AGTA Tutorial 1- Solutions

Lecturer: Aris Filos-Ratsikas Tutor: Charalampos Kokkalis

February 4, 2025

Exercise 1 (Dominant and Dominated Strategies)**.** Consider the following simple game. Alice and Bob have ten different game theory books to divide between them and each one has a strict preference over them, i.e. they are not indifferent between any two books. They come up with the following solution. They will both state their complete preferences over the books and then Alice will select her five favourite books first and Bob will get the rest.

A. What are the strategy spaces S_A and S_B of Alice and Bob respectively?

B. How many strictly dominant and how many weakly dominant strategies does Alice have?

C. How many strictly dominant and how many weakly dominant strategies does Bob have?

D. Does Alice have any strictly dominated strategies? What about Bob?

E. Does Alice have any weakly dominated strategies? What about Bob?

Solution 1. We provide the answers to A-E below.

A. Alice and Bob specify their preferences over all the possible books. The strategy spaces S_A and S_J for both Alice and Bob are all the permutations of the ten books. For example, if we use $\text{book}_1 \succ^A \text{book}_2$ to denote that Alice prefers book 1 to book 2, then one possible strategy of Alice would be $\text{book}_3 \succ^A \text{book}_8 \succ^A \text{book}_6 \succ^A \ldots \succ^A \text{book}_1 \succ^A \text{book}_7$.

B. Alice does not have any strictly dominant strategies. Note that as long as she makes sure that her five favourite books are ranked (in any order) in one of the first five spots, the ranking for the remaining books is irrelevant. Therefore, there are multiple strategies that give Alice the highest possible payoff, regardless on what Bob does. In other words, any strategy that ranks the five favourite books in the first five places in any order is a weakly dominating strategy. There are $5! \cdot 5!$ such strategies.

C. Notice that what Bob says actually does not matter at all, as the final allocation is a function of only Alice's strategy. Therefore, Bob does not have any strictly dominant strategies but all of her strategies are weakly dominant.

D. Alice has some strategies that are strictly dominated. Specifically, ranking one of her five favourite books in any position other than the first five positions is strictly dominated, because she will receive a worse book that she would, if she placed her five favourite books in the first five places. Bob does not have any strictly dominated strategies, since all of her strategies are weakly dominant.

E. Alice has several weakly dominated strategies. First of all, her strictly dominated strategies described above are also weakly dominated by definition. Also, any strategy that ranks her five favourite books in the first five positions is weakly dominated by any other strategy that does the same (regardless of the ordering of the first five positions and the ordering of the rest of the positions). All of Bob's strategies are weakly dominated.

Exercise 2 (Second price auction)**.** Consider the following auction scenario. There is an item for sale and n interested bidders; the *valuation* of bidder i for the item is v_i, which represents how many pounds the bidder would be willing to spend on buying the item.

In a *second price auction*, the auctioneer asks the bidders to report their valuations v_i and then sells the item to the bidder with the highest bid at a price p equal to the second highest bid (break ties arbitrarily). All other agents (who do not receive the item) are charged 0. The payoff of any bidder i is 0 if she does not receive the item and $v_i - p$ if she does.

A. What is the (pure) strategy space S_i of each bidder i?

B. Show that for any bidder i, there is a strategy s_i that weakly dominates any strategy $s'_i > v_i$.

C. Show that for any bidder i, there is a strategy s_i that weakly dominates any strategy $s'_i < v_i$.

D. Does this game has a weak dominant strategy equilibrium? If not, explain your answer. If yes, state the equilibrium profile.

Solution 2. We provide the answers to A-D below.

A. The pure strategy space S_i of each bidder i is the set of all non-negative real numbers $\mathbb{R}_+$ as the bidder is allowed to bid any positive number. This is an example of a *continuous strategy space*, which is different from the discrete strategy spaces that we saw in matrix games.

B. Consider the strategy $s_i = v_i$ in which bidder i bids his true value for the item. Consider any strategy $s'_i < v_i = s_i$. First, if the bidder does not win the item, his payoff is 0 (he does not pay anything). The payoff from $s_i = v_i$ is at most 0, since the bidder will never be asked to pay more than his valuation and therefore in that case $s_i = v_i$ is at least as good as any s'_i. Secondly, if the bidder wins the item with bid s'_i, then he will have to pay a price equal to the next highest bid b_j. If $b_j \geq v_i$, then the bidder has a payoff of at most 0 and therefore $s_i = v_i$ is at least as good. If $b_j < v_i$, then using strategy $s_i = v_i$, bidder i still wins the item and still pays a price of b_j, resulting in the same payoff. In all, the bidder can not get a higher payoff when playing any strategy $s'_i > v_i$ compared to playing v_i and therefore the strategy $s_i = v_i$ weakly dominates any such s'_i.

C. The argument is very similar and again we consider the strategy $s_i = v_i$. If the bidder does not win the item, his payoff is at most 0 which is also the payoff of $s_i = v_i$. If he wins the item, he either has payoff at most 0 or he would win the item if he had bid $s_i = v_i$ and he would have payed the same price. In all, the strategy $s_i = v_i$ weakly dominates any strategy $s'_i < v_i$.

D. The game always has a weak dominant strategy equilibrium in which all bidders bid their true values v_i, i.e. the equilibrium profile is $v = (v_1, v_2, \ldots, v_n)$. This follows from the previous two arguments applied to each player individually. In such games where telling the truth is a (weak) dominant strategy equilibrium, we will say that the implemented mechanism or auction is *truthful* or *strategyproof*. The truthfulness of the second price auction (also known as the Vickrey auction) is a celebrated result in economics.

Exercise 3 (Iteratively Removing Dominated Strategies)**.** Consider the game given by the utility bi-matrix. Reduce this game to a 1×1 game by iteratively removing strictly dominated strategies.

	C1	C2	C3	C4	C5
R1	4,-1	3,0	-3,1	-1,4	-2,0
R2	-1,1	2,2	2,3	-1,0	2,5
R3	2,1	-1,1	0,4	4,-1	0,2
R4	1,6	-3,0	-1,4	1,1	-1,4
R5	0,0	1,4	-3,1	-2,3	-1,-1

Solution 3. The order of elimination is R4 (by R3), C1 (by C5), R5 (by R2), C2 (by C3), R1 (by R3), C4 (by C3), R3 (by R2), C3 (by C5) and we are left with (R2,C5) with a payoff of $(2, 5)$.

Exercise 4 (Finding all Mixed Nash Equilibria)**.** Consider the game given by the following utility bi-matrix.

	C1	C2	C3	C4
R1	7,3	6,3	5,5	4,7
R2	4,2	5,7	8,6	5,8
R3	6,1	3,8	2,4	5,9

A. Consider the mixed strategies $x_1 = (1/4, 1/2, 1/4)$ and $x_2 = (2/3, 1/3, 0, 0)$ for Player 1 and Player 2, respectively. What is the expected utility $u_1(x_1, x_2)$ of Player 1 from the mixed strategy profile (x_1, x_2). Write the explicit formula for computing the utility and show the calculations in detail.

B. Find all the mixed Nash equilibria of this game. Which of those are pure Nash equilibria?

Solution 4. We provided the solutions to A and B below.

A. The expected utility of Player 1 from (x_1, x_2) is given by

$$\begin{aligned}
u_1(x_1, x_2) &= \sum_{i \in S_i,\ j \in S_2} x_1(i)\ x_2(j)\ u_1(i,j) = \sum_{i \in S_i} \sum_{j \in S_2} x_1(i)\ x_2(j)\ u_1(i,j) \\
&= \frac{1}{4}\left(\frac{2}{3}u_1(1,1) + \frac{1}{3}u_1(1,2)\right) + \frac{1}{2}\left(\frac{2}{3}u_1(2,1) + \frac{1}{3}u_1(2,2)\right) + \frac{1}{4}\left(\frac{2}{3}u_1(3,1) + \frac{1}{3}u_1(3,2)\right) \\
&= \frac{1}{4}\left(\frac{2}{3}\cdot 7 + \frac{1}{3}\cdot 6\right) + \frac{1}{2}\left(\frac{2}{3}\cdot 4 + \frac{1}{3}\cdot 5\right) + \frac{1}{4}\left(\frac{2}{3}\cdot 6 + \frac{1}{3}\cdot 3\right) \\
&= 61/12
\end{aligned}$$

B. We first observe that strategy C4 of Player 2 strictly dominates all other strateges of Player 2. This means that we can remove strategies C1,C2, and C3, and obtain the following reduced bimatrix.

	C4
R1	4,7
R2	5,8
R3	5,9

Furthermore, we now observe that strategy R1 of Player 1 is dominated by both strategies R2 and R3, so we can also remove it, and obtain the following reduced bimatrix.

	C4
R2	5,8
R3	5,9

Player 2 obviously has only one strategy (C4), so in any MNE, we will have $x_2(C4) = 1$, i.e., the strategy of Player 2 will be $x_2 = (0, 0, 0, 1)$. On the other hand, either strategy R2 or strategy R3 result in the same utility for Player 1, so any probability mixture of the two maximises the utility for the player. Therefore, any profile where x_2 is as above and $x_1 = (0, p, 1-p)$ for any $p \in [0, 1]$ is a MNE. Out of those, there are two PNE, namely (R2,C4) and (R3,C4).

Exercise 5 (Guess Half the Average)**.** Consider the following game called "Guess Half the Average", amongst n players, with $n > 1$: Each player independently chooses (guesses) a whole number between 1 and 1000. The person that guessed the number that is closest to *half the average* of the chosen numbers wins this game (for a utility of 1), and everyone else loses the game (for a utility of 0). If there are multiple players that have the same closest to half the average guess (call those *winners*), then we use one of the following tie-breaking rules:

- Each winner gets a utility of $1/k$, where k is the number of winners.
- Each winner gets a utility of 1.

A. Discuss how you would play each of those two versions of the game if you were playing them with your classmates from the AGTA course and why.

B. For both version of the game, find all the pure Nash equilibria. Argue why there are no other pure Nash equilibria besides those that you have found.

Solution 5. Let's first consider the tie-breaking rule in which all players who are closest to half the average split the payoff equally. We claim tha the unique PNE of the game is when all players choose 1. This profile is clearly a PNE, as any unilateral deviation to another choice would result in a payoff of 0 rather than $1/n$. To show that this PNE is unique, we reason as follows. Assume by contradiction that there exists a different PNE of this game $s = (s_1, \ldots, s_n)$, where s_i is the choice of Player i. Let $k > 1$ be the largest number that any player chooses; such a k exists by assumption on the profile s. Let i^* be a Player that chooses k. The only way for Player i^* to achieve non-zero utility is if all other players also choose k, i.e., $s_i = k$ for all players $i \in N$, in which case Player i achieves utility $1/n$. But in that case, Player i^* can switch to choosing $s_{i^*} = k - 1$ and increase her utility from $1/n$ to 1.

Now let's consider the second tie-breaking rule, in which each winner gets a utility of 1. Here we should expect to have more equilibria. Specifically, for any $k \in \{1, \ldots, 1000\}$, consider the profile $s = (s_1, \ldots, s_n)$, where $s_i = k$ for every Player i. Everyone receives a maximum utility of 1, so there is obviously no beneficial deviation to another strategy, and s is a PNE. Next we argue that these are the only PNE of this game. Assume by contradiction that we have a different PNE s', in which there exist at least two Players i and j such that $s'_i \neq s'_j$. Again, let k me the maximum number chosen, and let k' be the second-to-largest number; these numbers exist by the assumption $s'_i \neq s'_j$. We claim that k' must be strictly closer to half the average, and hence the player that chose k would prefer to deviate to choosing k' instead, contradicting the fact that s is a PNE. To see that, observe that, if we let h denote half the average, it clearly holds that $h < k/2$, since k is the largest chosen number. This means that any number $k' \in [1, k)$ is strictly closer to h than k.