# Algorithmic Game Theory and Applications

(Approximate) Mechanism Design on Restricted Domains

# The Gibbard-Satterthwaite Theorem

Theorem (Gibbard 73 - Satterthwaite 75): In the unrestricted domain, when there are $m \geq 3$ candidates, a voting rule is truthful and onto if and only if it is dictatorial.

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What if there are only 2 candidates?

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Majority is Threshold$(1)$.

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Proof: A simple monotonicity argument: Assume $a$ is the winner, and consider a voter $i$.

- If $a \succ_i b$, then $i$ already has its top choice elected.
- If $b \succ_i a$, then if voter $i$ misreports $a \succ_i' b$, $a$ would still be elected, since it is still "above the threshold".

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By the GS Theorem, this voting rule cannot be truthful, as it is onto, but not a dictatorship.

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$c$ would be the (Condorcet) winner.

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Theorem (Gibbard 73 - Satterthwaite 75): In the unrestricted domain, when there are $m \geq 3$ candidates, an ordinal, deterministic voting rule is truthful and onto if and only if it is dictatorial.

# Cardinal vs Ordinal (Randomised) Rules

In simple words: A cardinal voting rule is ordinal if it disregards the numbers and only keeps the information about the relative ranking between the candidates.

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- The GS Theorem does *not* apply if we have randomised voting rules which are truthful-in-expectation. There are however some other theorems that apply (maybe tutorial).

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# Unrestricted Domain

A *social choice function*, or *voting rule*, or *mechanism* is a function $f : (\succ)^n \rightarrow A$ mapping preference profiles to candidates,

where $\succ^n$ is the space of all possible preference profiles.

The unrestricted domain: $\succ^n$ can contain any preference profile.

i.e., for any voter $i \in N$, $\succ_i$ is the set of *all permutations* of $\{1,\ldots,m\}$.

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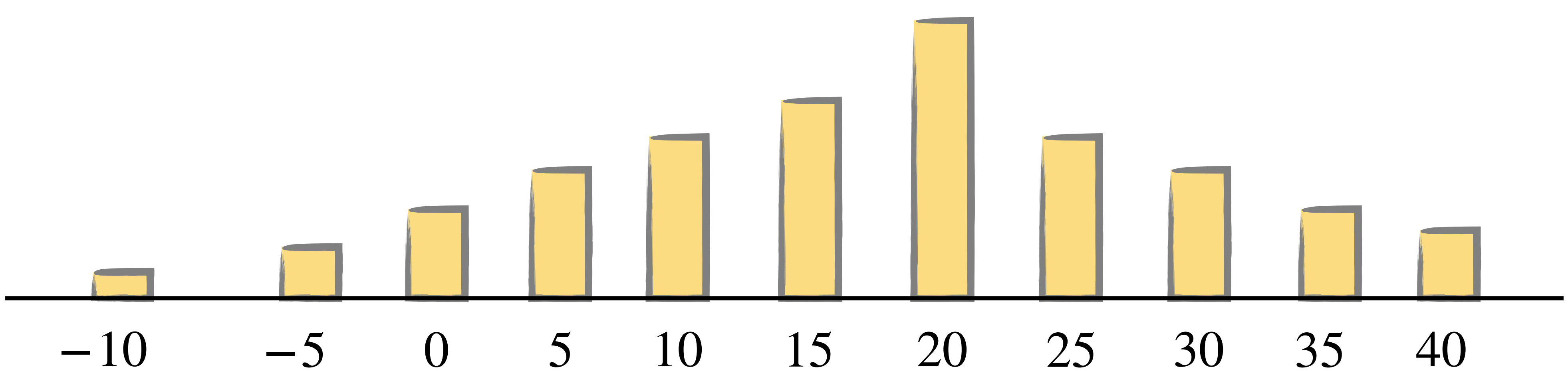

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Introduced by Black in 1948, as a domain for which Condorcet winners always exist.

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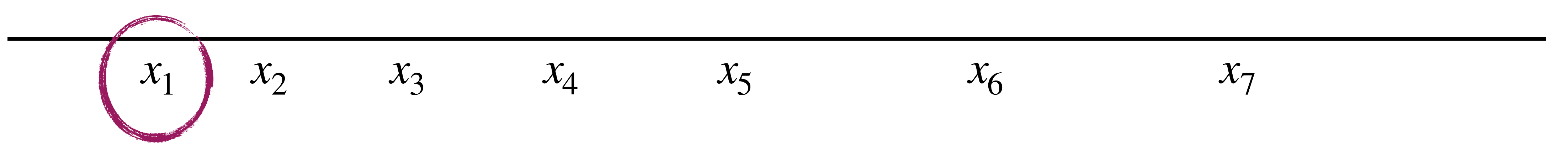

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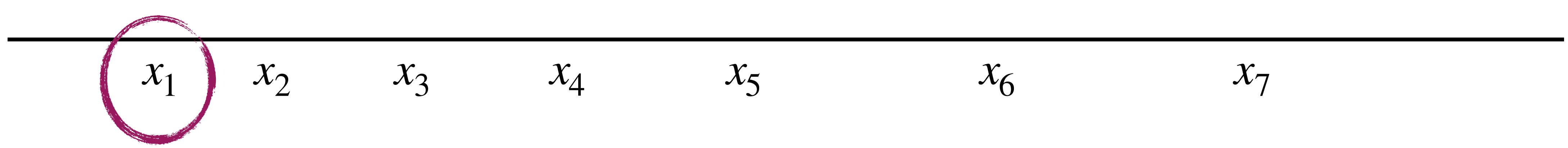

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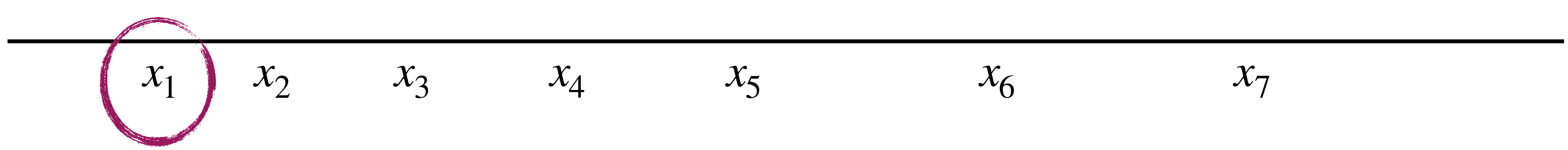

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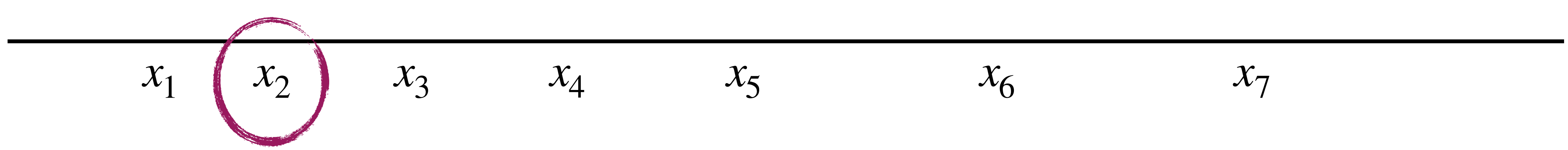

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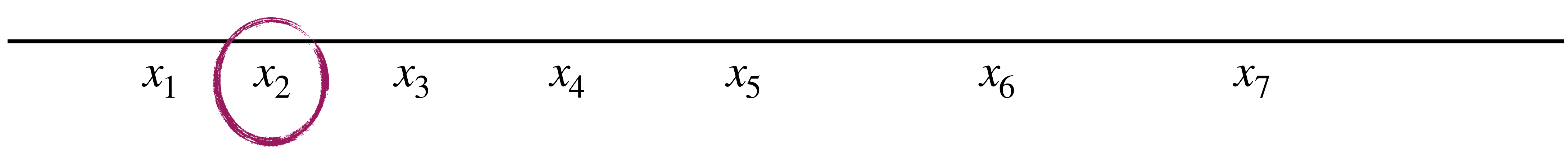

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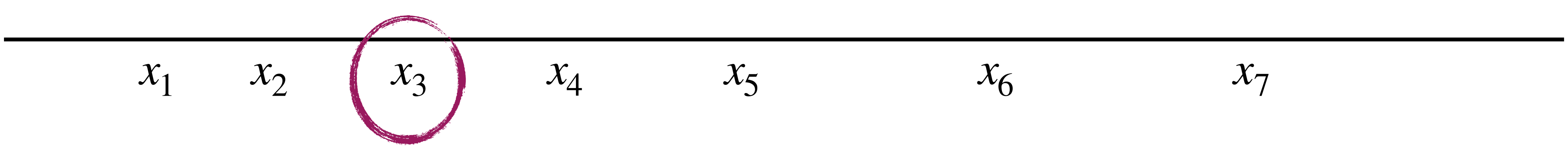

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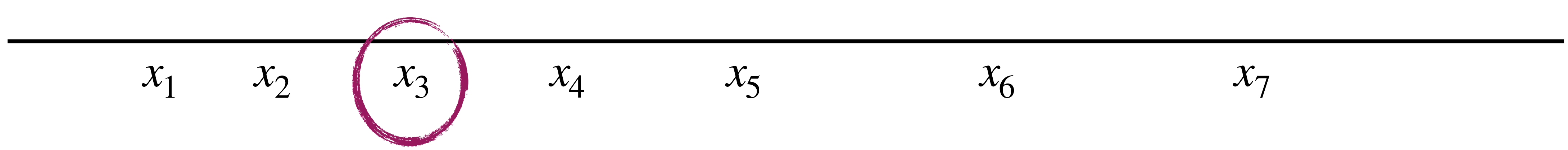

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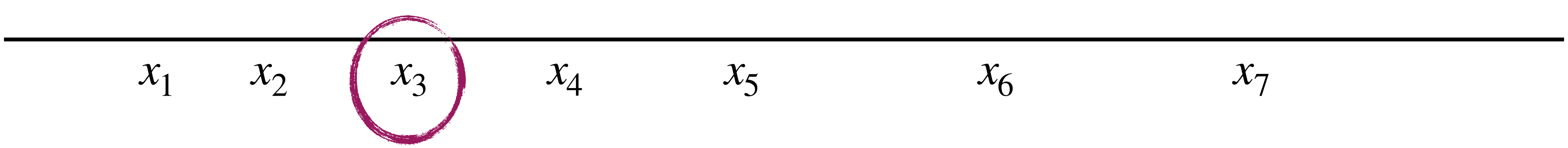

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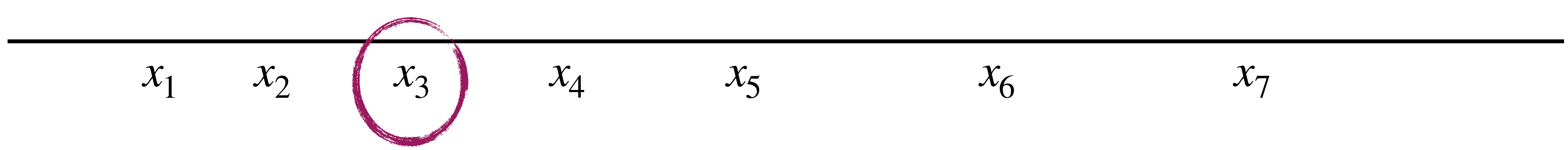

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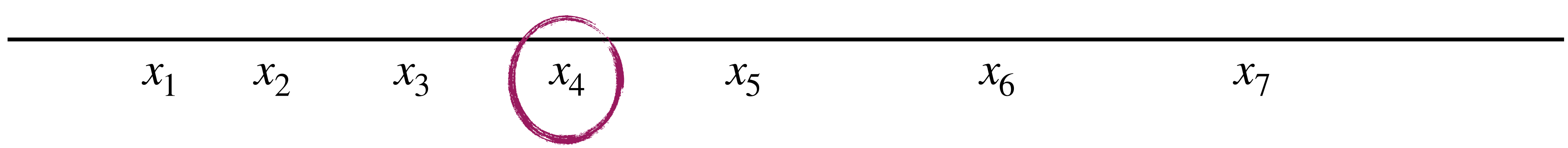

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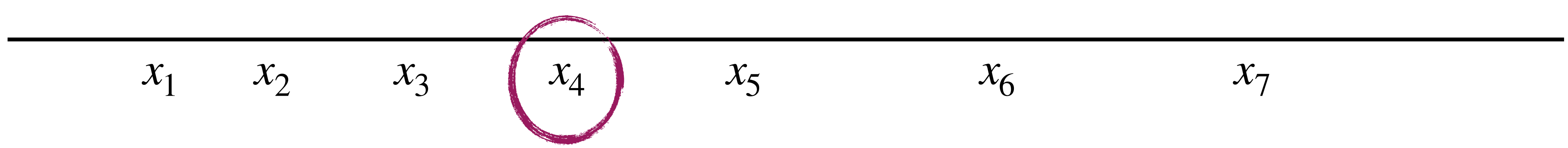

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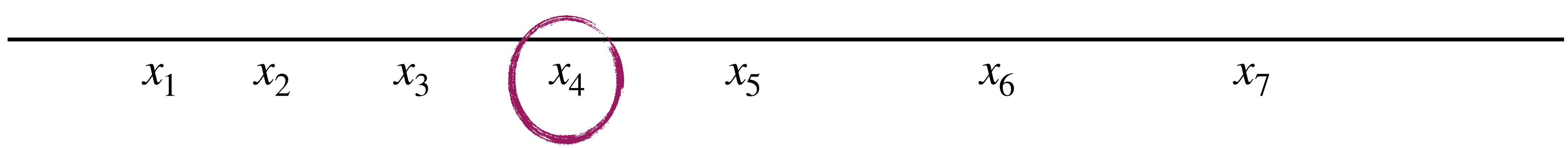

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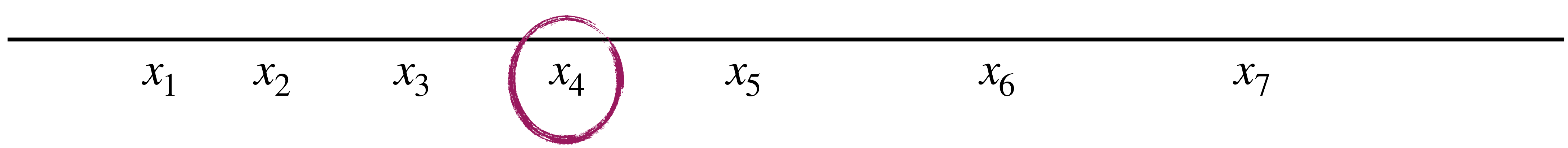

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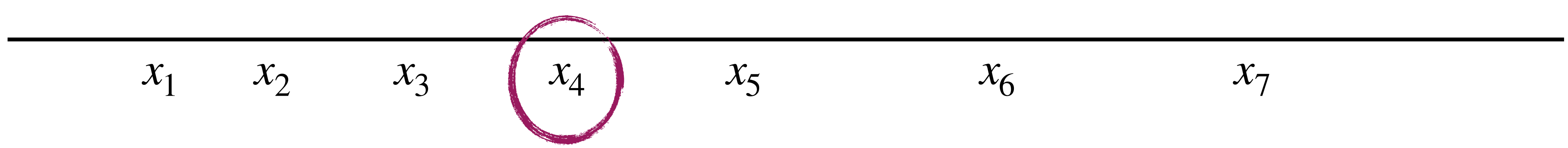

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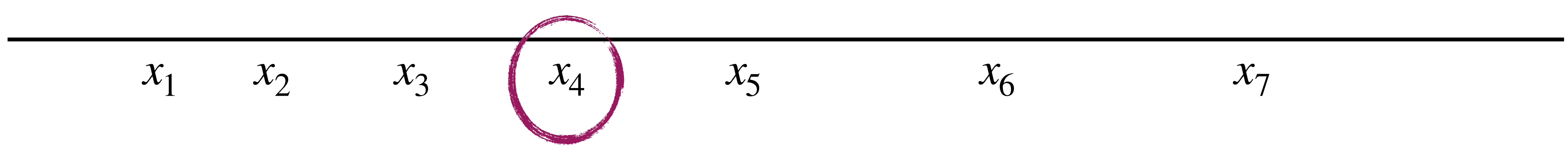

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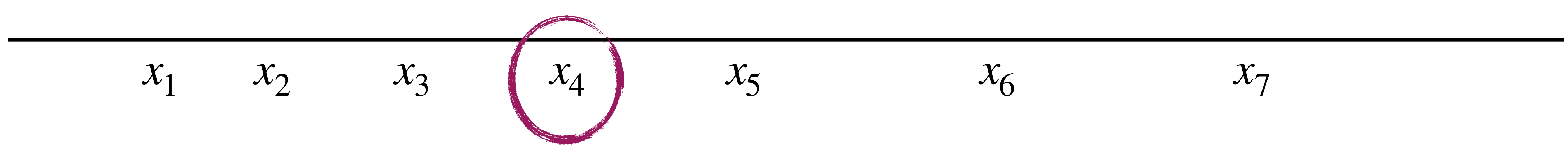

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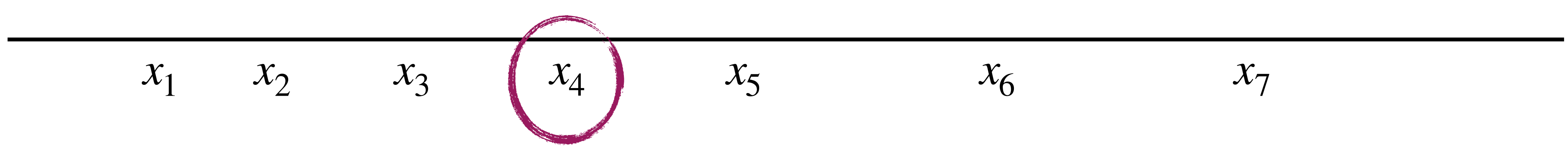

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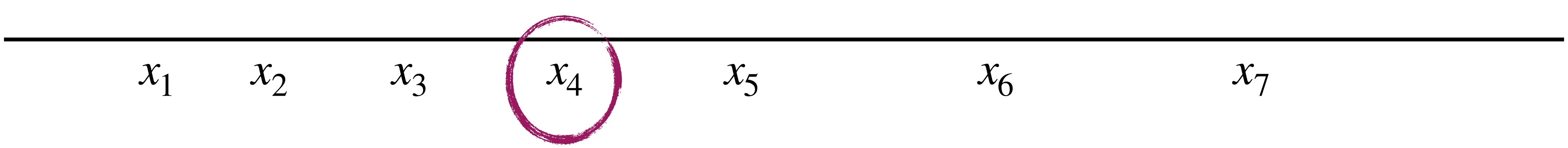

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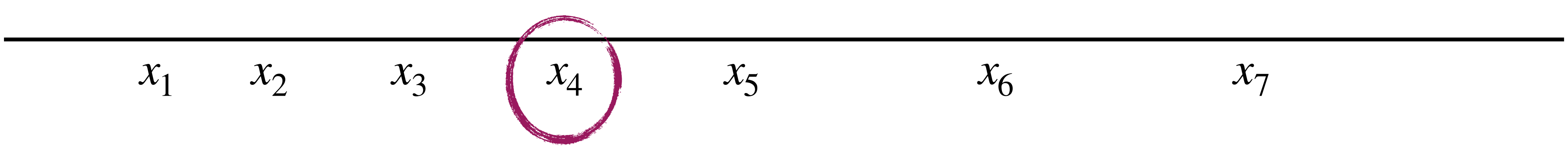


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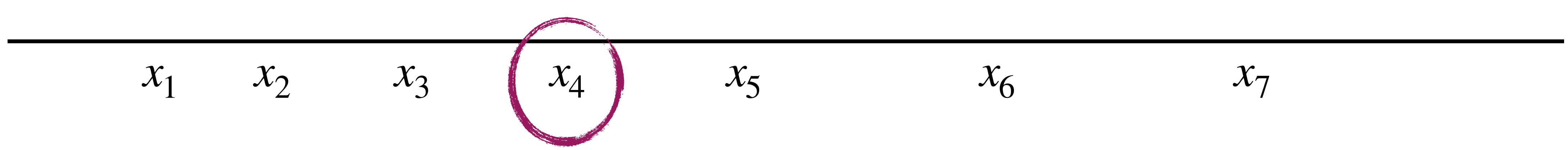


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What else is $x_4$?

# Median Voter Rule

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Theorem: The median voter rule is truthful.

Proof: Easy, we've seen it before.

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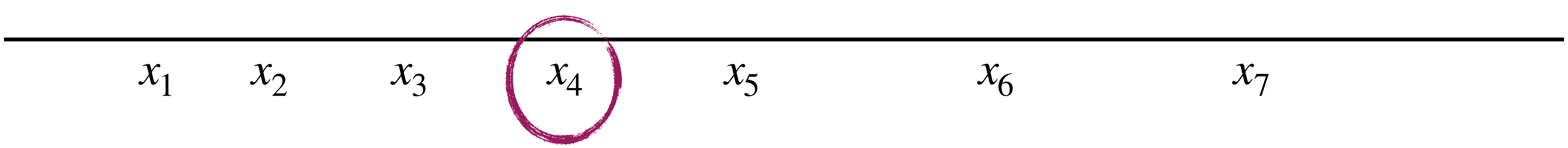
The proof follows from the same monotonicity argument: To affect the median, a voter with $p_i \leq f(\succ)$ must report something to the right of the median. But then the median will move farther away from $p_i$.



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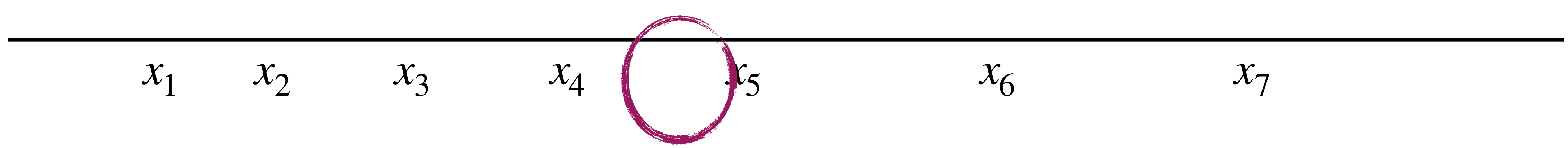

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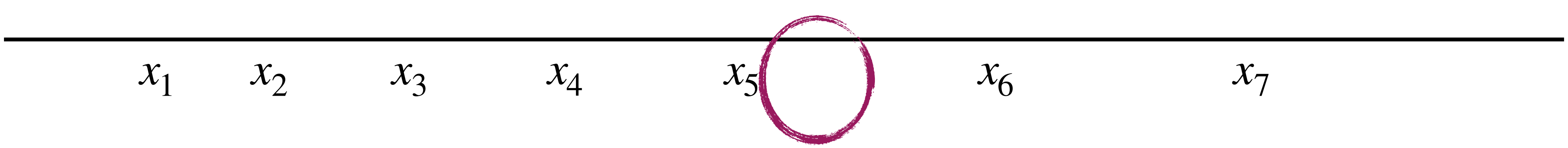

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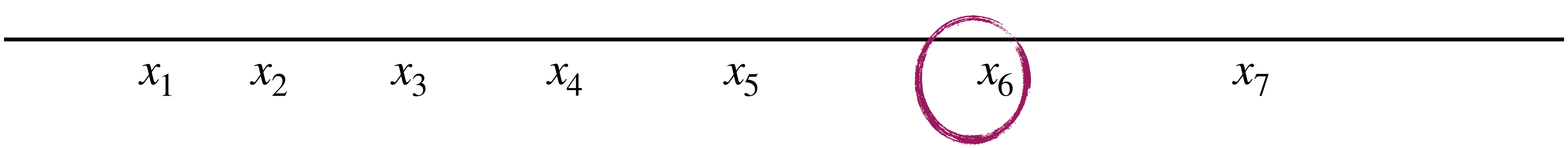

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Proof: Virtually identical to before, check at home.

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   e.g., *“A voting rule is truthful and onto if and only if it is a $k$-th order statistic voter rule.”*

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We can also have any $k$-th ordered statistic among all the candidates.

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Theorem (Moulin 1980): A voting rule $f$ is truthful, onto, and anonymous if and only if there exist $y_1, y_2, \ldots, y_{n-1}$ such that for all $\succ$, it holds that

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If you are interested, check the AGT book Definition 10.3.

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Obviously we still want the voting rule to be robust to incentives, so we are still interested in truthfulness.

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General Question: Among all the truthful voting rules, which one is the best with respect to the global objective?

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Refined Question: Among all the truthful voting rules, or, in this context, *mechanisms*, what is the one with the smallest possible approximation ratio?

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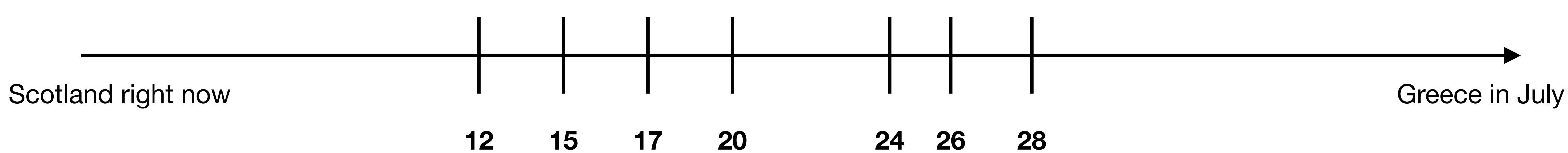
We want to design a truthful mechanism (voting rule) for the problem that has the minimum possible approximation ratio for the social cost objective, i.e., the sum of agents' costs $\sum_{i \in N} |y - x_i|$

# Example 2:
# Setting the temperature

The reports shown in the picture are the peaks, but any temperature is a possible outcome.



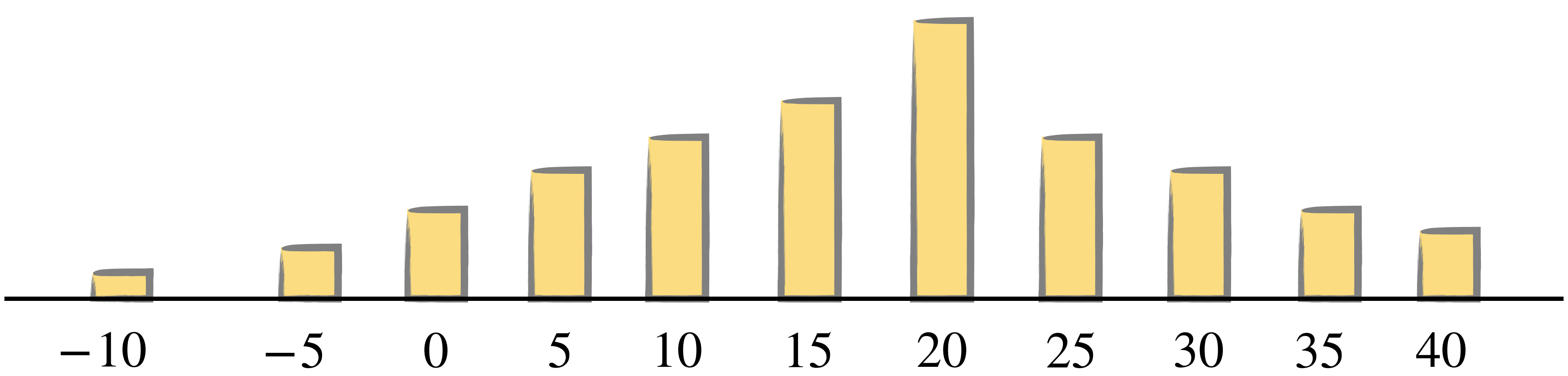
# Single-Peaked Preferences

Assume that we have a set of possible temperatures for the thermostat, e.g., $\{-10, -5, 0, 5, 10, 15, 20, 25, 30, 35, 40\}$.

Let's say that your ideal temperature would be $20$ degrees.

It is reasonable to assume that you would also prefer $25$ degrees to $30$ degrees, and likewise, $15$ degrees to $10$ degrees.

Generally, the "farther away" we move from your ideal temperature, the less happy you become.



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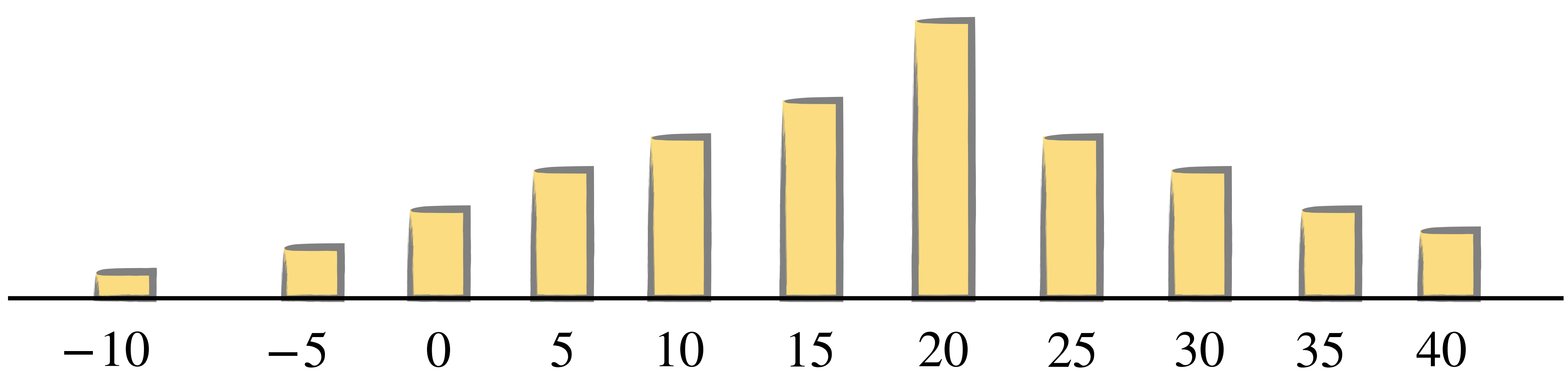

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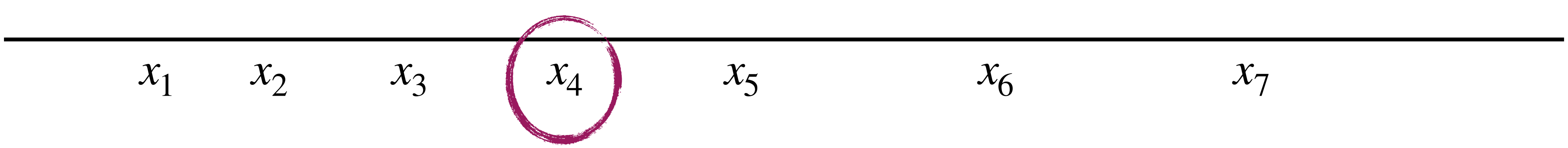

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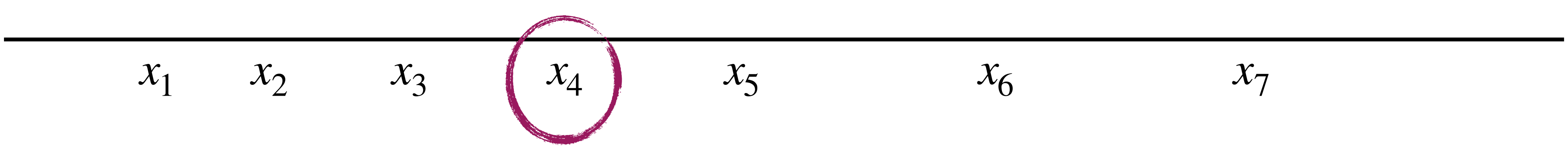

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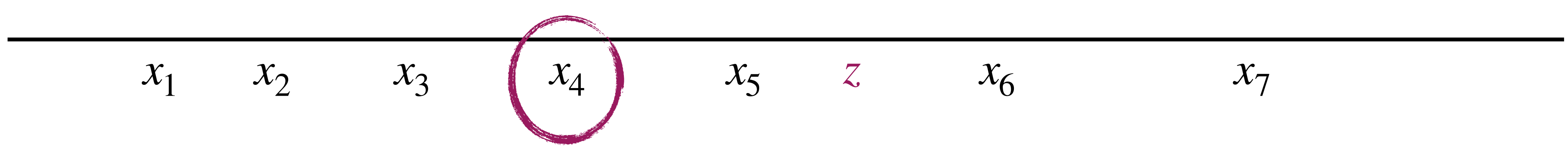

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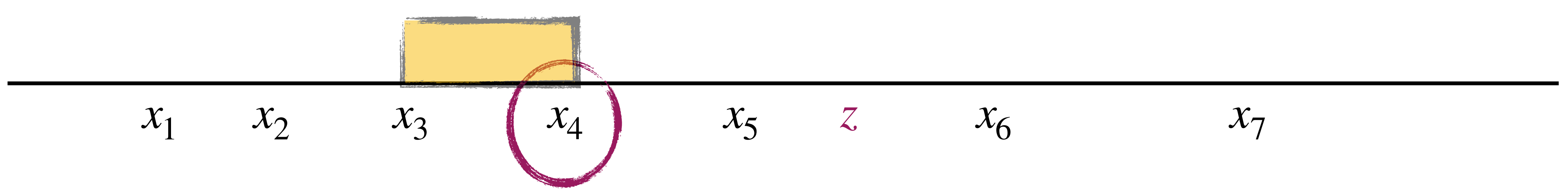

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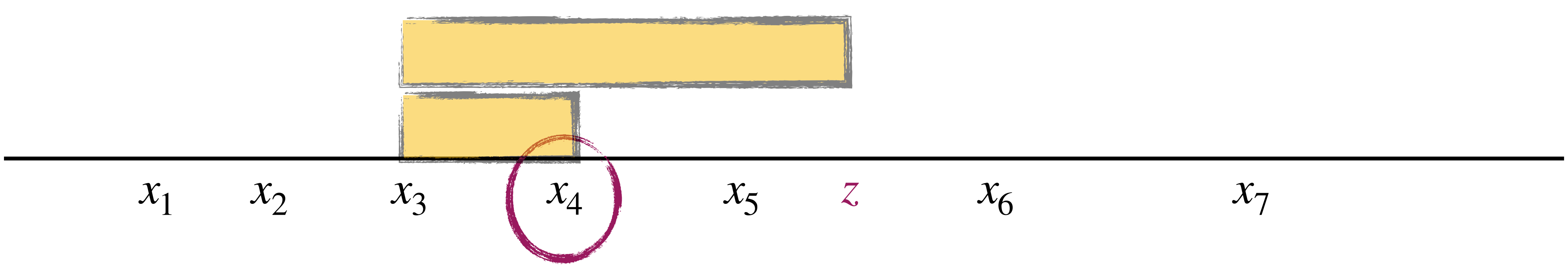

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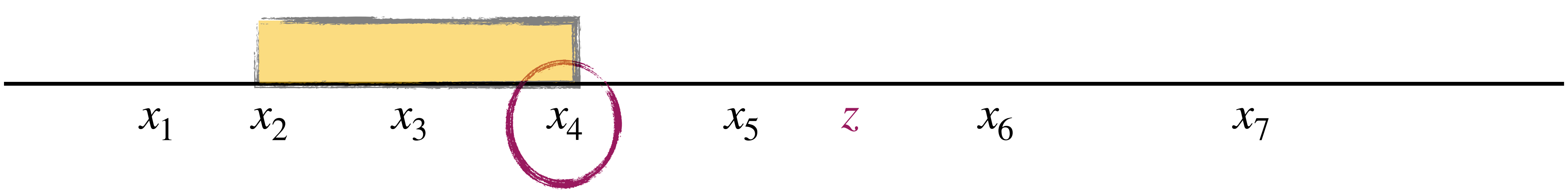

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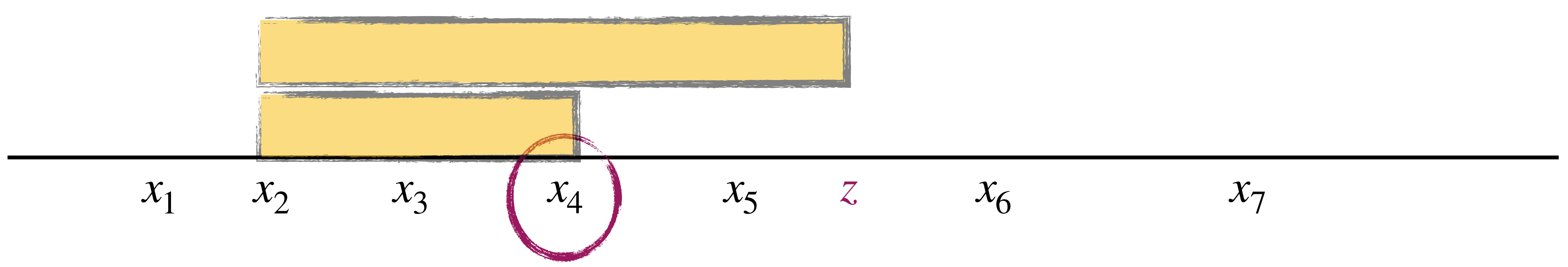

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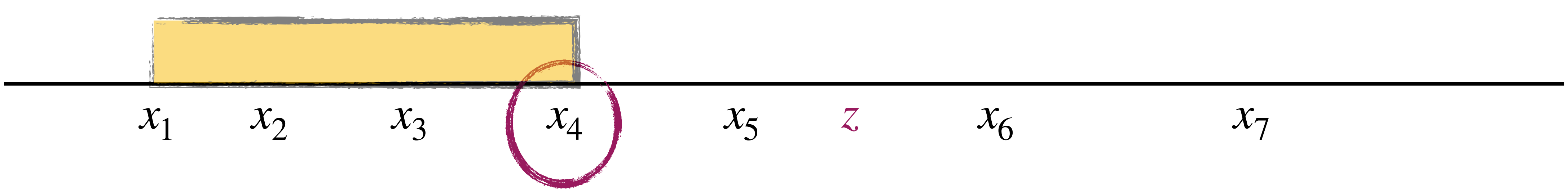

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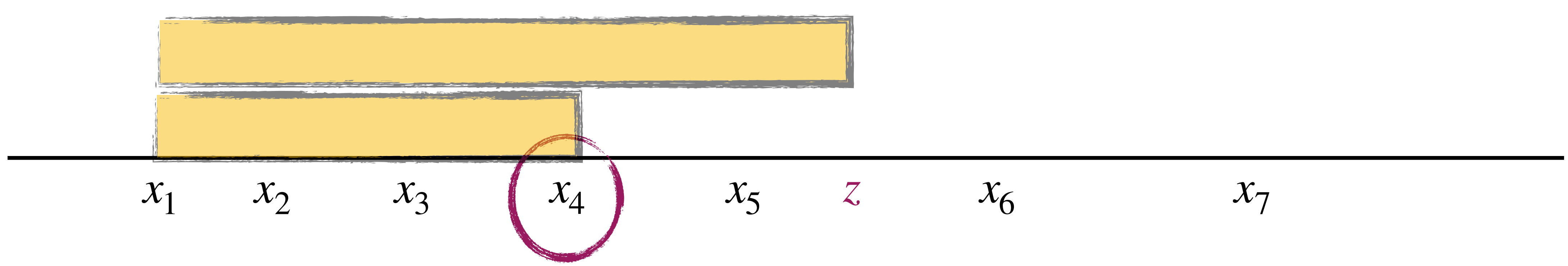

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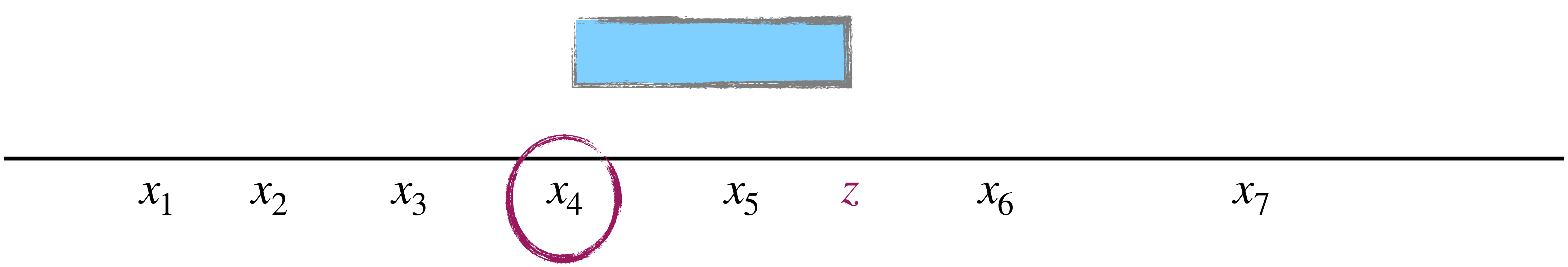

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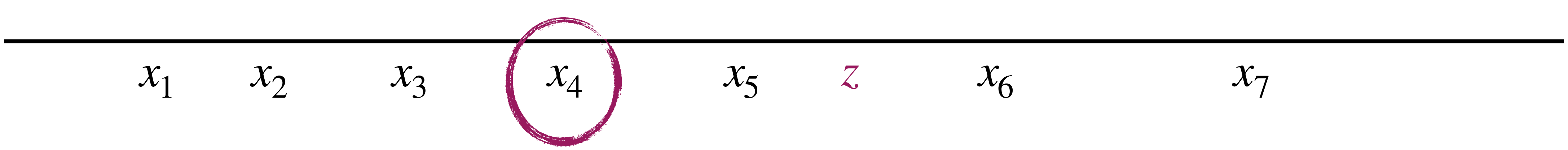

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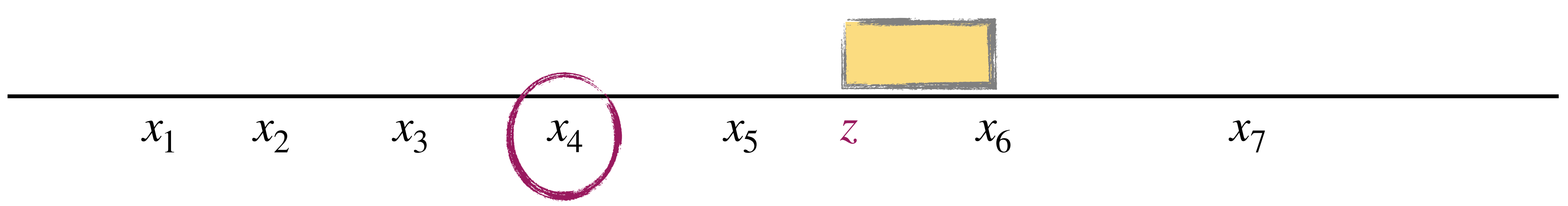

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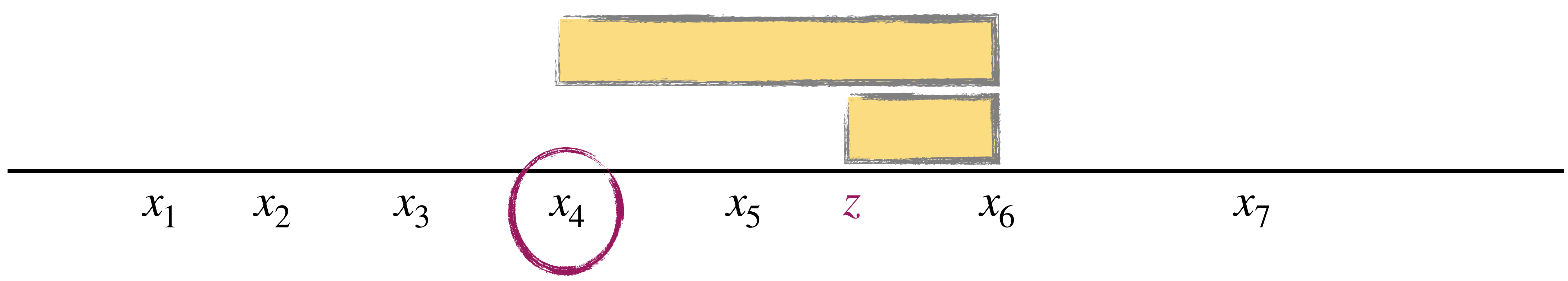

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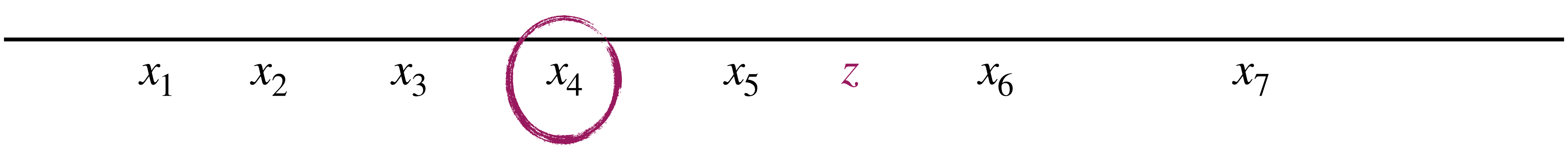

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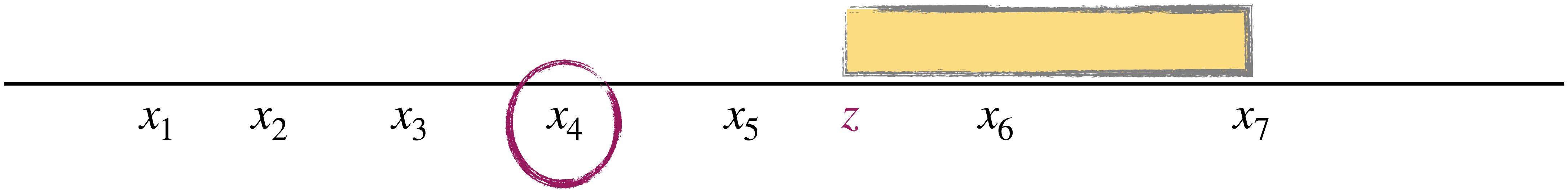

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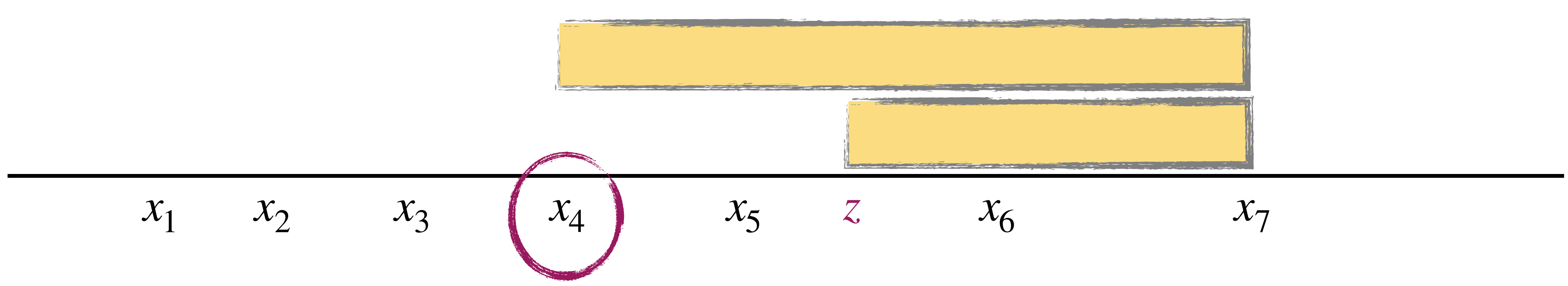

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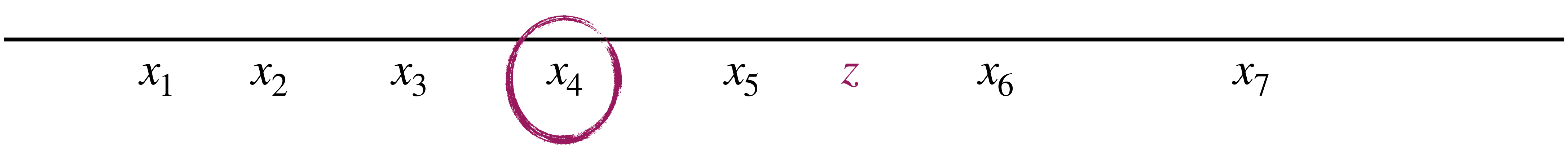

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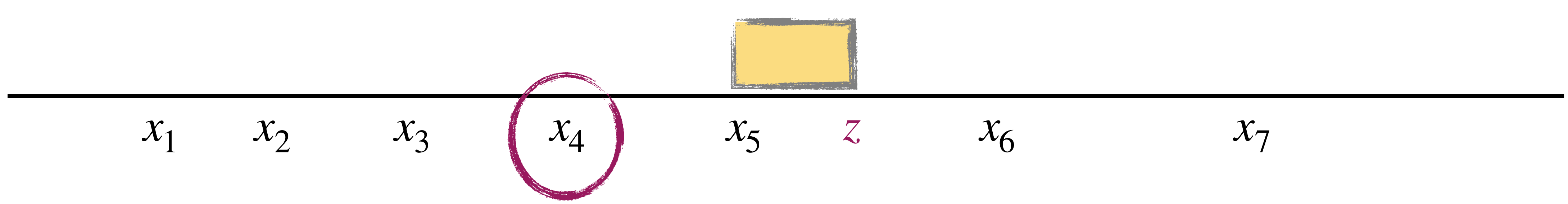

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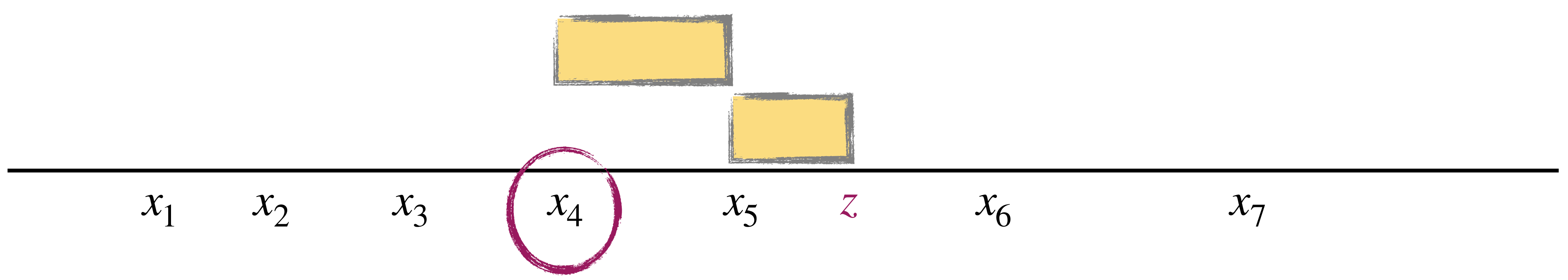

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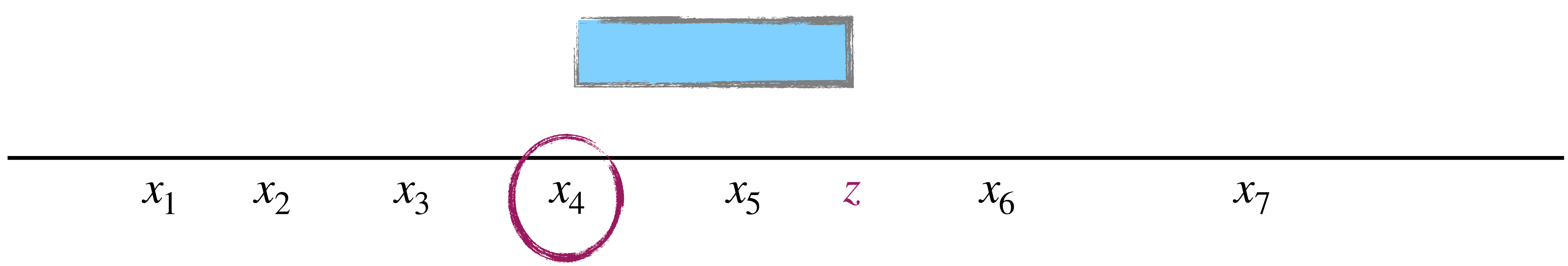
At least half of the agents pay an extra



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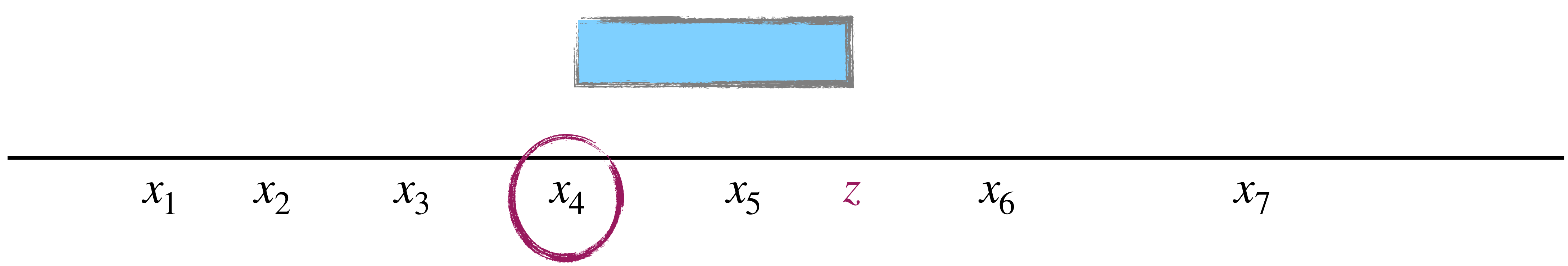


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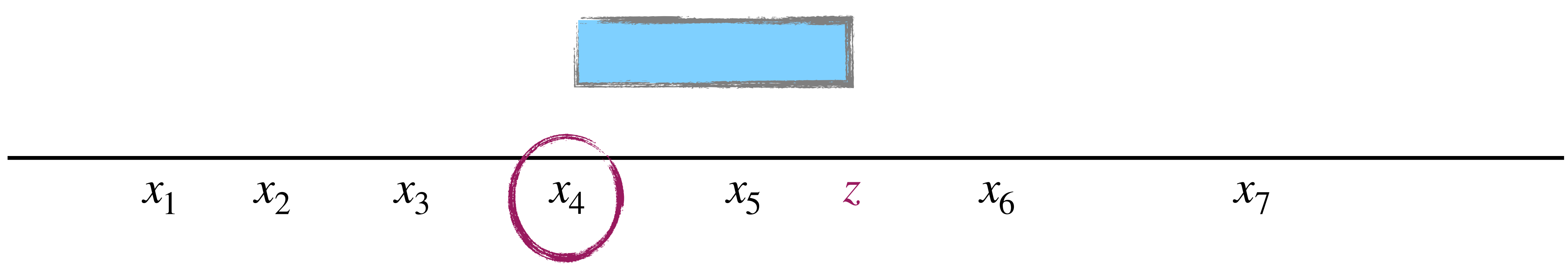


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What is the approximation ratio of the median voter mechanism?

# Truthful Facility Location (Procaccia and Tennenholtz 2010)

There is a set $N = \{1, \ldots, n\}$ agents (voters), each of which has an ideal location (the "peak") $x_i$ on the real line $\mathbb{R}$.

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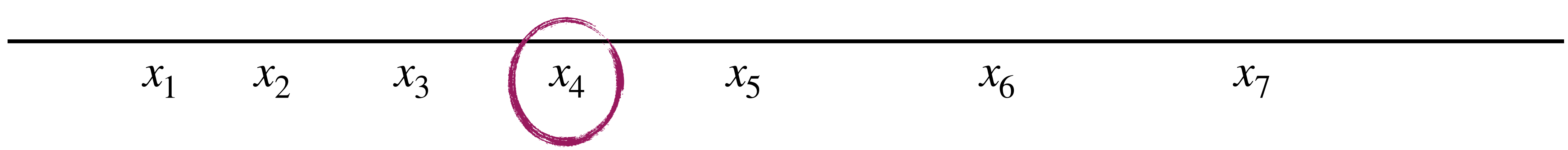

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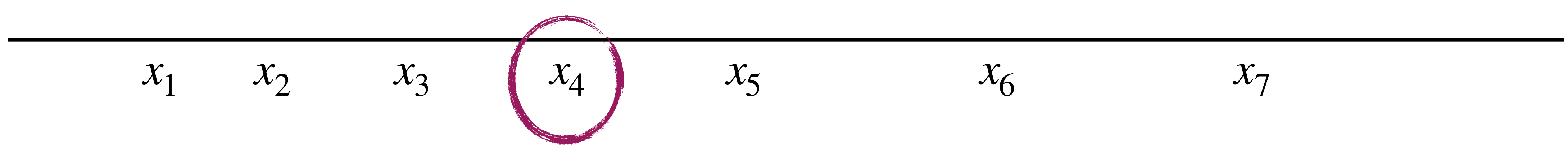

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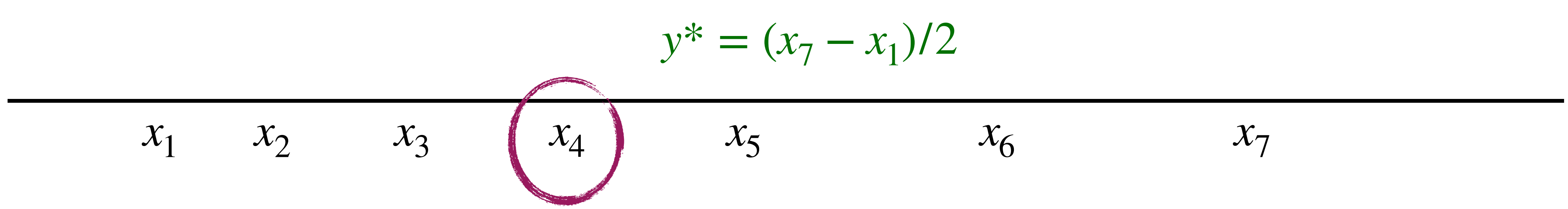

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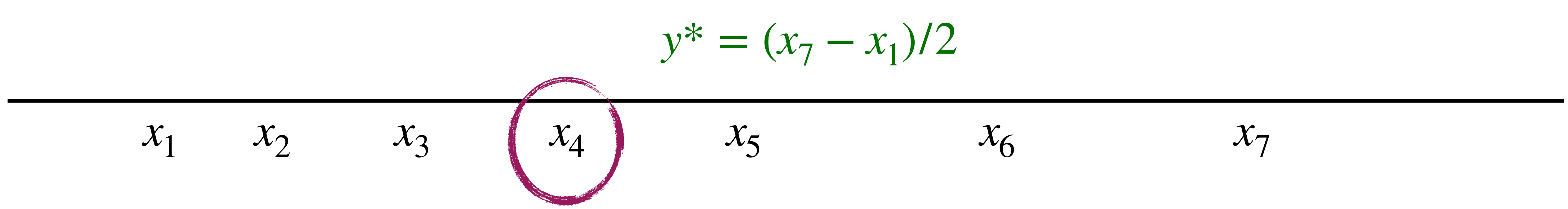
What is the optimal location (the one that minimises the maximum cost)?

How far can the median be from the optimal location?



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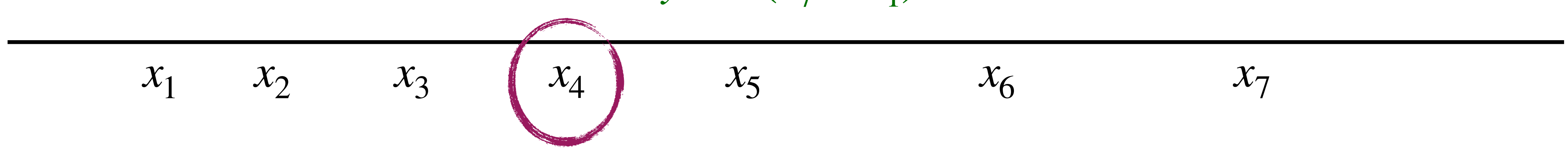
It looks fairly close in the example!

$y^* = (x_7 - x_1)/2$



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Consider this example though.

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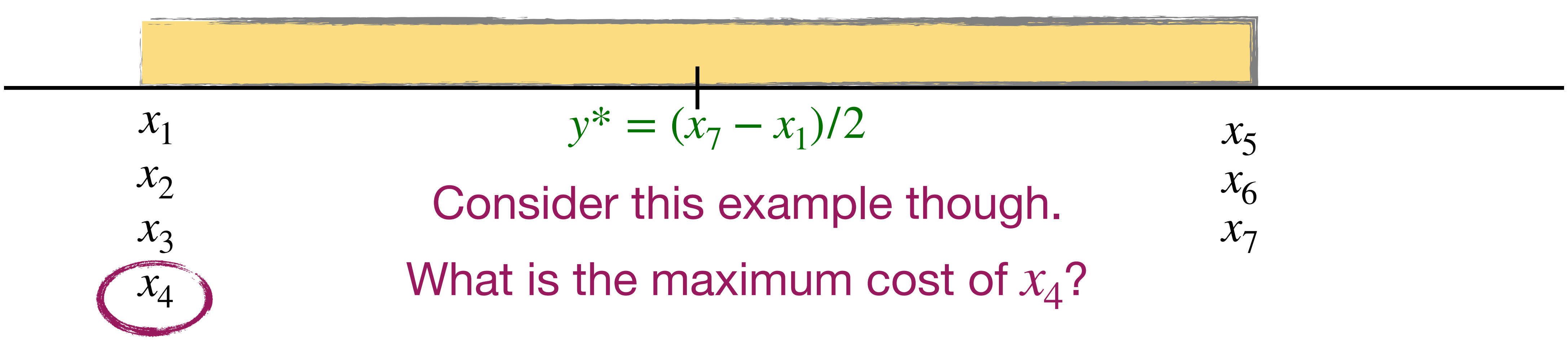


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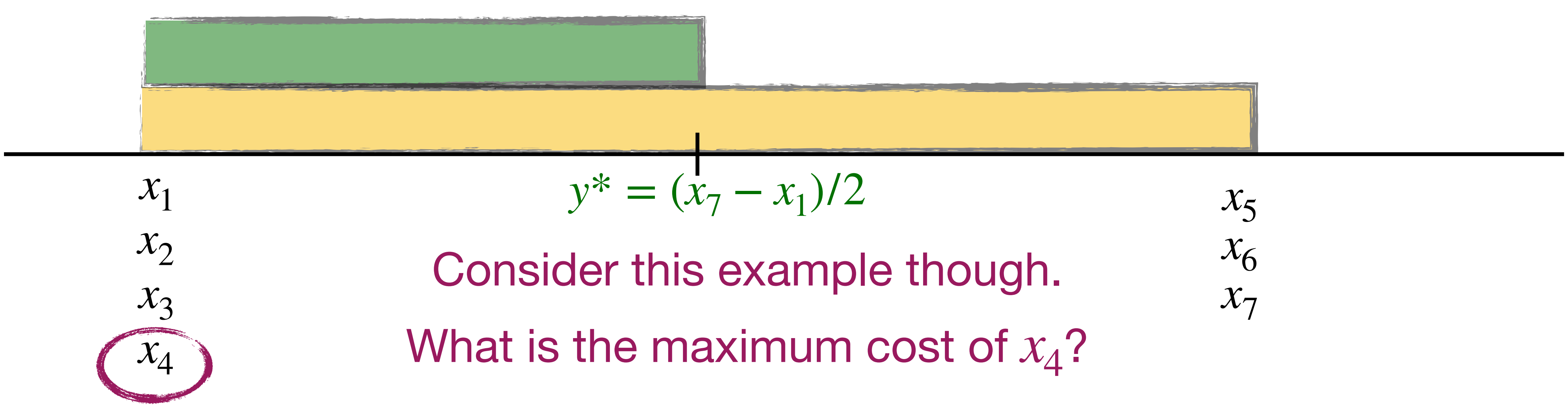


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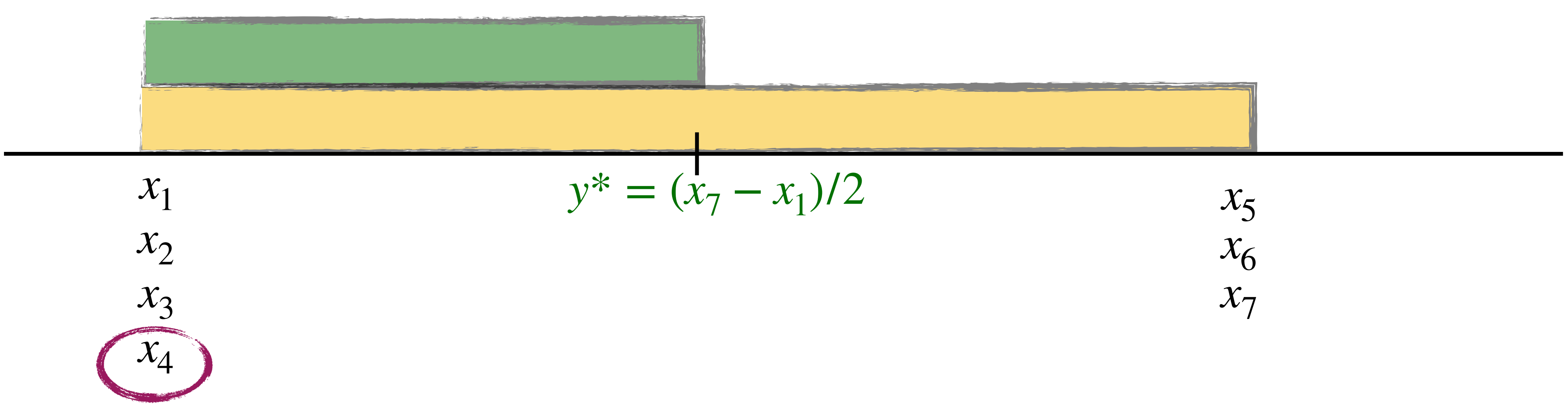

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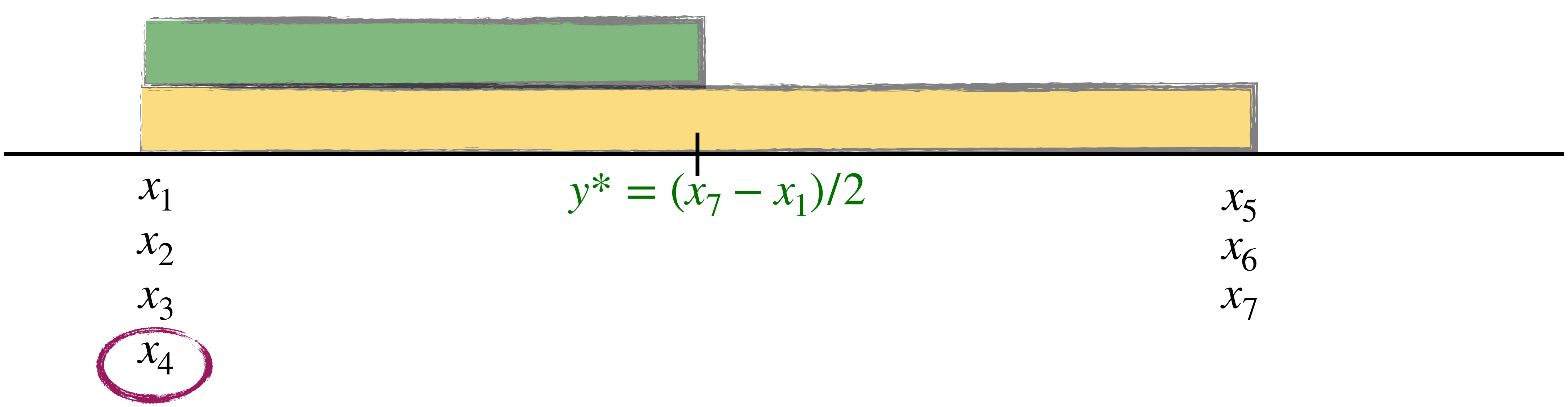
The approximation ratio of the MVM is at least 2.



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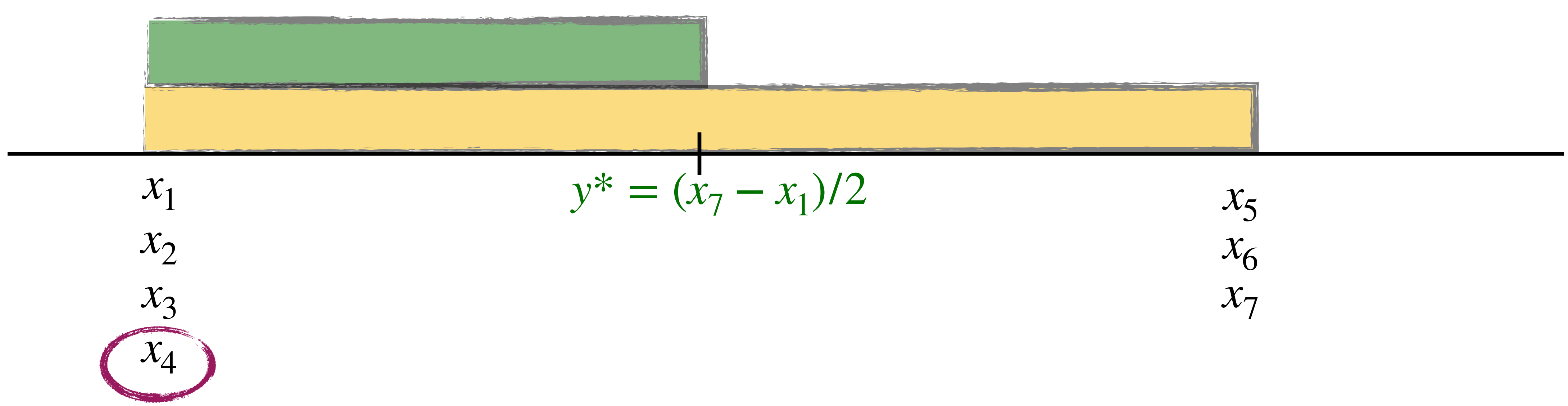

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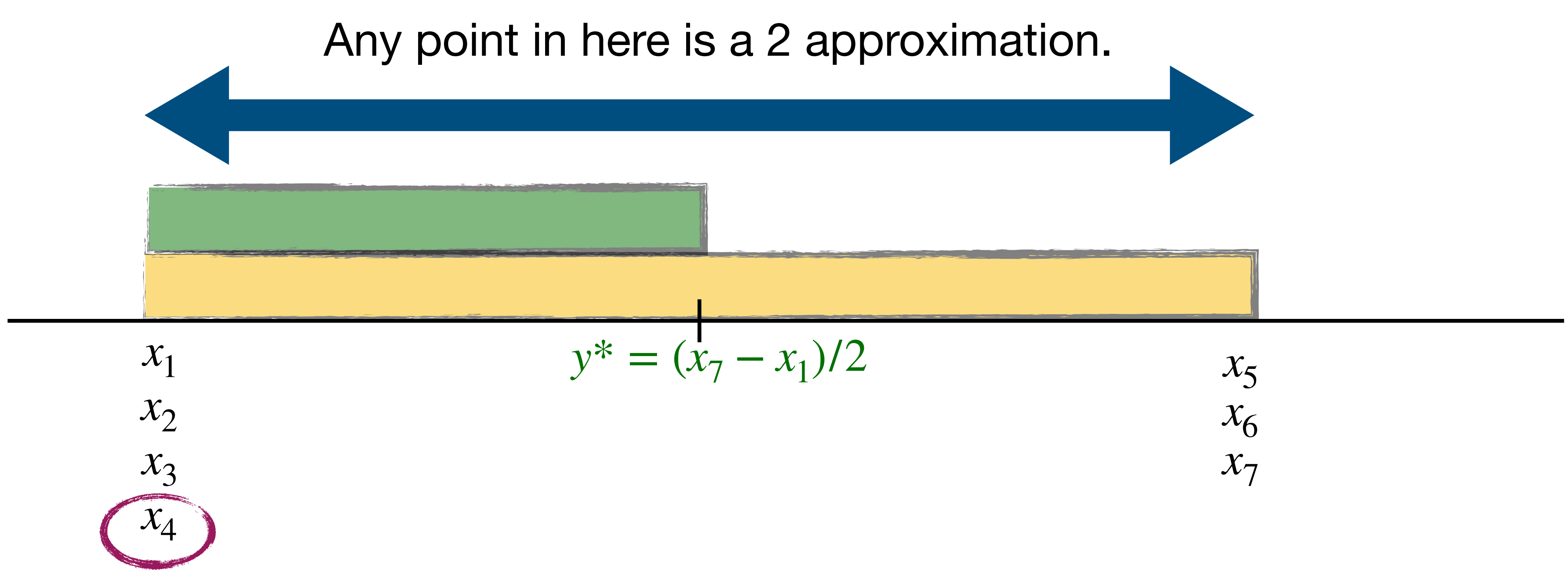

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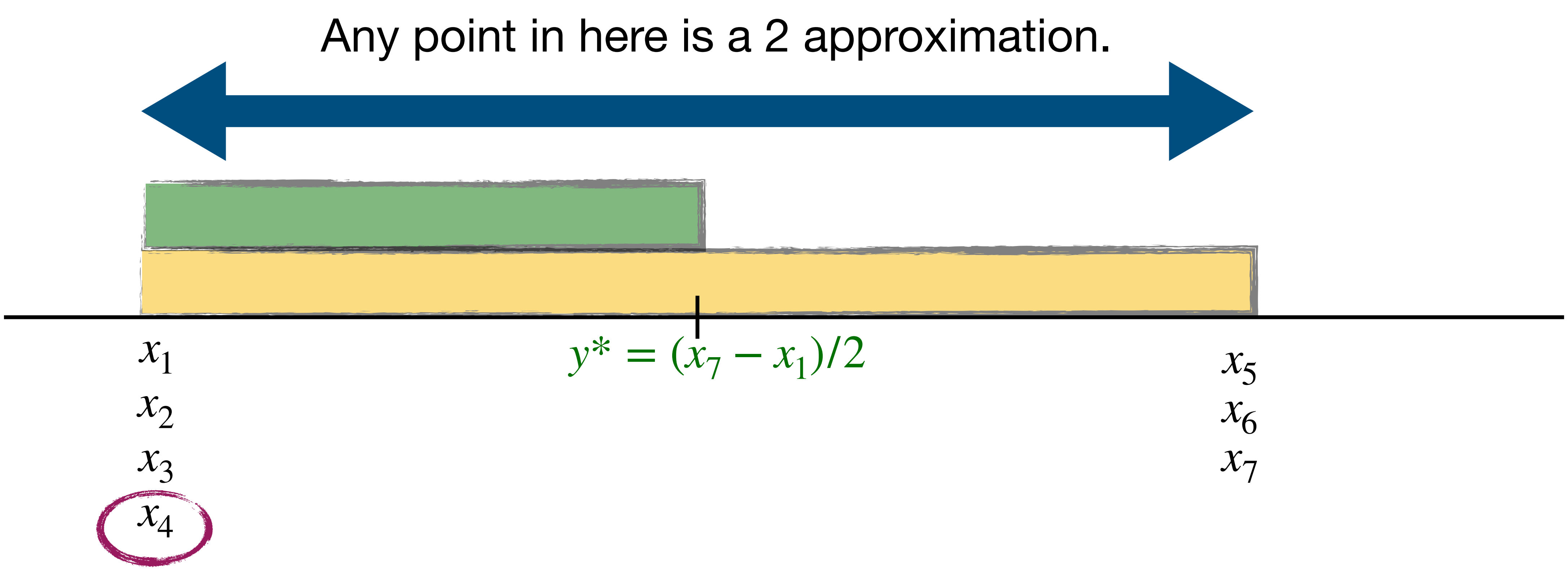
The approximation ratio of the MVM is at most 2.

The approximation ratio of any $k$-th order statistic is exactly 2.



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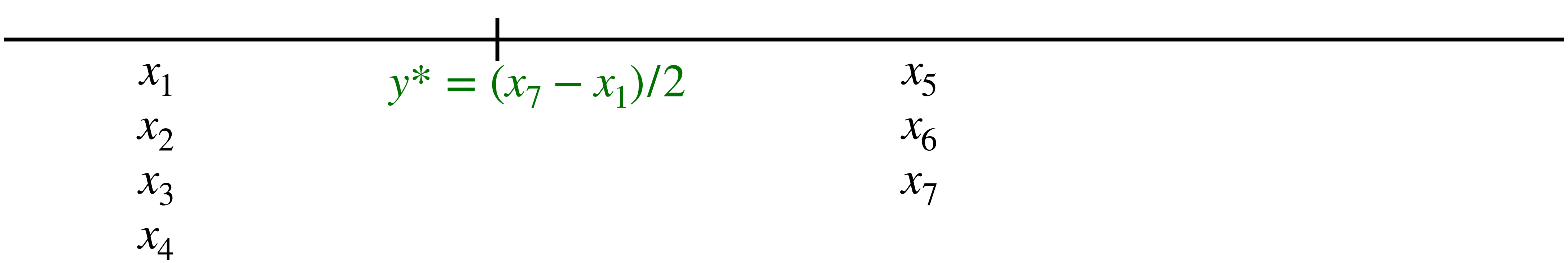
# A lower bound for all truthful mechanisms



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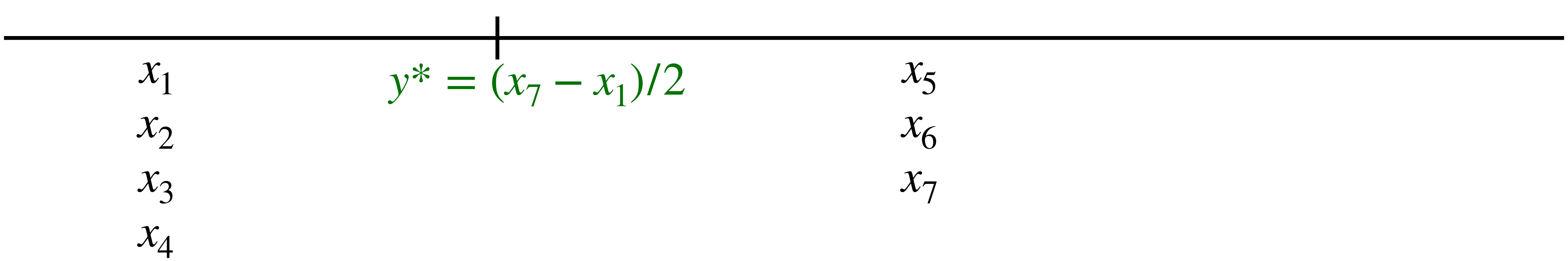

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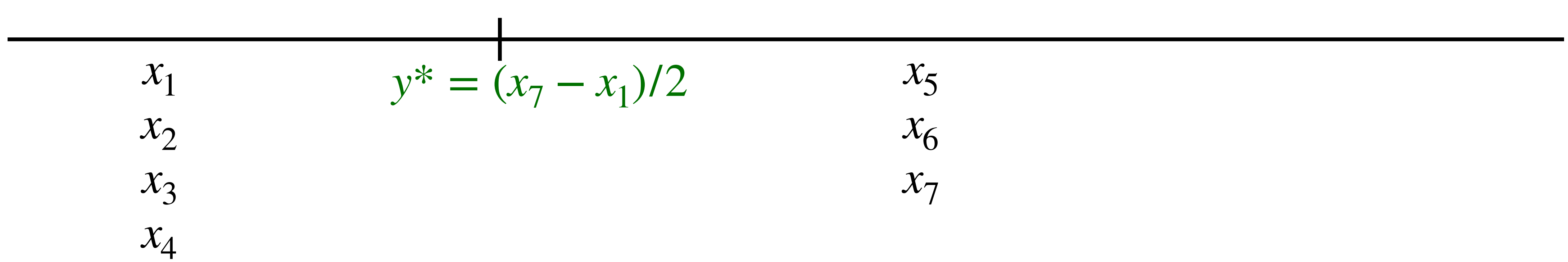
Assume by contradiction that there exists some truthful mechanism $M$ with approximation ratio $< 2$.



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Consider the instance shown below.

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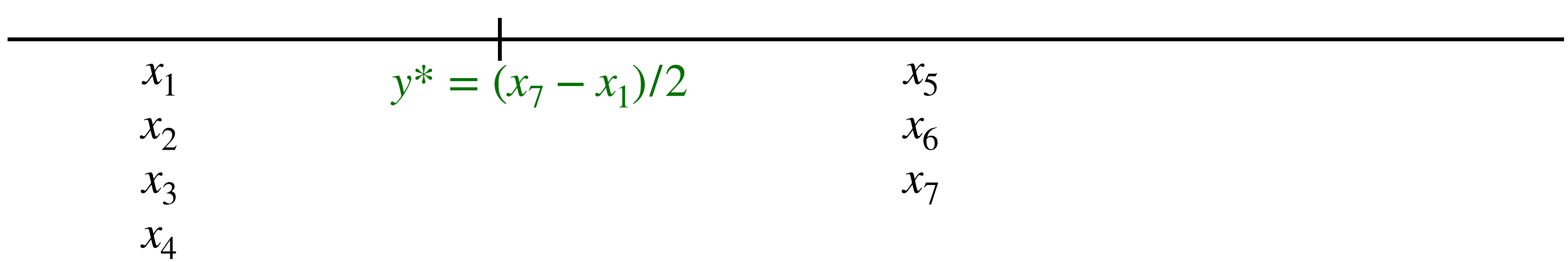
$\Rightarrow$ the facility needs to be placed in the interior of the interval, wlog, closer to the right endpoint.



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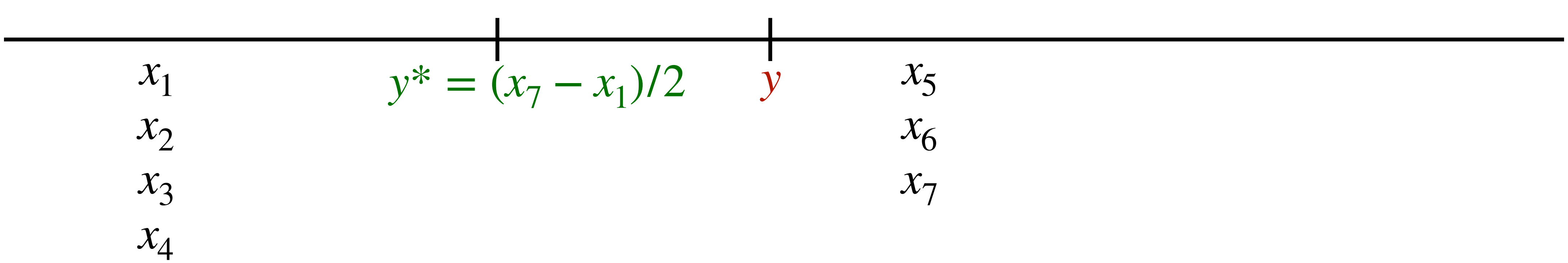

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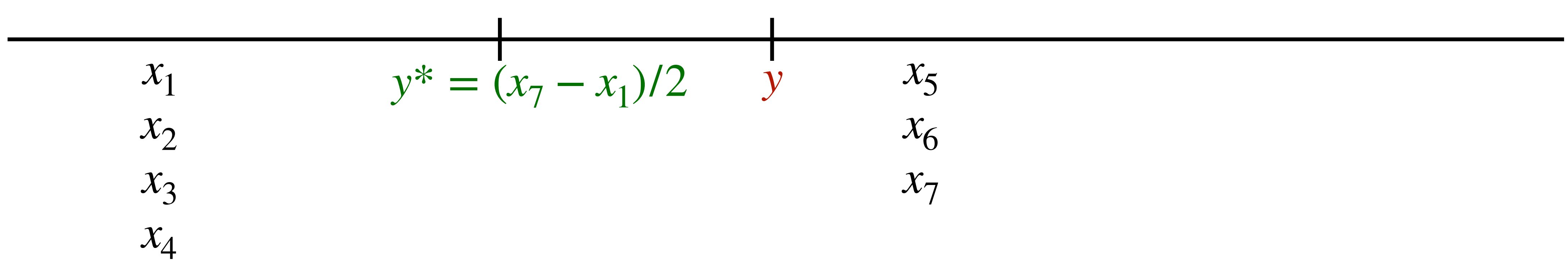
Assume now that agent with peak $x_5$ reports $x'_5 = y$



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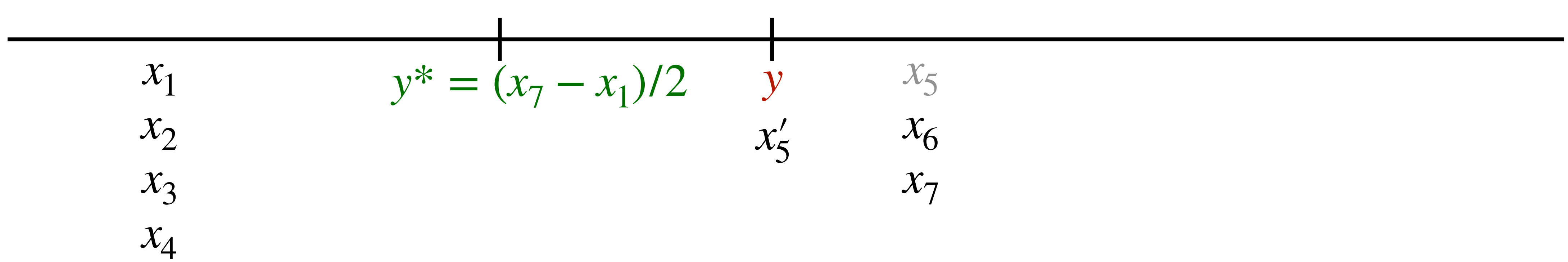

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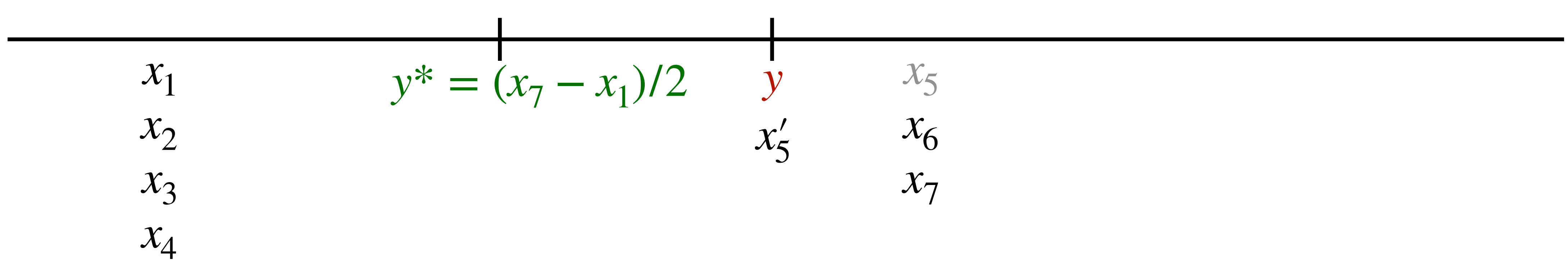

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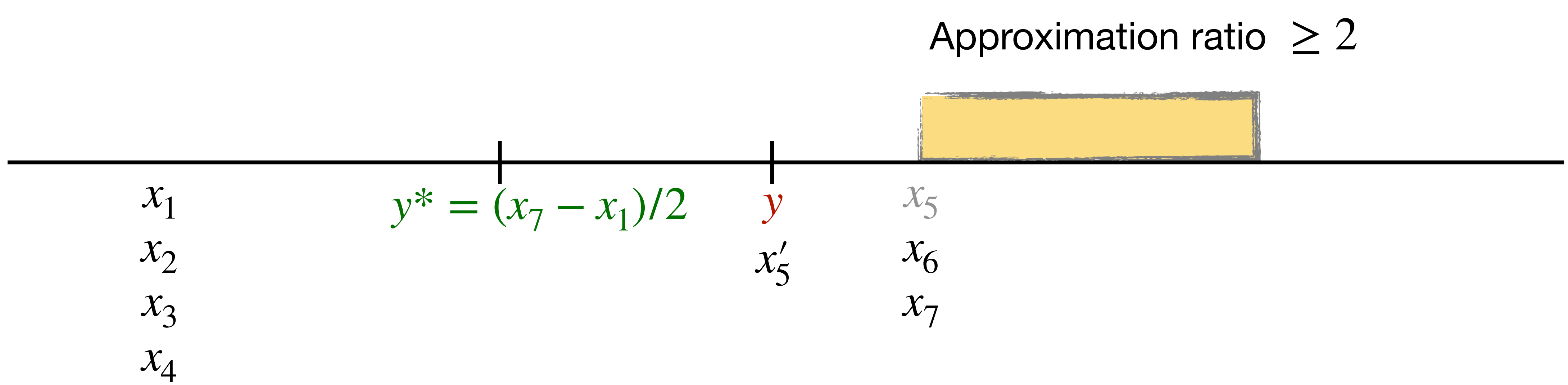

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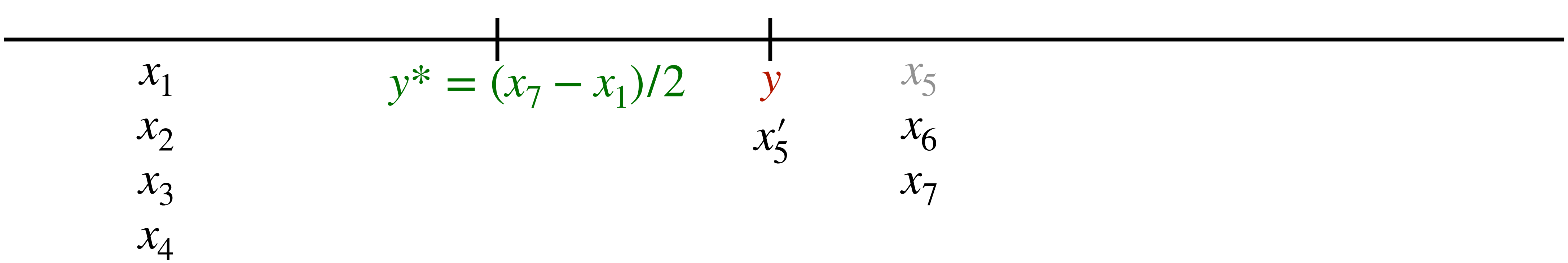

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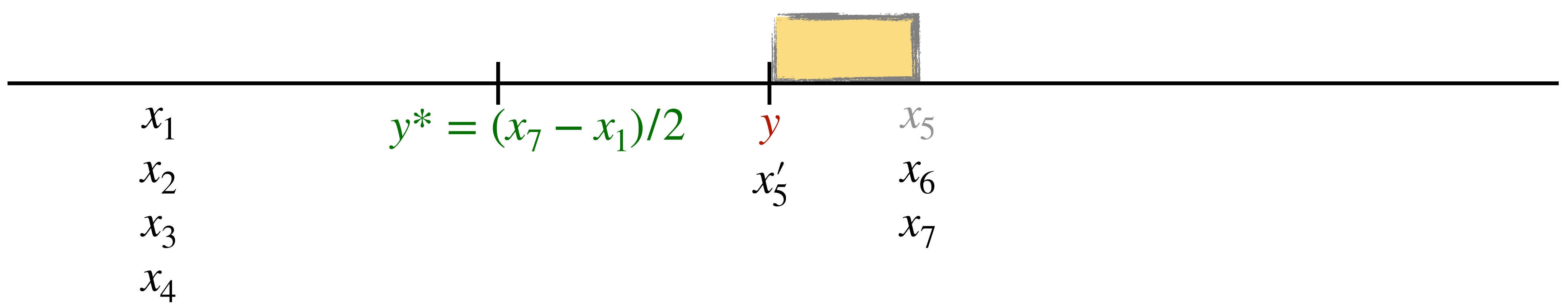
$x_5$ brings the facility closer to its true peak.



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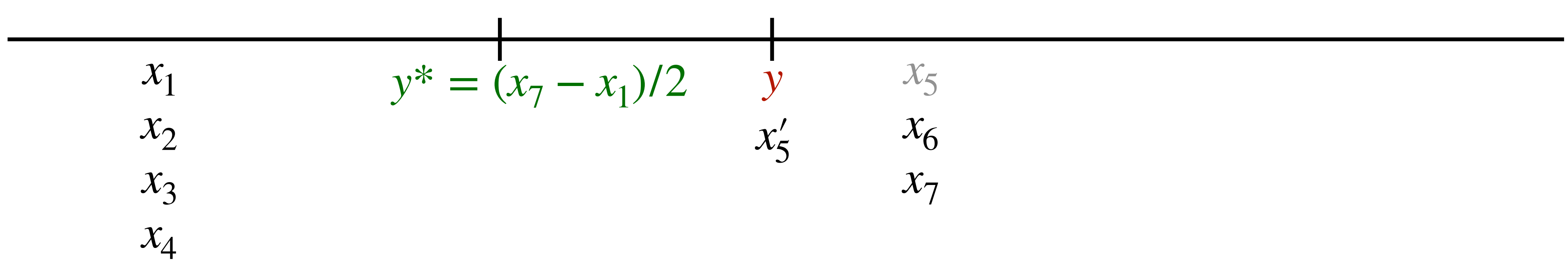

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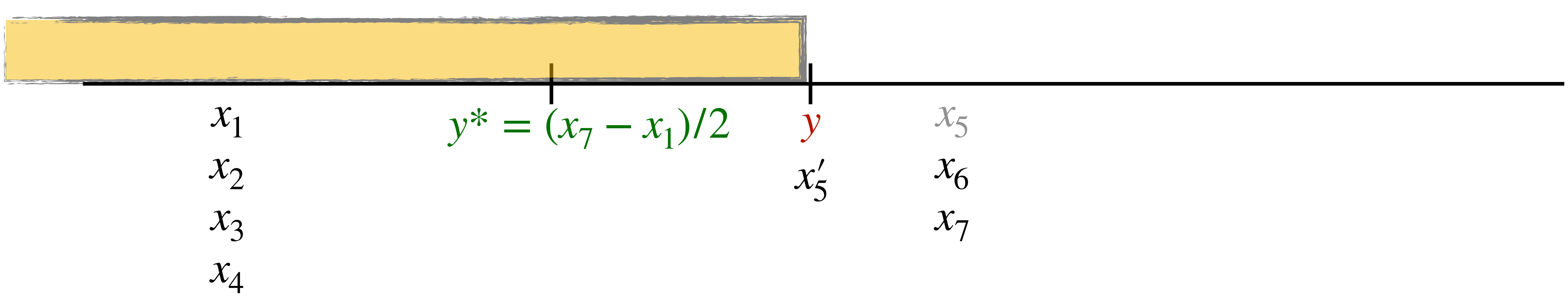
It could be the case that $x_5'$ is the true peak and $x_5$ is the misreport.



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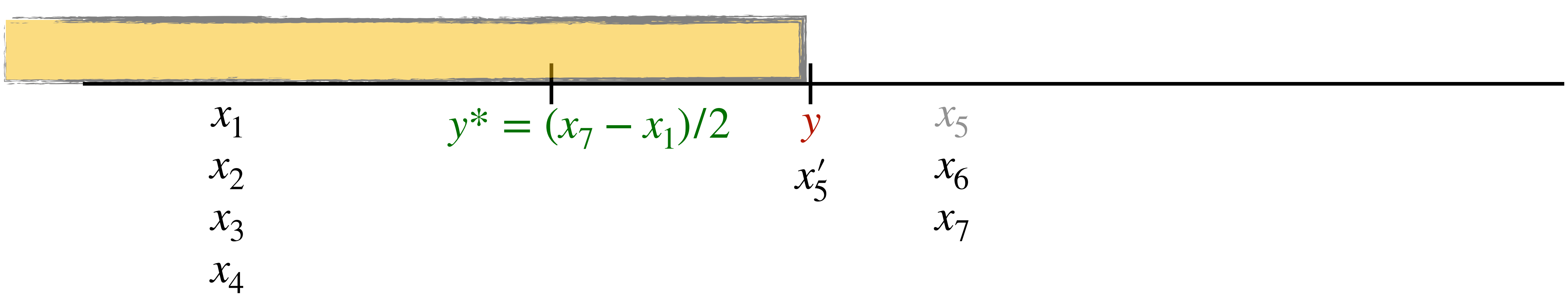
In that case the misreport would bring the facility exactly on the true peak.



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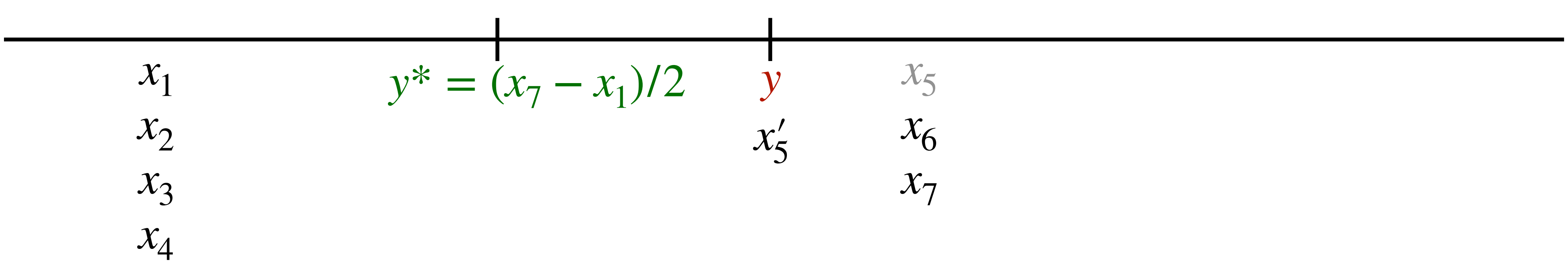
The facility cannot change position, let’s see why.



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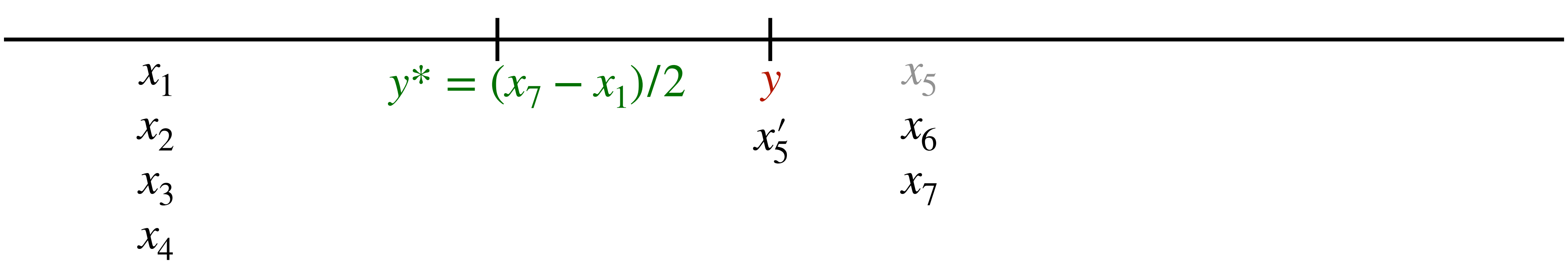
We can use the same argument for $x_6$ and $x_7$.



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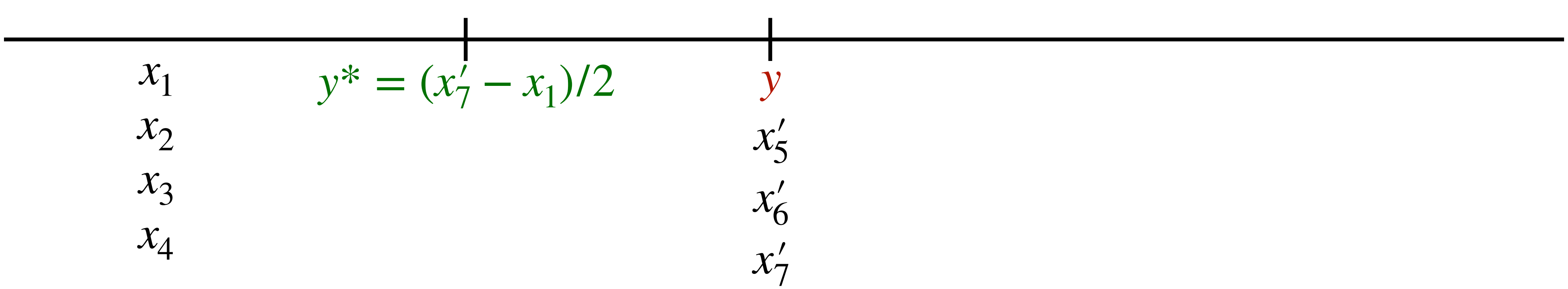

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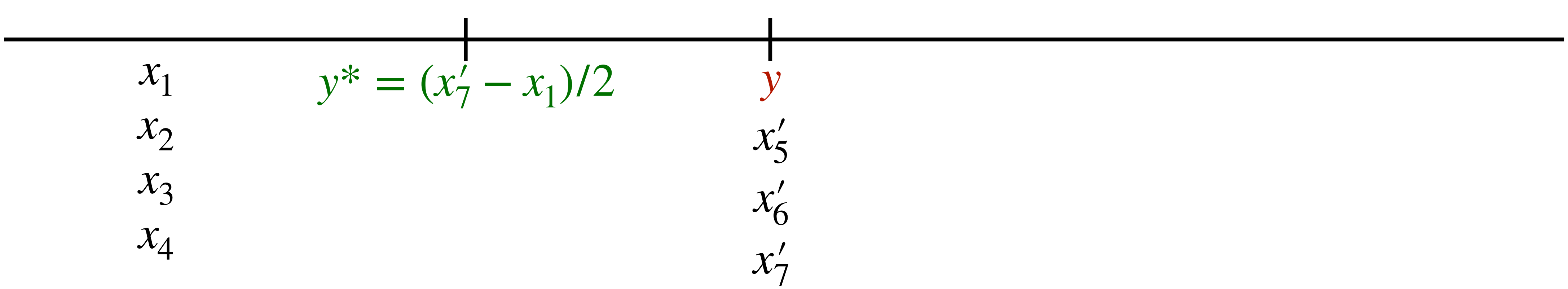
What is the ratio on this instance?



# Truthful Facility Location, max cost objective

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Theorem (Procaccia and Tennenholtz 2010): The best possible approximation ratio achieved by any truthful mechanism for the maximum cost objective is 2. This is achieved by any $k$-th ordered statistic mechanism.