AGTA Tutorial 7 - Solutions

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Exercise 1. By inspection, we can see that the maximum achievable social welfare is 15, arising from bidder B getting 2 of the items and bidder C getting the remaining 2 items. To compute the payment of bidder i, we have to compute the difference between the maximum social welfare had i not participated in the auction and the social welfare of the outcome above, where i has participated in the auction. Using this, we calculate

$$\begin{aligned} p_A &= 15 - 15 = 0 \\ p_B &= 14 - 8 = 6 \\ p_C &= 14 - 7 = 7 \end{aligned}$$

Exercise 2.

A. The second price auction allocates the item to the highest bidder, therefore it is also welfare maximizing. Additionally, we can see that the payments of the second-price auction are consistent with those of the VCG mechanism, since all bidders except the highest one pay 0, and the highest bidder pays the value of the second highest bidder v_2, which is exactly the maximum social welfare achievable if we remove the highest bidder.

B. It is straightforward to see that the only possible monotone allocation f gives the item to the highest bidder. To compute the payments, we can then use the definition of the critical value:

$$c_i(\mathbf{v_{-i}}) = \sup_{v_i : f(v_i, \mathbf{v_{-i}}) \neq W_i} v_i$$

We can then easily see that the critical value for the winning bidder is exactly the second highest value, i.e., v_2, therefore the second condition of Myerson's characterization is satisfied, since the winner pays her critical value.

C. Consider a second-price auction with two bidders having values v_1, v_2 such that $v_1 > v_2$. Then, the strategy profile where bidder 1 bids 0 and bidder 2 bids $b > v_1$ is a pure Nash equilibrium (although bidder 1's strategy is weakly dominated). At this equilibrium, the social welfare is v_2 (the worst possible social welfare, since by the description of the second-price auction at least one of the bidders will get the item and $v_1 > v_2$). Hence, the resulting Price of Anarchy is:

$$\text{PoA} = \frac{\text{OPT(SW) at equilibrium}}{\text{worst SW at equilibrium}} = \frac{v_1}{v_2}$$

which is unbounded (we could have $v_2 = 0$).

Exercise 3.

A. The expected revenue can alternatively be seen as the expectation of the minimum of the two random variables:

$$\mathbb{E}[\text{Rev}] = \mathop{\mathbb{E}}_{v_1,v_2\sim U[0,1]}[\min\{v_1,v_2\}] = \int_0^1 \Pr(\min\{v_1,v_2\} > x)\ \mathrm{d}x =$$
$$= \int_0^1 \Pr(v_1 > x)\cdot \Pr(v_2 > x)\ \mathrm{d}x = \int_0^1 (1-x)^2\ \mathrm{d}x = \left[x - x^2 + \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

B. We can express the expected revenue as the sum of the following expectations, conditioning on the relation between v_1, v_2 and the reserve price r (we will show the result for arbitrary r and then we can substitute $r = 1/2$ as required):

$$\mathbb{E}[\text{Rev}] = \mathbb{E}[\text{Rev} \mid v_1,v_2 < r] + \mathbb{E}[\text{Rev}|v_1 \geq r, v_2 < r] + \mathbb{E}[\text{Rev}|v_1 < r, v_2 \geq r] + \mathbb{E}[\text{Rev} \mid v_1,v_2 \geq r] =$$
$$= 0 + 2\cdot \int_0^r \int_r^1 r\ \mathrm{d}x\ \mathrm{d}y + \int_r^1\int_r^1 \min\{x,y\}\ \mathrm{d}x\ \mathrm{d}y = 2r^2(1-r) + \int_r^1 \left(\int_r^y x\ \mathrm{d}x\ + \int_y^1 y\ \mathrm{d}x\right)\ \mathrm{d}y =$$
$$= 2r^2(1-r) + \int_r^1 \left[\frac{x^2}{2}\right]_r^y + y(1-y)\ \mathrm{d}y = 2r^2(1-r) + \frac{1}{2}\left[\frac{y^3}{3} - r^2 y\right]_r^1 + \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_r^1 = \frac{1}{3} + r^2 - \frac{4}{3}r^3$$

We can then use the first order condition to see that the maximum revenue is achieved when

$$\frac{d}{dr}\left(\frac{1}{3} + r^2 - \frac{4}{3}r^3\right) = 0 \Rightarrow 2r - 4r^2 = 0 \Rightarrow r = \frac{1}{2}$$

which we can then substitute into the formula for the expected revenue to get an expected revenue of $5/12$.

C. To compute the maximum possible expected revenue that can be extracted in this case we can use Myerson's optimal auction. We first need to compute the virtual valuations:

$$\phi_i(v_i) = v_i - \frac{1 - F(v_i)}{f(v_i)} = 2v_i - 1$$

From the optimal auction, we know that the expected revenue is equal to the expectation of the maximum $\phi_i(v_i)^+$, where we define $z^+ := \max\{z, 0\}$. Let M be the random variable corresponding to $\max \phi_i(v_i)^+$. To compute the expected value of M, we begin by calculating its cdf:

$$F_M(m) = \Pr(M \leq m) = \Pr(2v_1 - 1 \leq m)^2 = \Pr\left(v_1 \leq \frac{m+1}{2}\right)^2 = \left(\frac{m+1}{2}\right)^2$$

Taking the derivative of the cdf, we get that the pdf of M is:

$$f_M(m) = \frac{m+1}{2}$$

We can now compute the expected value of M:

$$\mathbb{E}[M] = \int_0^1 m\cdot\frac{m+1}{2}\ \mathrm{d}m = \frac{5}{12}$$

where the integral is taken from 0 to 1 instead of -1 to 1 because of the $(\cdot)^+$ operator, which forces m to be nonnegative. Notice that this matches the expected revenue we computed in **B** with a second-price auction with reserve price $1/2$.