Introduction to Algorithms and Data Structures

Tutorial 2

your tutor

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Q1: arithmetic operations of $\mathbf{Expmod}$

The task is to evaluate

$$a^n \bmod m \quad \text{as} \quad \mathbf{Expmod}(a, n, m) \text{ (different methods)},$$

and then from that

$$\mathbf{ProbablePrime}(n) \ = \ (\mathbf{Expmod}(2, n-1, n) \stackrel{?}{=} 1)$$

How many arithmetic operations $(\times, +, -, \mathrm{div}, \mathrm{mod})$ will be used?

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Method B:

Expmod (a,n,m):
    b=1
    for i = 1 to n
        b = (b × a) mod m
    return b

Method C:

Expmod (a,n,m):
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- We do n iterations $\Rightarrow$ exactly $3n$ arithmetic operations. So $\Theta(n)$.

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For **ProbablePrime**(n) (we have $a = 2$), we also have to subtract 1 from n to set up the computation $\mathbf{Expmod}(2, n-1, n)$ ... $\Rightarrow$
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note 1: What about the other operations?
note 2: Constants for the $O(\cdot)$ and the $\Omega(\cdot)$?

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- 3-4 arithmetic operations at each "level" ($\times$, mod, div, maybe a 2nd $\times$)
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note: Constants?

Q2: Bubblesort

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BubbleSort (A):
    for i = 1 to |A| − 1
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            if A[j] > A[j+1]
                swap A[j] and A[j+1]
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- Run an example with (say) $18, 11, 2, 9, 8$
- Discuss the "sweeps"

Bubblesort example

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  - If $k < |\mathrm{A}| - 1$, then when $j = k$, this larger element will be swapped into position $k + 1$.
    For *every* $j = k + 2, \ldots$, this largest value gets swapped right until j reaches $|\mathrm{A}| - 2$, when the element gets swapped into $|\mathrm{A}| - 1$.

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  - If $k < |\mathsf{A}| - 1$, then when $j = k$, this larger element will be swapped into position $k + 1$. For *every* $j = k + 2, \ldots$, this largest value gets swapped right until j reaches $|\mathsf{A}| - 2$, when the element gets swapped into $|\mathsf{A}| - 1$.

claim: *After $|A| - 1$ sweeps (ie $n - 1$ sweeps), the array will be fully sorted.*

"Invariant" is that after i sweeps, the largest i elements $x_1 \leq x_2 \leq \cdots \leq x_i$ are in sorted order in the top i positions.

note: Talk about invariants.

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- Each "sweep" is of length n, with $n-1$ comparisons, maybe some swaps.
- As "sweeps" progress, the big items gather (in sorted order) up the top.
- In the later sweeps, most comparisons are redundant, few swaps happen.

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Best-case is already-sorted order, worst-case reverse-sorted order.

Only affects swaps though. And overall both cases are $\Theta(n^2)$

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```
BubbleSort2(A):
    i = 1
    repeat
        i = i+1
        flg = false
        for j = 0 to |A|−i
            if A[j] > A[j+1]
                swap A[j] and A[j+1]
                flg = true
    until flg = false
```

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- We will always do $|A| - 1 - i$ swaps on the i-th iteration (the highest item for 0, 1, . . ., $|A| - 1 - i$ is in position 0).
- So we run every "sweep" (length n, $n-1$, . . ., 2)
- $\sum_{i=1}^{n-1} i = \Theta(n^2)$ like Insertsort.

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best case occurs when A is already sorted.

- Now the *first* sweep does 0 swaps, so we can stop immediately.
- $\Theta(n)$ best-case running-time.

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So for any "inversion" i,j we will see a specific comparison of these items (and it can only be this exact inversion). Therefore

$$\text{number of comparisons} \;\geq\; \text{number of inversions.}$$

Q3: Write a version of MergeSort that uses just two arrays A and B of size n.

We require two subroutines:

- **MergeAtoB**(m,p,n): merges the segment A[m],...,A[p−1] with the segment A[p],...,A[n−1]
  (assuming these segments are themselves already sorted),
  and writes the result to B[m],...,B[n−1].
- **MergeBtoA**(m,p,n): merges the segment B[m],...,B[p−1] with the segment B[p],...,B[n−1], and writes the result to A[m],...,A[n−1].

Minor variants of the **Merge** procedure from lectures (except do not return a value). Should work correctly even when one of the segments has length 0.

Q3 cont'd.

The following recursive procedure for MergeSort will then work:

```
MergeSort(m,n):
    if n−m > 1
        q = ⌊(m+n)/2⌋
        p = ⌊(m+q)/2⌋
        r = ⌊(q+n)/2⌋
        MergeSort(m,p)
        MergeSort(p,q)
        MergeSort(q,r)
        MergeSort(r,n)
        MergeAtoB(m,p,q)
        MergeAtoB(q,r,n)
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- values of m,n,q,p,r,
- plus a record of which line of code we've got to in that call, so that we know where to return to.

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So the total memory requirement is $\Theta(n) + \Theta(\lg n) + \Theta(1) = \Theta(n)$.