AGTA Tutorial 5

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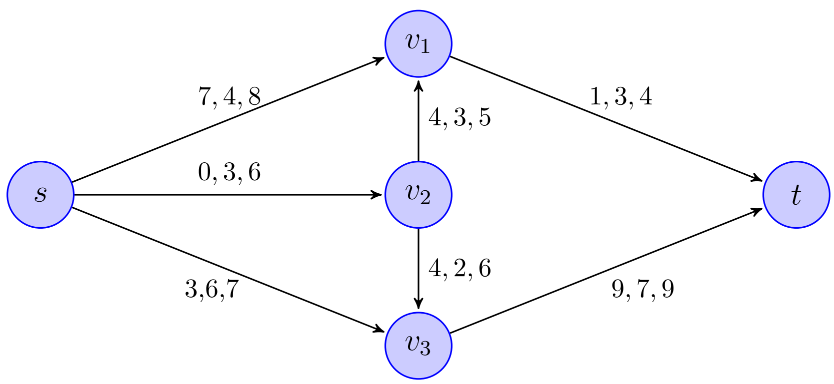
Exercise 1. Consider the *atomic network congestion game*, with three players, described by the following directed graph.



In this game, every player i (for $i = 1, 2, 3$) needs to choose a directed path from the source s to the target t. Thus, every player i's set of possible actions (i.e., its set of pure strategies) is the set of all possible directed paths from s to t.

Each edge e is labeled with a sequence of three numbers (c_1, c_2, c_3). Given a profile $\pi = (\pi_1, \pi_2, \pi_3)$ of pure strategies (i.e., s-t-paths) for all three players, the *cost* to player i of each directed edge, e, that is contained in player i's path π_i, is c_k, where k is the total number of players that have chosen edge e in their path. The total cost to player i, in the given profile π, is the sum of the costs of *all* the edges in its path π_i from s to t. Each player of course wants to minimize its own total cost.

Compute a pure strategy Nash Equilibrium in this atomic network congestion game.

Exercise 2. Consider the class of linear network congestion games, i.e., network congestion games with linear cost functions $c_e(x) = a_r \cdot x + b_r$. For these games, the cost functions can be represented by providing a_e and b_e for each edge $e \in E$ by its binary representation. The strategy sets can be represented *implicitly*, by using the graph G as an input (e.g., represented as an adjacency list) and the origin-destination (source-target) pair (o_i, d_i) for each player i.

- Explain why the best response dynamics algorithm for finding a pure Nash equilibrium of such a game is still a pseudopolynomial time, under this representation of the cost functions.
- Now that the strategies are implicitly represented rather than explicitly, can the best response of an agent be computed in polynomial time? Justify your answer.

Exercise 3.

A. Consider any linear congestion game (i.e., a congestion game with linear cost functions), and any strategy profile $s \in S_1 \times \ldots \times S_n$ in this game. Let $\Phi(s)$ be the value of Rosenthal's potential function on input s, and let $\mathrm{SC}(s)$ be the social cost of s. Show that

$$\frac{1}{2}C(s) \leq \Phi(s) \leq C(s)$$

Recall that Rosenthal's potential function is defined as:

$$\Phi(s) = \sum_{r \in R} \sum_{j=1}^{n_r(s)} c_r(j),$$

where $n_r(s) = \#(r, s)$ is the number of players that use resource r under strategy profile s.

B. Consider a congestion game for which we have the following guarantee: The cost functions c_r are such that no resource is every used by more than λ players. Use the Potential Method (and Rosenthal's potential function) to show that the Price of Stability of any such game is at most λ.

Exercise 4. Consider the class of *singleton congestion games*, i.e., congestion games in which the strategies of the players consist of single resources. Show that in singleton congestion games, a pure Nash equilibrium can be computed in polynomial time.

Hint: Starting from a singleton congestion game G, construct an "equivalent" game $\tilde{G}$ with $\tilde{c}_{\max} = poly(n, m)$, and argue that the best response dynamics algorithm converges in $\tilde{G}$ in polynomial time.

Exercise 5 (Cut Games)**.** Consider the following class of games, called *cut games*. We have an undirected graph $G = (V, E)$ in which each player i controls a vertex $v_i \in V$. Each edge $e \in E$ is associated with a *nonnegative* weight w_e. A *cut* is a partition of the set of vertices into two disjoint sets LEFT and RIGHT. Each player selects the set in which his contorolled vertex will be, i.e. the strategy space of each player i is $s_i = \{\text{LEFT, RIGHT}\}$ and let $s = (s_1, \ldots, s_n)$ be the corresponding strategy profile.

Let $\text{CUT}(s)$ be the set of edges that have one endpoint in each set, and let N_i be the set of neighbours of v_i in the graph G. The utility of player i is defined as

$$u_i(s) = \sum_{e \in \text{CUT}(s) \cap N_i} w_e.$$

A. Design a cut game in which the Price of Anarchy is 2.

Hint: Use a graph with four vertices and unit weights.

B. Prove that the Price of Anarchy of cut games is at most 2.

Hint: First argue that in a pure Nash equilibrium, the total weight of the neighbouring vertices of a vertex v_i in the cut is at least half the total weight of all the neighbouring vertices of the vertex.