AGTA Tutorial 2 - Solutions

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Exercise 1. Consider the (finite) 2-player zero-sum game given by the following utility matrix A, where the utilities of the maximiser (the row player) are indicated.

	C1	C2	C3	C4	C5
R1	4	2	9	2	5
R2	6	3	5	9	7
R3	1	4	8	5	7
R4	5	1	3	5	6

A. Specify the linear programs that compute the optimal strategies for the maximiser (the primal) and for the minimiser (the dual).

B. Using the "educated guess" approach that we saw in the lectures, solve this game by hand to find the optimal strategies for the two players and the value of the game.

Hint: You may want to first try to simplify the game by eliminating redundant pure strategies.

Solution 1. Before we answer questions A and B, we will first perform elimination of strictly dominated strategies to reduce the game to a smaller game. We observe that strategy C5 of the minimiser is strictly dominated by strategy C2, therefore it can be eliminated. We have the following matrix:

	C1	C2	C3	C4
R1	4	2	9	2
R2	6	3	5	9
R3	1	4	8	5
R4	5	1	3	5

Similarly, the strategy R4 of the maximiser is strictly dominated by the strategy R2, so we can also eliminate it, to obtain the matrix:

	C1	C2	C3	C4
R1	4	2	9	2
R2	6	3	5	9
R3	1	4	8	5

Next, we observe that now strategy C3 of the minimiser is strictly dominated by C2, so we also eliminate it:

	C1	C2	C4
R1	4	2	2
R2	6	3	5
R3	1	4	5

Now R1 is strictly dominated by R2, so we eliminate it:

	C1	C2	C4
R2	6	3	5
R3	1	4	5

Finally, C4 is strictly dominated by C2 now, so we eliminate it to obtain the final 2×2 matrix:

	C1	C2
R2	6	3
R3	1	4

A. We write the linear programs for the maximiser (the primal) and the minimiser (the dual) respectively:

$$\begin{array}{ll} \text{maximise} & v \\ \text{subject to} & 6x_1 + x_2 \geq v \\ & 3x_1 + 4x_2 \geq v \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{array} \qquad \begin{array}{ll} \text{minimise} & u \\ \text{subject to} & 6y_1 + 3y_2 \leq u \\ & y_1 + 4y_2 \leq u \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0 \end{array}$$

B. Our educated guess is that each player is going to play all strategies in (R1,R2) and (C1,C2) in their support. Using this, we can solve the system of linear inequalities for each player.

For the maximiser, we have

$$6x_1 + x_2 = 3x_1 + 4x_2$$
$$x_1 + x_2 = 1$$

By substituting $x_2 = 1 - x_1$ above, we obtain $5x_1 + 1 = 4 - x_1 \Rightarrow x_1 = 1/2$ and then $x_2 = 1/2$. In that case $v = 7/2$.

For the minimiser, we have

$$6y_1 + 3y_2 = y_1 + 4x_2$$
$$y_1 + y_2 = 1$$

By substituting $y_2 = 1 - y_1$ above, we obtain $3y_1 + 3 = 4 - 3y_1 \Rightarrow y_1 = 1/6$ and then $y_2 = 5/6$. In that case $u = 7/2$.

Since $v = u$, our educated guess was correct, and the value of the game is 7/2.

Exercise 2. Alice and Bob are playing a "match the card colour" game. Alice holds a black Ace card and a red 8 card. Bob holds a red 2 card and a black 7 card. They simultaneously choose on of their cards. If the chosen cards are of the same colour, Alice wins the game, otherwise Bob wins the game. The amount won is a number of GBP equal to the number on the winner's card (the Ace card counts as 1).

A. Model this game as a (finite) 2-player zero-sum game with Alice being the maximiser and Bob being the minimiser.

B. Using the "educated guess" approach that we saw in the lectures, solve this game by hand to find the optimal strategies for the two players and the value of the game.

Solution 2. We provide the answers to A and B below.

A. The game can be modelled using the following utility matrix:

	R2	B7
BA	-2	1
R8	8	-7

B. Our educated guess is that each player is going to play all strategies in their support. Using this, we can solve the system of linear inequalities for each player.

For Alice we have

$$-2x_1 + 8x_2 = x_1 - 7x_2$$
$$x_1 + x_2 = 1$$

By substituting $x_2 = 1 - x_1$ above and solving, we obtain $x_1 = 15/18$ and $x_2 = 3/18$.

For Bob we have

$$-2y_1 + y_2 = 8y_1 - 7y_2$$
$$y_1 + y_2 = 1$$

By substituting $y_2 = 1 - y_1$ above and solving, we obtain $y_1 = 10/18$ and $y_2 = 8/18$.

To verify our guess, we compute the expected payoff of the maximiser / loss of the minimiser when using the strategies above, and find that it is $-1/3$ for Alice and $-1/3$ for Bob. Since these are equal, our guess was correct, and the value of the game is $-1/3$.

Exercise 3. Consider the (finite) 2-player game given by the following utility bimatrix.

	C1	C2	C3	C4
R1	7,3	6,4	5,5	4,7
R2	4,2	7,9	8,6	8,8
R3	6,1	9,7	2,4	6,9

A. First, simplify the game by eliminating redundant pure strategies, if any.

B. Compute all the MNE of the game (by hand), by formulating an appropriate system of linear equations and solving them. Justify why your system of linear equations actually computes MNE of the game.

Solution 3. We provide the answers to A and B below.

A. We observe first that C1 is strictly dominated by C2 for Player 2. This gives us the following bimatrix:

	C2	C3	C4
R1	6,4	5,5	4,7
R2	7,9	8,6	8,8
R3	9,7	2,4	6,9

Next, we observe that R1 is strictly dominated by R2 for Player 1. This gives us the following bimatrix:

	C2	C3	C4
R2	7,9	8,6	8,8
R3	9,7	2,4	6,9

Now we observe that C3 is strictly dominated by C4 for Player 2. This gives us the following bimatrix:

	C2	C4
R2	7,9	8,8
R3	9,7	6,9

B. We first verify that there are no PNE in this game. Indeed, if any player plays a pure strategy, then it can be seen by inspection of the utility matrix that the unique best response to that would be a pure strategy as well. Therefore, we will guess that any MNE of the game is fully mixed, i.e., all pure strategies for both players are in the support of the mixed equilibrium strategies. Let (x_1, x_2) be the equilibrium strategy of Player 1 and (y_1, y_2) be the equilibrium strategy of Player 2.

We know from the lectures (Proposition 2) that every pure strategy in the support of a mixed equilibrium strategy must yield the same payoff to the player. From this, we obtain that, from the perspective of Player 1, we have:

$$7y_1 + 8y_2 = 9y_1 + 6y_2$$
$$y_1 + y_2 = 1$$

Solving this system, we obtain that $y_1 = 1/2$, $y_2 = 1/2$. Similarly, using Proposition 2 from the perspective of Player 2, we have:

$$9x_1 + 7x_2 = 8x_1 + 9x_2$$
$$x_1 + x_2 = 1$$

Solving this system, we obtain that $x_1 = 2/3$, $x_2 = 1/3$.