AGTA Tutorial 3

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Exercise 1. Assume that you have developed 10 different algorithms for finding mixed Nash equilibria in games. You would like to test these algorithms on 100 instances. In particular, the running time of an algorithm on the set of 100 instances is its maximum running time over all instances in the set. You may assume that for an (algorithm, instance) pair, the running time is always the same.

A randomised algorithm is a probability distribution over your 10 algorithms, i.e., a probability vector $(p_1, \ldots, p_{10})$, where p_i for $i \in \{1, \ldots, 10\}$ is the probability of selecting algorithm i out of your 10 algorithms. Every randomised algorithm naturally has an expected running time on an instance, and its running time is the worst-case expected running time over all instances in the set.

Describe a polynomial-time algorithm for finding the best randomised algorithm for the 100 instances, i.e., the one with the smallest expected running time.

Exercise 2. Consider the following linear program:

$$
\begin{array}{lll}
\text{maximise} & v & \\
\text{subject to} & v - 2x_1 - 7x_2 & \leq 0 \\
& v - 9x_1 & \leq 0 \\
& v - 4x_1 - 3x_2 & \leq 0 \\
& x_1 + x_2 & = 1 \\
& x_1, x_2 & \geq 0
\end{array}
$$

A. Write down the dual to this linear program.

B. Describe a 2-player zero-sum game (by writing down its utility matrix) that is consistent with this linear program computing an optimal strategy of the maximiser and the dual computing an optimal strategy of the minimiser.

Exercise 3. Solve the following linear program using the simplex method.

$$
\begin{array}{ll}
\text{maximise} & 6x_1 + 6x_2 + 5x_3 + 9x_4 \\
& \\
\text{subject to} & 2x_1 + x_2 + x_3 + 3x_4 \leq 5 \\
& x_1 + 3x_2 + x_3 + 2x_4 \leq 3 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

Show each dictionary and each basic feasible solution produced during the execution of the algorithm. Explain which variable is the entering variable and which one is the leaving variable and why.

Exercise 4. Consider the following linear program.

$$
\begin{array}{ll}
\text{maximise} & 2x_1 + x_2 \\
& \\
\text{subject to} & 2x_1 + x_2 \leq 4 \\
& 2x_1 + 3x_2 \leq 3 \\
& 4x_1 + x_2 \leq 5 \\
& x_1 + 5x_2 \leq 1 \\
& x_1, x_2 \geq 0
\end{array}
$$

A. Solve the LP above using the simplex method. Show each dictionary and each basic feasible solution produced during the execution of the algorithm. Explain which variable is the entering variable and which one is the leaving variable and why.

B. Solve the LP above by drawing the feasible region in two dimensions and checking the objective function value on each of its corners.

Exercise 5 (Game of Chicken)**.** Consider the following game of *Chicken*, where two cars are headed towards each other in opposite directions at full speed. Each driver can choose to either swerve, (or "chicken") or to continue going straight, (or "dear"). If they both dear, they crush and their cars get destroyed. If they both chicken, they both leave unharmed, but they don't get the satisfaction of besting their opponent. If one chickens and the other dares, then the daring driver is the winner. The utility bimatrix of the game is given below.

Driver 1 / Driver 2	**Chicken**	**Dare**
Chicken	6,6	2,7
Dare	7,2	0,0

A. Find all the pure Nash equilibria of the game.

B. Find a mixed Nash equilibrium of the game which is not a pure Nash equilibrium.

C. Assume that some trusted party (e.g. a traffic warden or a traffic light) is giving the two players some *advice* on how to play the game. In particular, the party first announces to the players that the possible strategy profiles of the game are either (C,D), (D,C) or (C,C) and each one happens with equal probability 1/3. Then, the party chooses one of the three profiles at random and lets the players know of their strategies in that profile, without letting them know of the strategy of the other player. We will consider some different kind of equilibrium, where the the players do not want to deviate from their prescribed advice, assuming their opponent is following the advice.

- Assume that Player 1 witnesses the advice "Dare". Prove that assuming that Player 2 sticks to his prescribed strategy (whatever that is), Player 1 does not want to deviate from playing "Dare".
- Assume that Player 1 witnesses the advice "Chicken". Prove that assuming that Player 2 sticks to his prescribed strategy (whatever that is), Player 1 does not want to deviate from playing "Chicken". Here, not wanting to deviate from playing "Chicken" means that his expected payoff from playing "Chicken" (over the randomness of the profile distribution of the trusted party) is higher than that of playing "Dare".

D. Compare the total utility (sum of players' utilities) of all the equilibria that you have computed. What do you observe?

Exercise 6 (Bonus Coding Exercise)**.** In your programming language of choice, code a solver for 2-player zero-sum games. Your solver should take as input a utility matrix in appropriate format (e.g., in Python this could be a list of lists) and output the optimal strategies for the maximiser and the minimiser, as well as the value of the game.