)$

✓ $(\cancel{x_1} \vee x_2 \vee x_3 \vee x_4)$

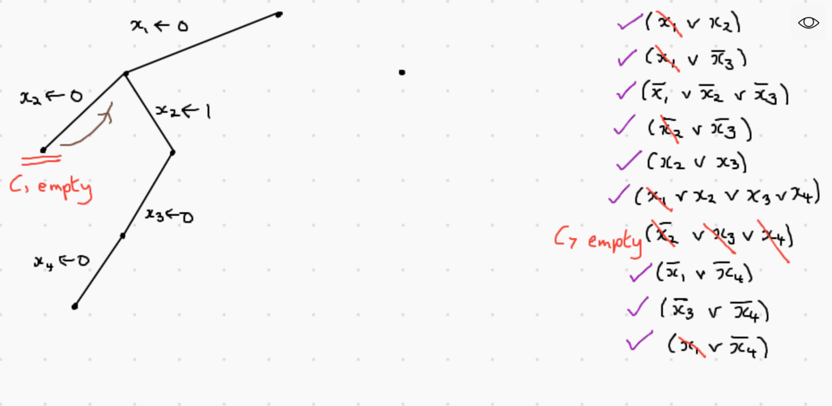
$(\cancel{\bar{x}_2} \vee \cancel{x_3} \vee x_4)$

✓ $(\bar{x}_1 \vee \bar{x}_4)$

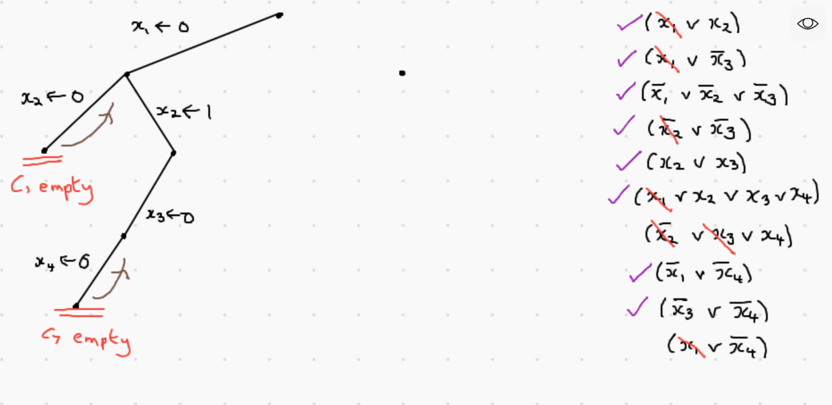
✓ $(\bar{x}_3 \vee \bar{x}_4)$

$(\cancel{x_1} \vee \bar{x}_4)$

Example: recursive backtracking



Example: recursive backtracking



Example: recursive backtracking

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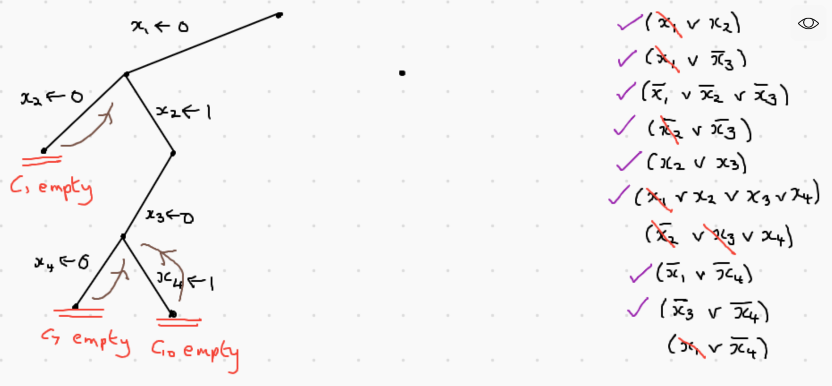
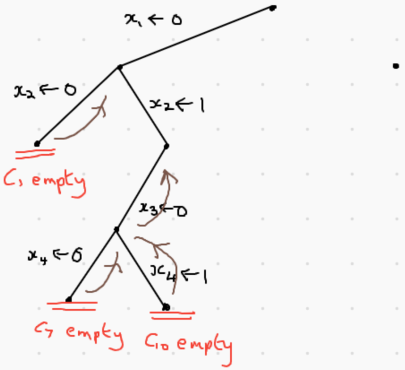
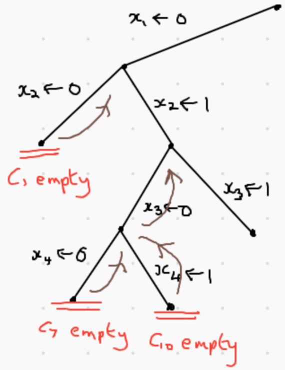


Example: recursive backtracking



$(\cancel{x_1} \vee x_2)$ ✓
$(\cancel{x_1} \vee \bar{x}_3)$
$(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$ ✓
$(\cancel{\bar{x}_2} \vee \bar{x}_3)$
$(x_2 \vee x_3)$ ✓
$(\cancel{x_1} \vee x_2 \vee x_3 \vee x_4)$ ✓
$(\cancel{\bar{x}_2} \vee x_3 \vee x_4)$
$(\bar{x}_1 \vee \bar{x}_4)$ ✓
$(\bar{x}_3 \vee \bar{x}_4)$
$(\cancel{x_1} \vee \bar{x}_4)$

Example: recursive backtracking



✓ $(\cancel{x_1} \vee x_2)$

C_2 empty $(\cancel{x_1} \vee \cancel{\overline{x_3}})$

✓ $(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

C_4 empty $(\cancel{\overline{x_2}} \vee \cancel{\overline{x_3}})$

✓ $(x_2 \vee x_3)$

✓ $(\cancel{x_1} \vee x_2 \vee x_3 \vee x_4)$

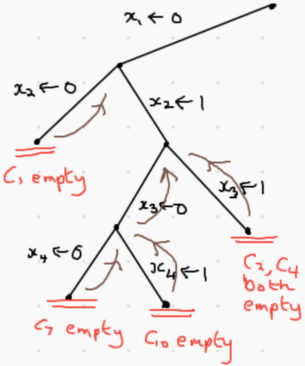
✓ $(\cancel{\overline{x_2}} \vee x_3 \vee x_4)$

✓ $(\overline{x_1} \vee \overline{x_4})$

$(\cancel{\overline{x_3}} \vee \overline{x_4})$

$(\cancel{x_1} \vee \overline{x_4})$

Example: recursive backtracking



$(\cancel{x_1} \vee x_2)$

$(\cancel{x_1} \vee \bar{x}_3)$

$(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

$(\cancel{\bar{x}_2} \vee x_3)$

$(x_2 \vee x_3)$

$(\cancel{x_1} \vee x_2 \vee x_3 \vee x_4)$

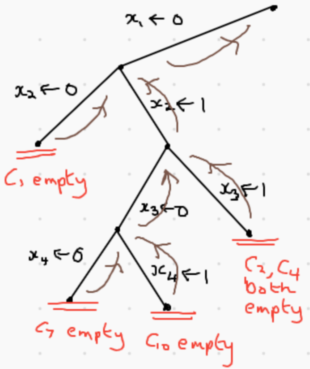
$(\cancel{\bar{x}_2} \vee x_3 \vee x_4)$

$(\bar{x}_1 \vee \bar{x}_4)$

$(\bar{x}_3 \vee \bar{x}_4)$

$(\cancel{x_1} \vee \bar{x}_4)$

Example: recursive backtracking



$(x_1 \vee x_2)$

$(x_1 \vee \overline{x_3})$

$(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

$(\overline{x_2} \vee \overline{x_3})$

$(x_2 \vee x_3)$

$(x_1 \vee x_2 \vee x_3 \vee x_4)$

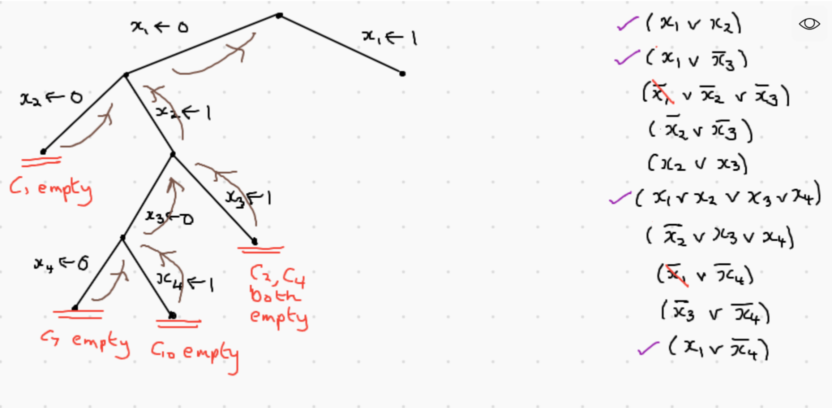
$(\overline{x_2} \vee x_3 \vee x_4)$

$(\overline{x_1} \vee \overline{x_4})$

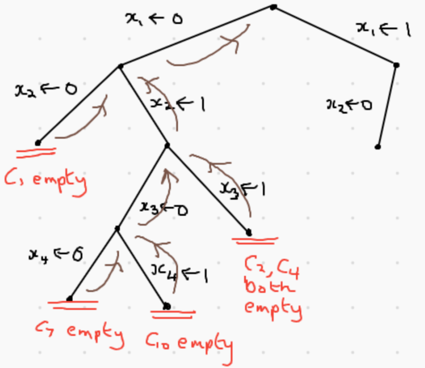
$(\overline{x_3} \vee \overline{x_4})$

$(x_1 \vee \overline{x_4})$

Example: recursive backtracking

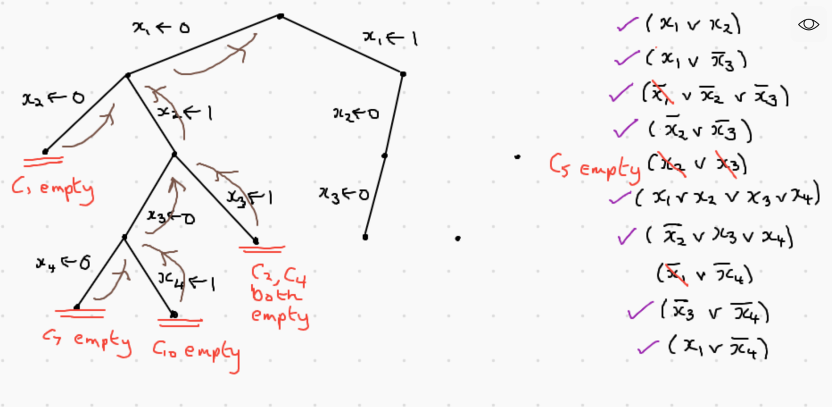


Example: recursive backtracking

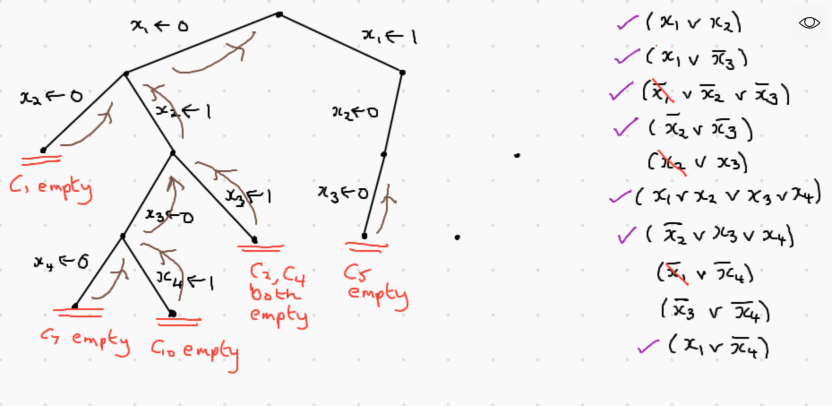


✓ $(x_1 \vee x_2)$
✓ $(x_1 \vee \overline{x}_3)$
✓ $(\cancel{\overline{x}_1} \vee \overline{x}_2 \vee \overline{x}_3)$
✓ $(\overline{x}_2 \vee \overline{x}_3)$
$(\cancel{x_2} \vee x_3)$
✓ $(x_1 \vee x_2 \vee x_3 \vee x_4)$
✓ $(\overline{x}_2 \vee x_3 \vee x_4)$
$(\cancel{\overline{x}_1} \vee \overline{x}_4)$
$(\overline{x}_3 \vee \overline{x}_4)$
✓ $(x_1 \vee \overline{x}_4)$

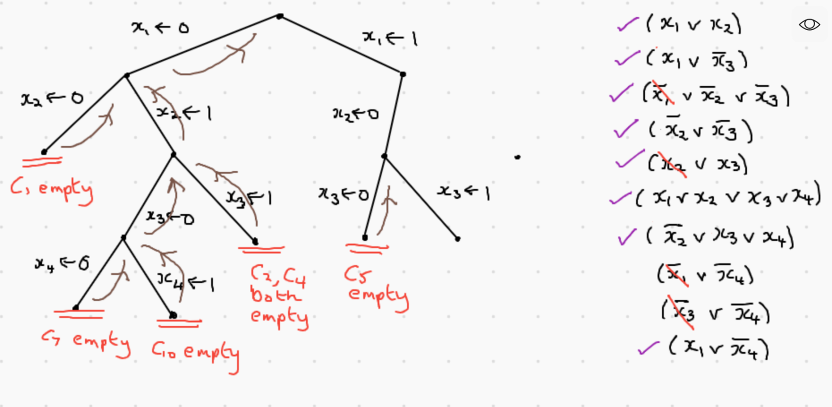
Example: recursive backtracking



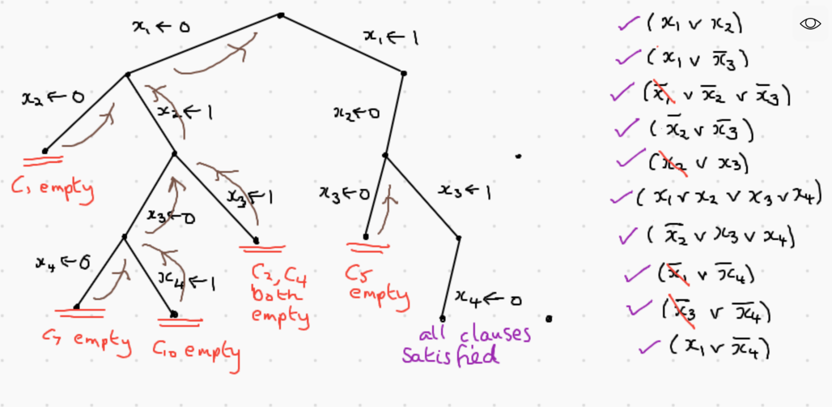
Example: recursive backtracking



Example: recursive backtracking



Example: recursive backtracking



Definitions

We work with respect to a general SAT formula $\Phi = C_1 \wedge \ldots C_m$ where each of the C_j contains between 1 and n literals over the variables $\{x_1, \ldots, x_n\}$.

- A unit clause is a clause which contains *exactly one literal*

  (for example, $C_j = (x_2)$ or $C_j = (\bar{x_7})$)
- A pure literal is defined with respect to the whole collection of clauses: the situation when a logical variable x_i appears with only one polarity, either
  - *only appearing* as the positive literal x_i ...
  - or alternatively *only appearing* as the negative literal $\bar{x}_i$

  (but not with both signs)
- Both definitions apply with respect to "active" clauses/literals in a partial assignment (some variables set), as well as the original CNF.

Some observations

We can make some observations to improve recursive backtracking:

unit clause
: If Φ contains a unit clause (ℓ), then there is only *one possible option* for setting the underlying variable in a fully Satisfying assignment:

  If ℓ is x_i, then set $x_i \leftarrow 1$, else if ℓ is $\bar{x_i}$, set $x_i \leftarrow 0$.

pure literal
: If we have some variable x_i that always appears with the same polarity (always as x_i, or alternatively always as $\bar{x_i}$) ... then we may assume we set x_i is set to its polarity in a satisfying assignment.

  There might also be a satisfying assignment setting x_i in conflict with its polarity, but it cannot *hurt* to match the polarity.

Applying these rules will alter the Φ, removing some clauses (unit clause) or reducing the number of literals in other clauses.
We should iterate

Davis, Putnam, Logemann, Loveland (DPLL)

Algorithm DPLL($\Phi = C_1 \wedge \ldots \wedge C_m$)

if every literal in Φ is "pure" **then return** T
else if Φ contains an empty clause **then return** F
else
    while we have some "unit clause" (ℓ) in Φ
        if (ℓ is x_i) **then** $\Phi \leftarrow \Phi(x_i \leftarrow 1)$
        else if (ℓ is $\bar{x}_i$) **then** $\Phi \leftarrow \Phi(x_i \leftarrow 0)$
    while we have some "pure literal" ℓ in Φ
        if (ℓ is x_i) **then** $\Phi \leftarrow \Phi(x_i \leftarrow 1)$
        else if (ℓ is $\bar{x}_i$) **then** $\Phi \leftarrow \Phi(x_i \leftarrow 0)$
    Choose a undetermined variable x_i of Φ how?
    return (DPLL($\Phi(x_i \leftarrow 0)$) **or** DPLL($\Phi(x_i \leftarrow 1)$))

$\Phi(x_i \leftarrow 0)$ eliminates the clauses that contain $\bar{x}_i$ (now satisfied), and deletes any x_i literals inside individual clauses of Φ (clauses become harder to satisfy). $\Phi(x_i \leftarrow 1)$ is symmetric.

Choosing a variable to "split" on

The process of forking-off $\mathrm{DPLL}(\Phi(x_i \leftarrow 0))$ and $\mathrm{DPLL}(\Phi(x_i \leftarrow 1))$ is called splitting on x_i.

There are different ways used to choose the next x_i:

- Any variable still active in the "as-yet-unsatisfied" clauses (our approach).
- Variable that appears in the most clauses.
- Variable x_1 that a lot, *mostly* with one polarity (match that polarity)
- Any literal in the shortest clause.
- The variable x_i with the highest weighted sum of clause sizes $\sum_{k=2}^{n} 2^{-k} \left|\{C_j : x_i \in C_j, |C_j| = k\}\right|$.

These are all *heuristics* - will work well on some instances of SAT, but may be poor choices for other examples.

Impact

Running time of DPLL:

- The bound of $O(2^n \cdot |\phi|)$ still applies.
- Running-time in practice is *much much better* than this.
- Proving improved upper bounds:
  Can't really hope for significant improvements (SAT being NP-complete).

DPLL said to be "a collection of algorithms" (the different heuristics for splitting).

DPLL forms the basis of many practical "SAT solvers" like GRASP and Chaff.

Further study

Viewing:
The “Sudoku problem” can also be addressed via recursive backtracking.

Doing:
Run the DPLL backtracking algorithm on the example from these slides.

You will notice it is much quicker than the naïve backtracking algorithm (especially on the $x_1 \leftarrow 0$ branch).