

Algorithms and Data Structures

Upper and Lower Bounds for Sorting, Matrix
Multiplication

Worst-case Running Times, Upper Bounds

Algorithm **Mergesort**(**A**[*i*, ..., *j*])

If *i=j*, return *i*

q=(*i+j*)/2

A_{left}=**Mergesort**(**A**[*i*, ..., *q*])

A_{right}=**Mergesort**(**A**[*q*+1, ..., *n*])

return **Merge**(**A**_{left} , **A**_{right})

Recurrence relation:

$$T(n) = 2T(n/2) + f(n)$$

where $f(n) = O(n)$

If we solve the recurrence relation we obtain

$$T(n) = O(n \lg n)$$

To be exactly precise

Recurrence relation: For some constant c ,

$$T(n) = \begin{cases} T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn, & \text{when } n > 2. \\ T(2) \leq c, & \text{otherwise.} \end{cases}$$

**How do we solve this
recurrence relation?**

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How do we solve this recurrence relation?

**We can use the
Master Theorem!**



The Master Theorem

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Suppose $T(n) \leq \alpha T(\lceil n/b \rceil) + O(n^d)$
for some constants $\alpha > 0$, $b > 1$ and $d \geq 0$.

$$\text{Then, } T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b \alpha \\ O(n^d \log_b n), & \text{if } d = \log_b \alpha \\ O(n^{\log_b \alpha}), & \text{if } d < \log_b \alpha \end{cases}$$

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Example: For MergeSort, $\alpha = b = 2$ and $d = 1$, we get $O(n \log n)$

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Wait a minute John!
This is ADS, not IADS.



How do we solve this recurrence relation?



How do we solve this recurrence relation?

Ah, that's right!



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“Guess and verify”: We guess the solution, and verify that it works after substituting in the recurrence relation. Often the argument is by *induction on n* .

“Unrolling the recursion”: Figure out the solution for the first few *levels*, and then identifying a pattern. In the end we sum the solutions over all the levels. Often also called the method of *“recursion trees”*.

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For $n = 8$, we have $T(8) = 2T(4) + 8c \leq 12c + 8c = 20c$.
 $8c \log_2 8 = 24c > 20c$.

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Assume that $T(2^{m-1}) \leq c \cdot 2^{m-1} \cdot \log_2 2^{m-1}$ holds for $n = 2^{m-1}$.

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Let's use the “unrolling” technique on this recurrence relation

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Solving the Mergesort recursion

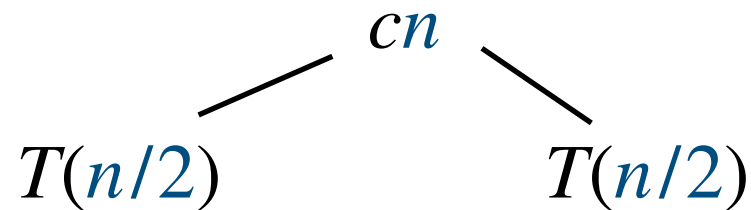
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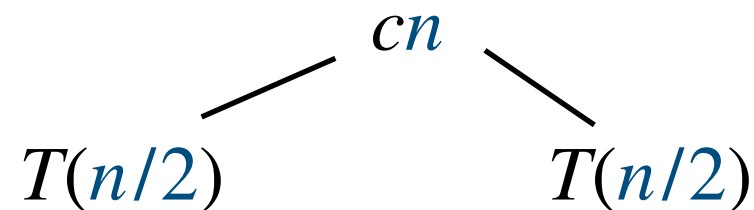
First iteration: Price of cn plus the cost of two subproblems of size $n/2$.



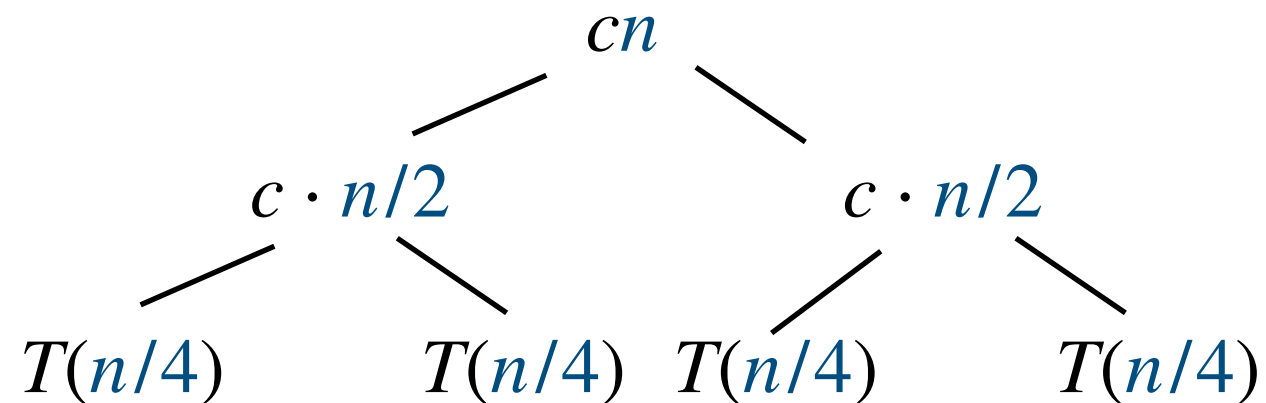
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Second iteration: Price of $c \cdot n/2$ for each subproblem, plus the cost of two subproblems of size $n/4$



Solving the Mergesort recursion

In total, there will be $\log n + 1$ levels (input halved every time).

Level 0 has cost $C_0(n) = cn$

Level 1 has cost $C_1(n) = 2c \cdot n/2 = cn$

Level 2 has cost $C_2(n) = 4c \cdot n/4 = cn$

Level j has cost $C_j(n) = 2^j c \cdot n/2^j = cn$

The last level has cost cn

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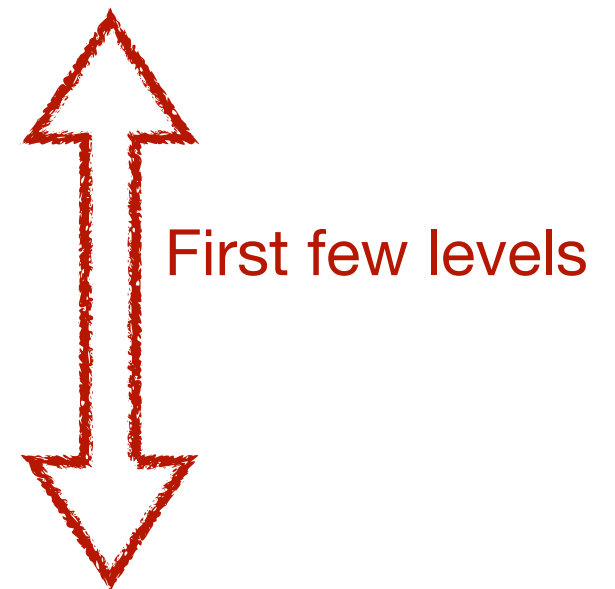
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First few levels

Identifying a pattern

Solving the Mergesort recursion

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Recurrence relation:

$$T(n) = \sum_{j=1}^{\log n+1} C_j(n) + cn = \sum_{j=1}^{\log n+1} cn + cn$$

Solving the Mergesort recursion

Recurrence relation:

Sum the solutions

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Solving the Mergesort recursion

Recurrence relation:

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$$T(n) = \sum_{j=1}^{\log n + 1} C_j(n) + cn = \sum_{j=1}^{\log n + 1} cn + cn$$

The overall running time is $O(n \log n)$.

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For a formal proof, those techniques could be used together.

The Quicksort algorithm

Algorithm **Quicksort**($\mathbf{A}[i, \dots, j]$)

$y = \text{Partition}(\mathbf{A}[i, \dots, j])$

Quicksort($\mathbf{A}[i, \dots, y-1]$)

Quicksort($\mathbf{A}[y+1, \dots, j]$)

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When $n_1 = n_2$, the running time is the same as **Mergesort**.

Running time of Quicksort

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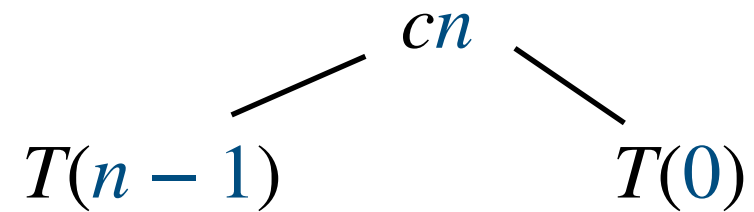
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What is the worst possible running time?

“Unrolling”

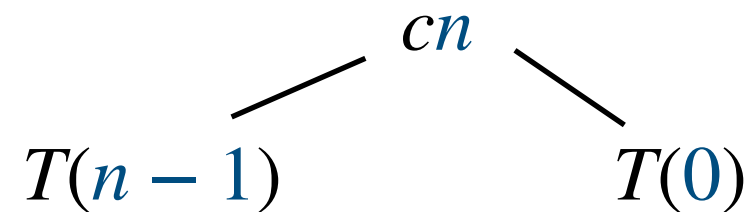
“Unrolling”

First iteration: Price of cn plus the cost of two subproblems of size $n - 1$ and 0 .

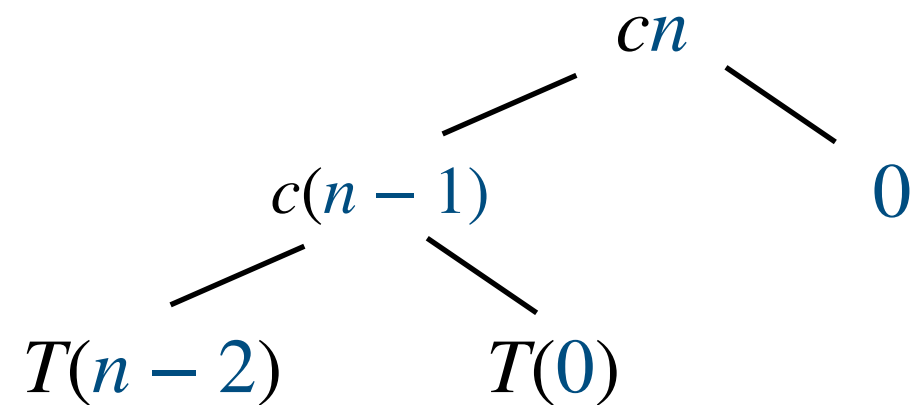


“Unrolling”

First iteration: Price of cn plus the cost of two subproblems of size $n - 1$ and 0 .



Second iteration: Price of $c(n - 1)$, plus the cost of two subproblems of size $n - 2$ and 0 .

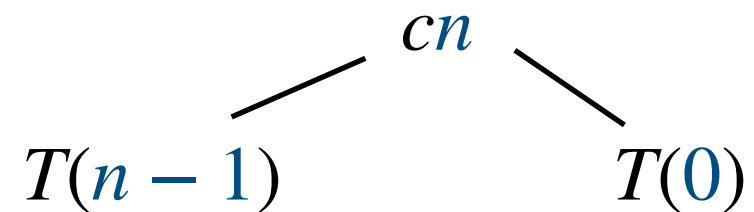


Solving the Quicksort recursion

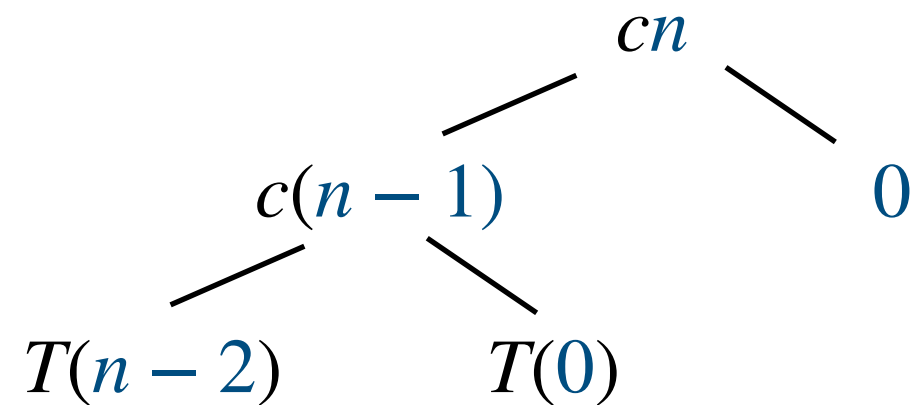
How many levels do we have in total?

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Solving the Quicksort recursion

How many levels do we have in total? n levels.

Level 0 has cost $C_0(n) = cn$

Level 1 has cost $C_1(n) = c(n - 1)$

Level 2 has cost $C_2(n) = c(n - 2)$

Level j has cost $C_j(n) = c(n - j)$

The last level has cost c

Solving the Quicksort recursion

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Recurrence relation:

$$\begin{aligned} T(n) &= \sum_{j=1}^{n-1} C_j(n) = c(n + n - 1 + n - 2 + \dots + 1) \\ &= c \frac{n(n+1)}{2} \leq cn^2 \end{aligned}$$

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Can we show a lower bound of $\Omega(n^2)$ on the running time of Quicksort?

Upper and Lower (Worst-Case) Bounds

Upper Bound $O(g_1(n))$: On *any possible input* to the problem, our algorithm will take time (at most) $O(g_1(n))$.

Lower Bound $\Omega(g_2(n))$: There *exists at least one input* to the problem, on which our algorithm will take time (at least) $\Omega(g_2(n))$.

When $g_1(n) = g_2(n)$, we say that our running time analysis is *tight*, and we have fully understood the (asymptotic, worst-case) running time of the algorithm.

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Q: Can you think of an input where Quicksort takes time $\Omega(n^2)$?

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Upper and Lower (Worst-Case) Bounds *for algorithms*

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Upper Bound $O(n \log n)$: We have identified an algorithm (Mergesort) that has worst-case running time $O(n \log n)$.

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Lower Bound $\Omega(n \log n)$: We need to prove that *every algorithm* has worst-case running time $\Omega(n \log n)$.

For us here concretely

Upper Bound $O(n \log n)$: We have identified an algorithm (Mergesort) that has worst-case running time $O(n \log n)$.

Lower Bound $\Omega(n \log n)$: We need to prove that *every algorithm* has worst-case running time $\Omega(n \log n)$.

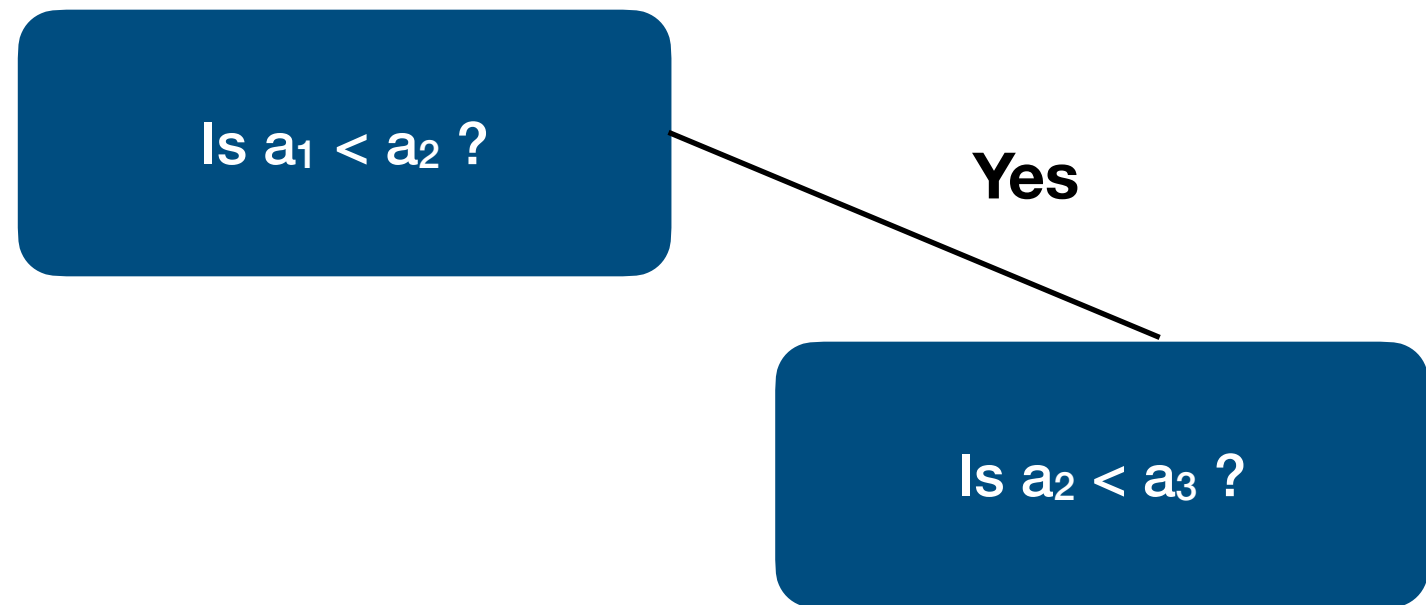
In other words, we will prove that there is no algorithm that is *asymptotically better* than Mergesort.

Lower bound for sorting

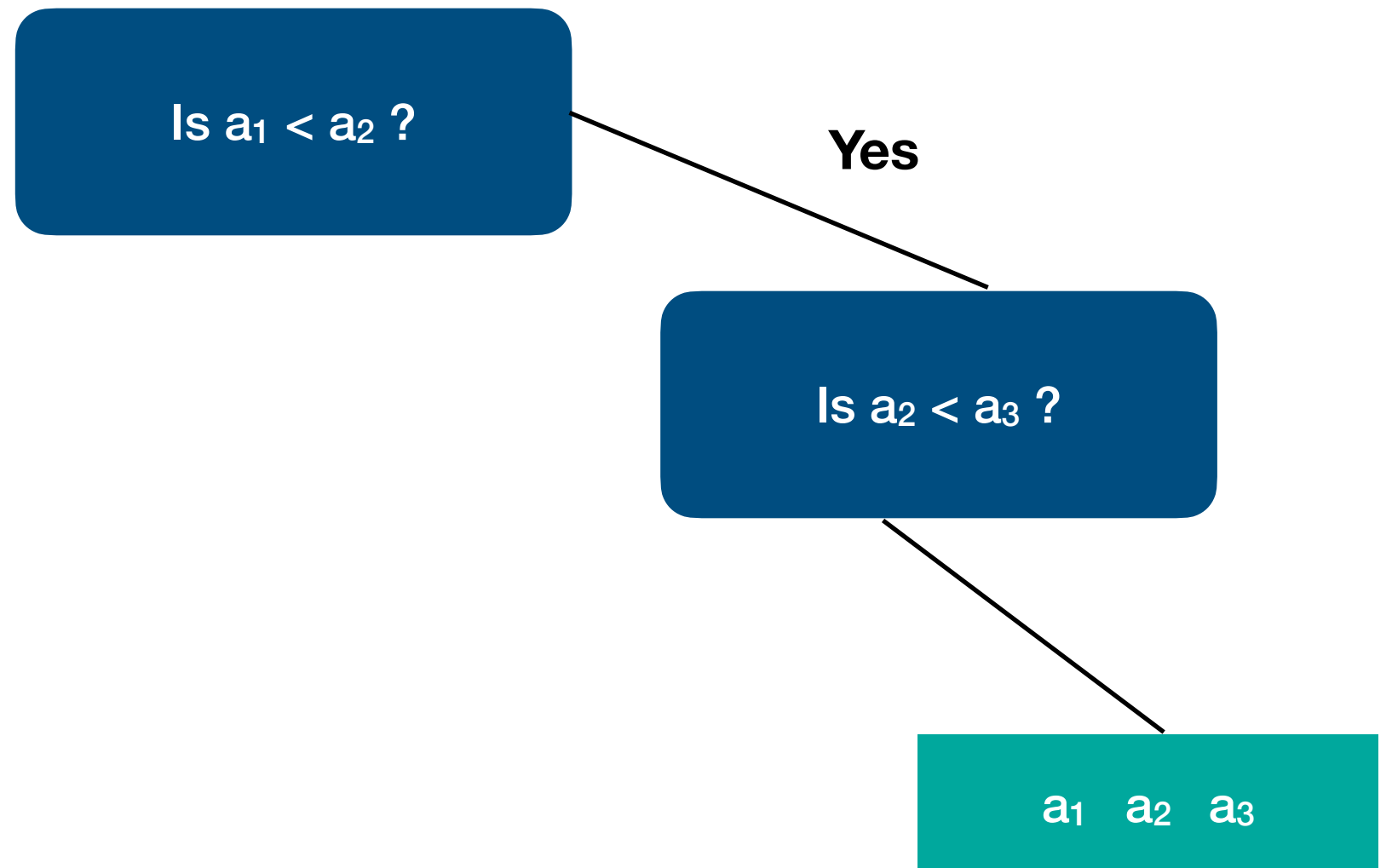
Lower bound for sorting

Is $a_1 < a_2$?

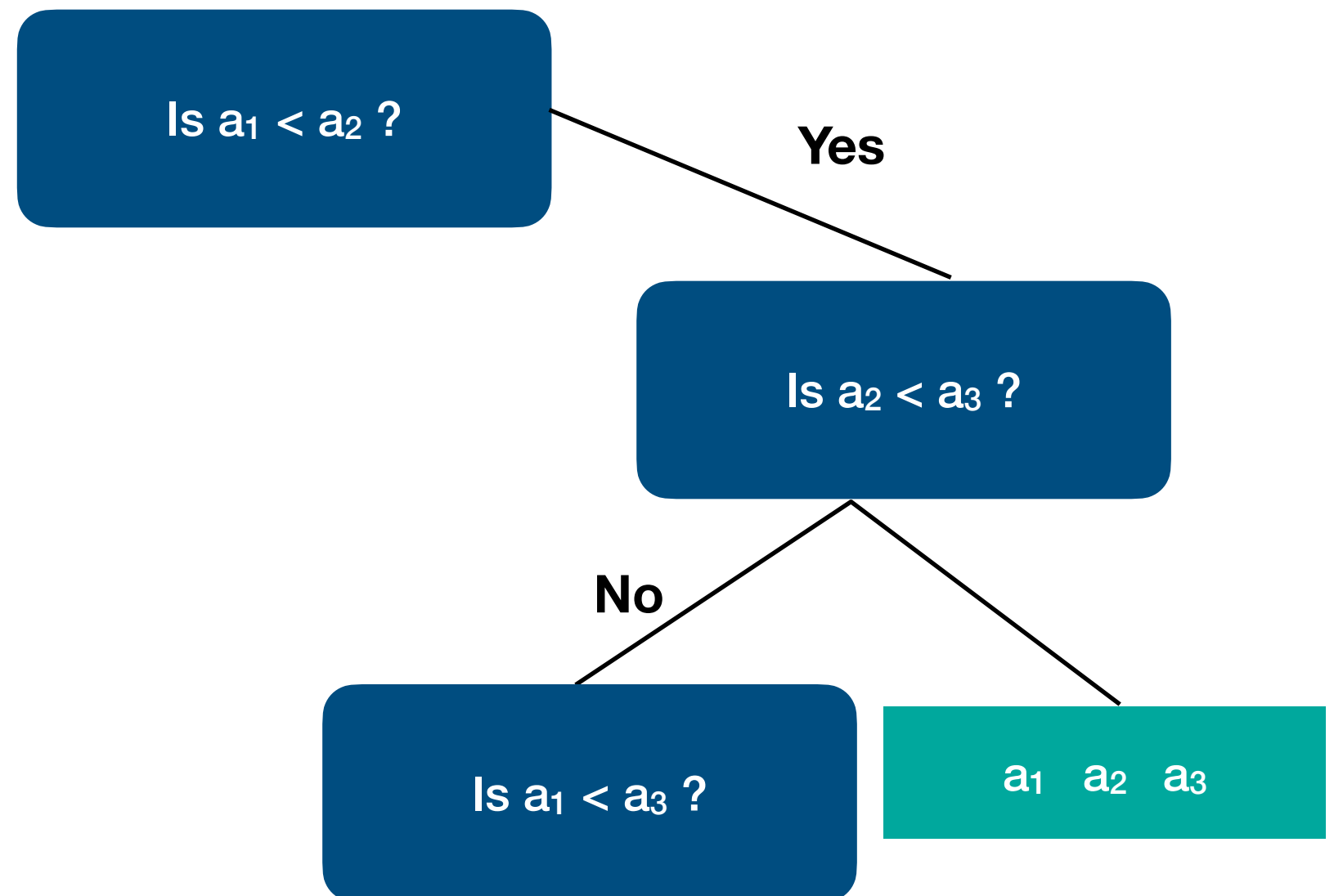
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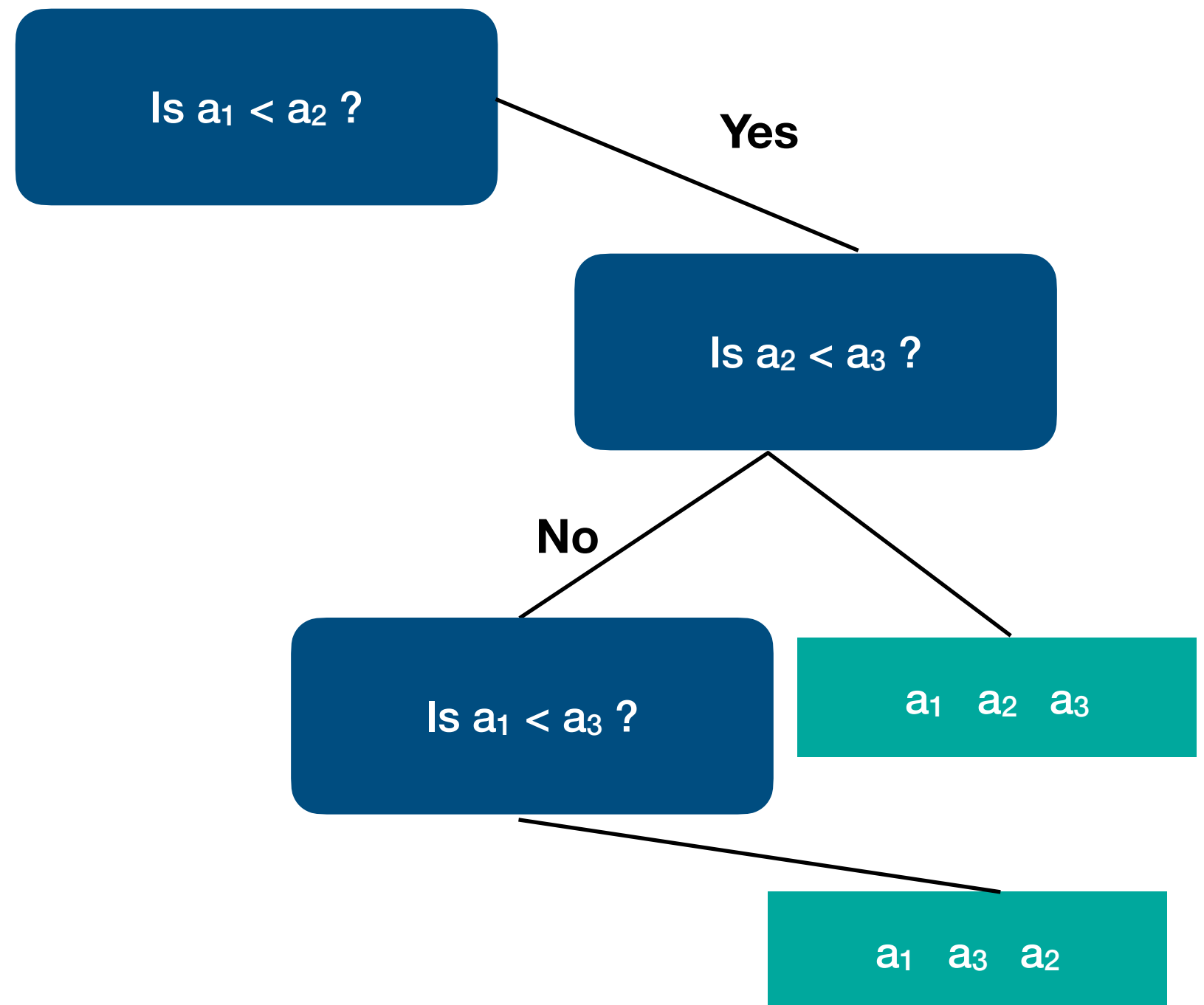
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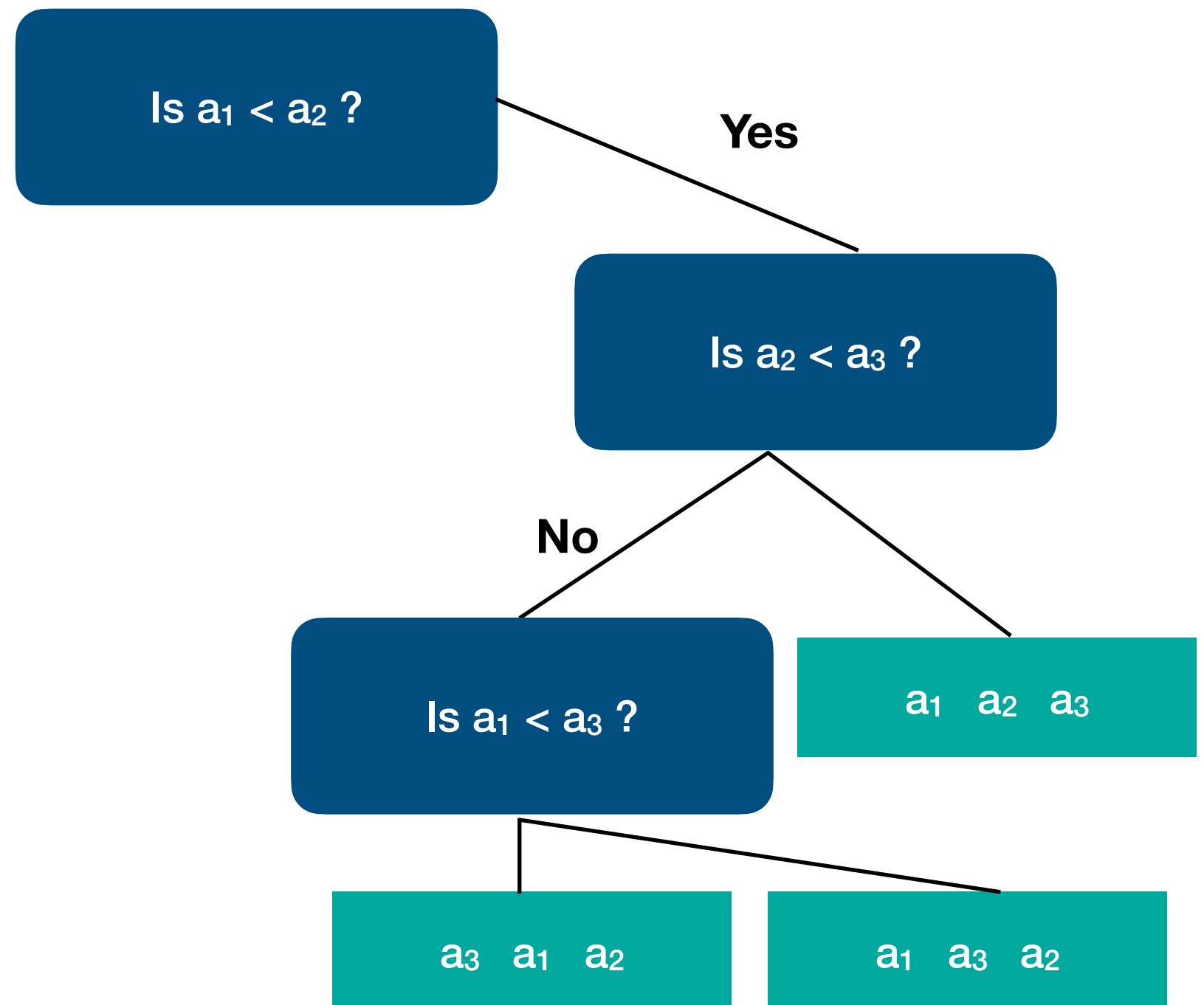
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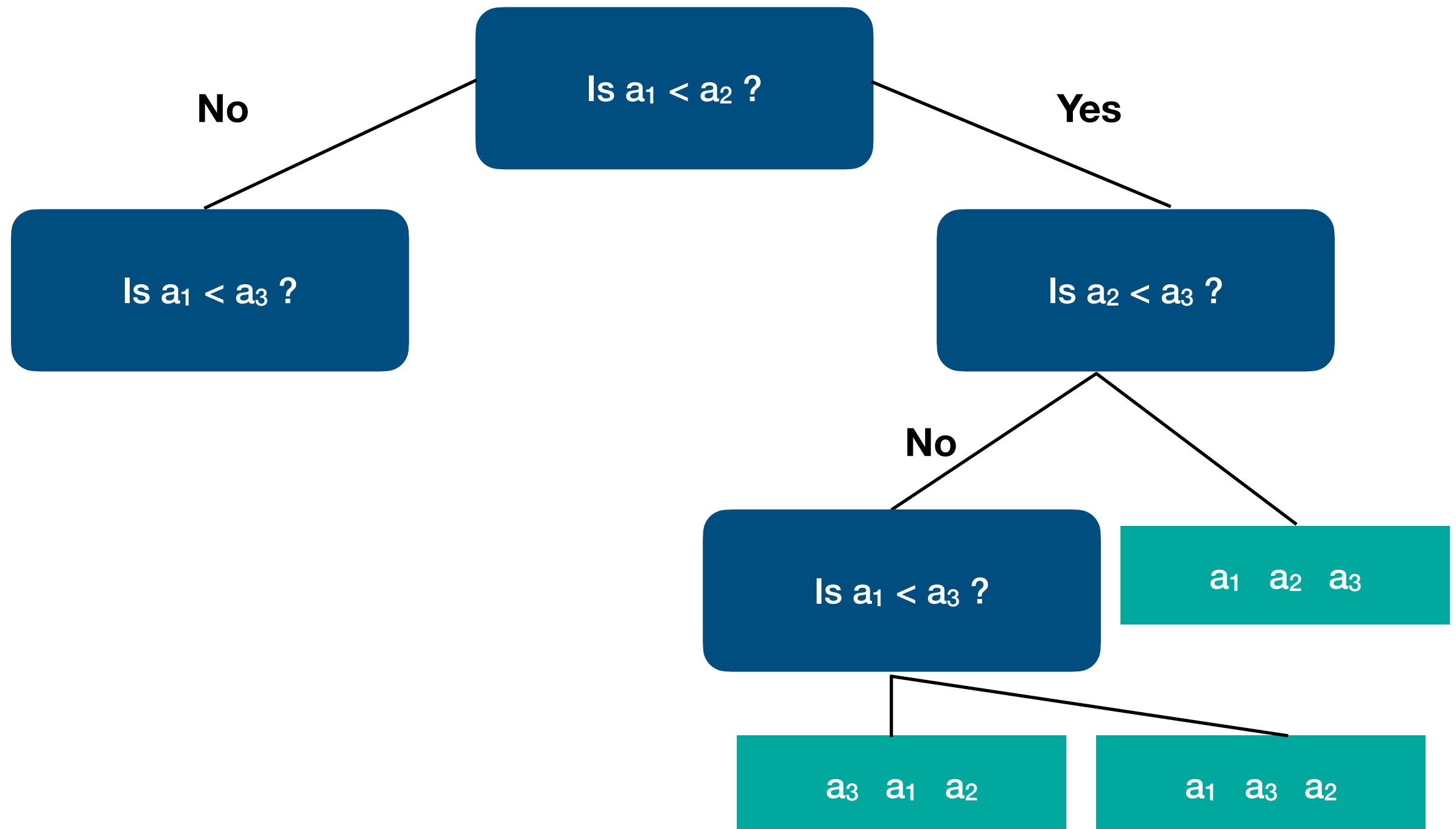
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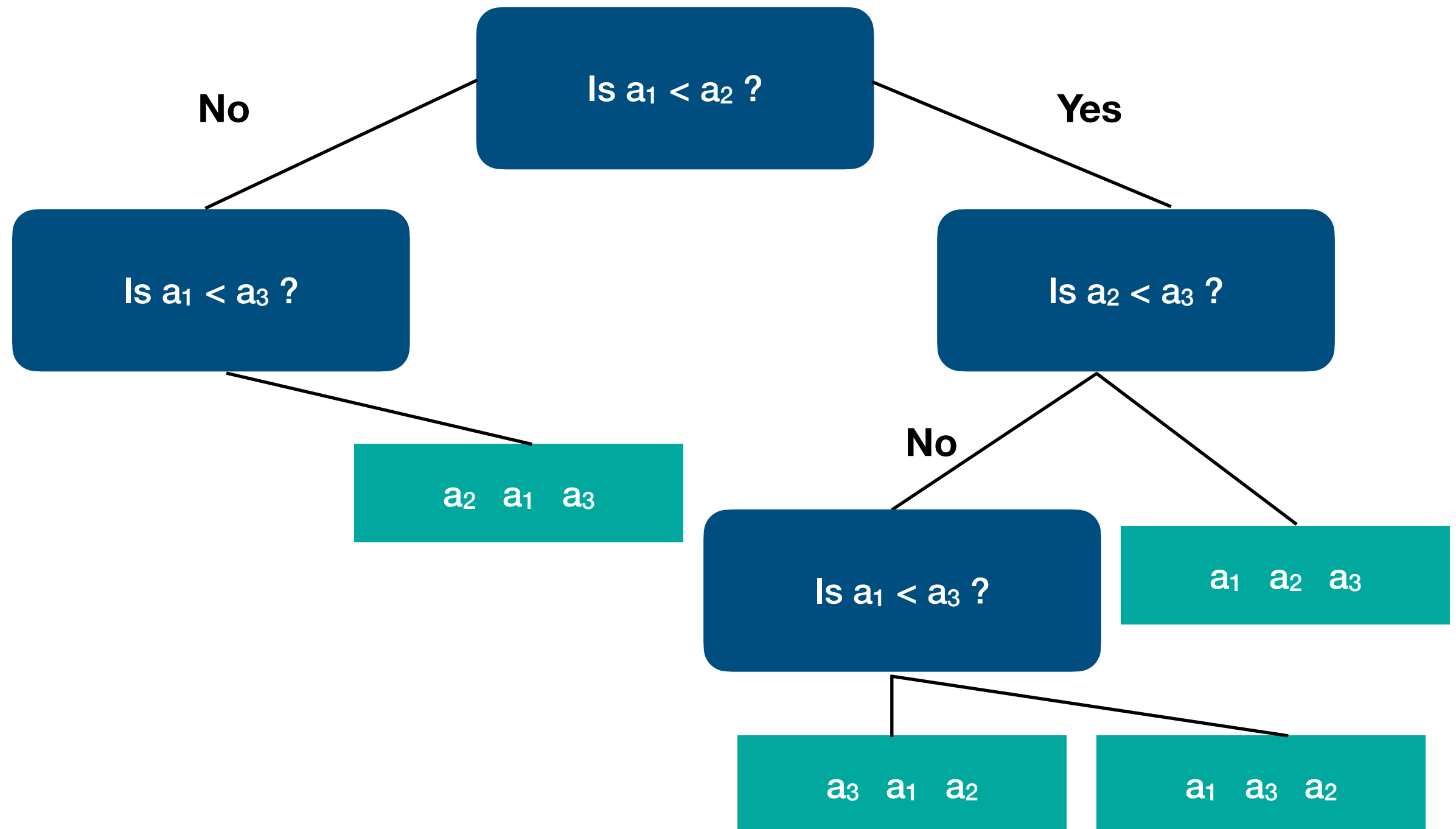
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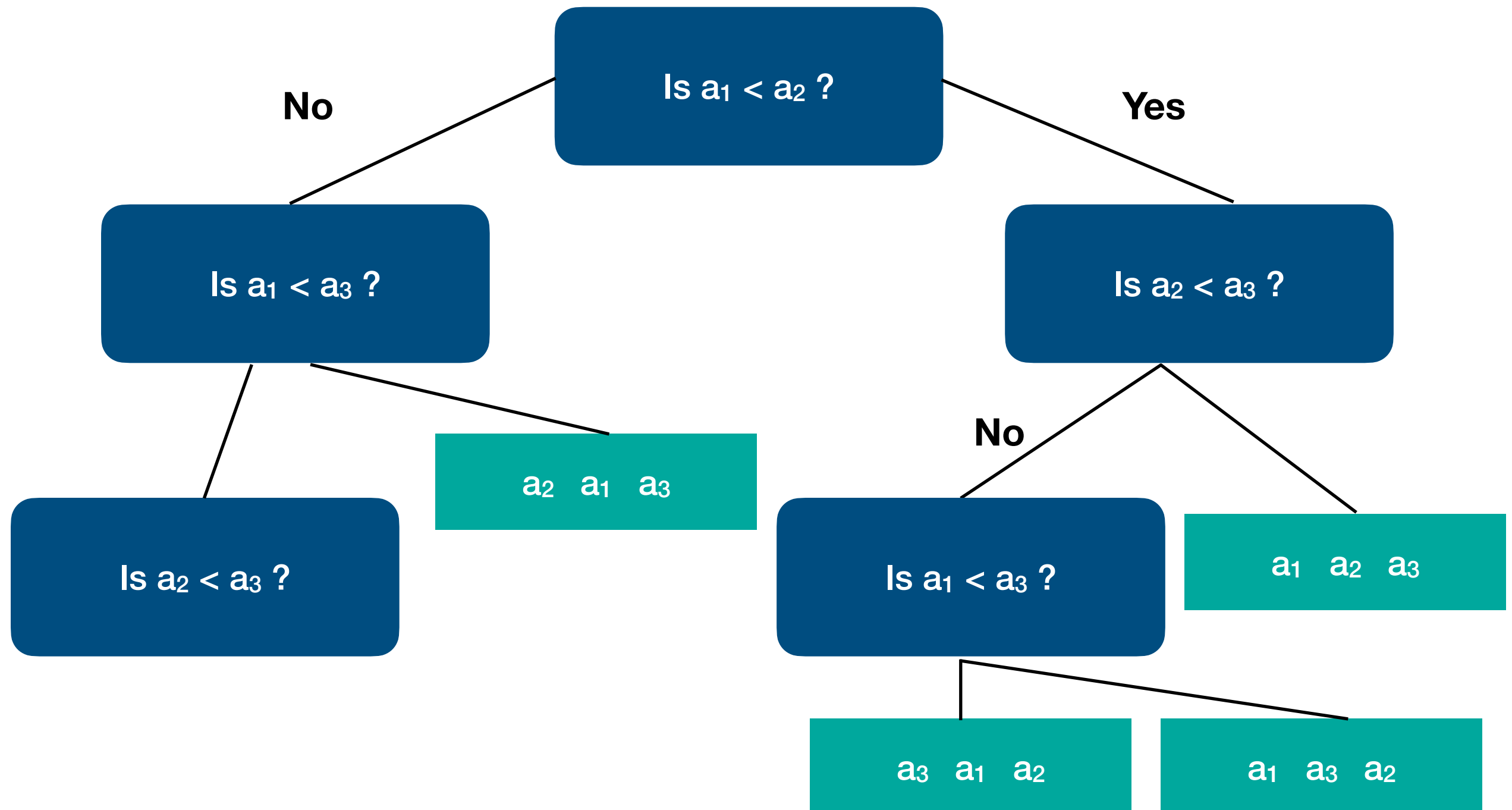
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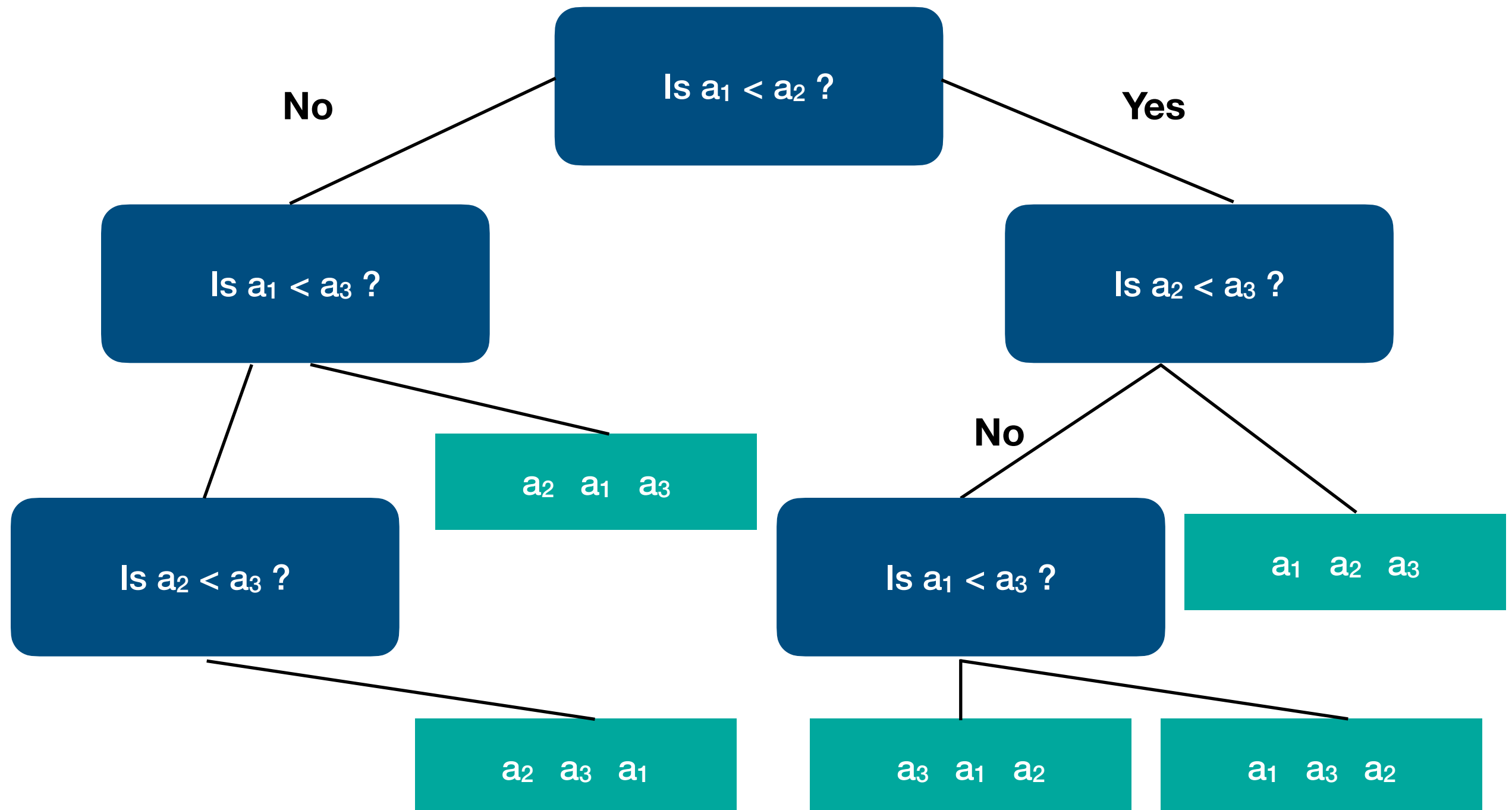
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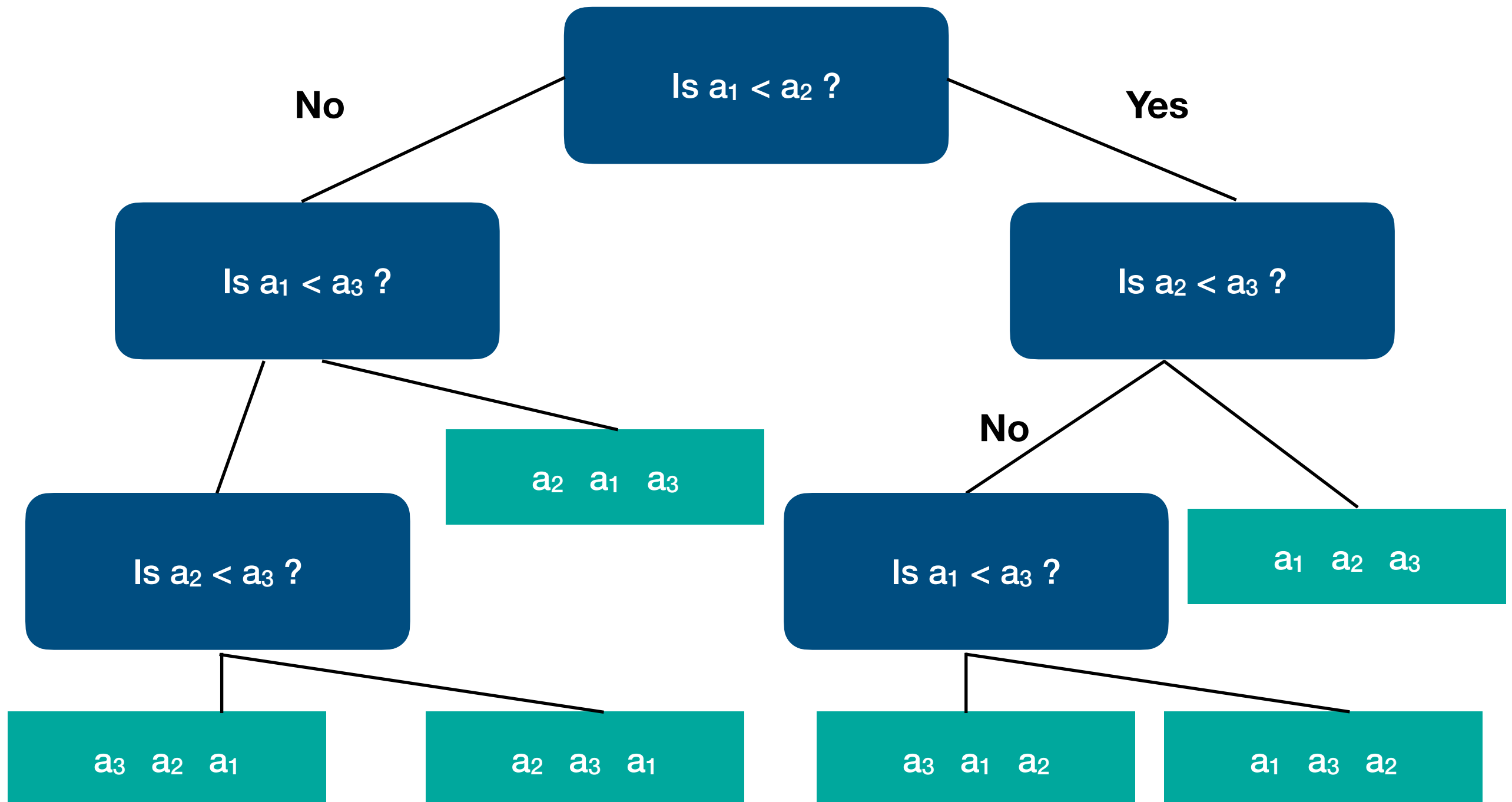
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Every possible permutation can appear as a leaf, since every possible permutation is a valid output.

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e.g., CountingSort

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Upper bound: We construct an algorithm that has performance $O(g(n))$ for **criterion A**.

Lower bound: We show that for any algorithm, the performance for **criterion A** is $\Omega(g(n))$.

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We try to design algorithms which are as good as possible and when we feel that we can not improve more, we try to prove the *matching* lower bound.

Matrix Multiplication

Matrix Multiplication

Assume that we have two square $(n \times n)$ -matrices

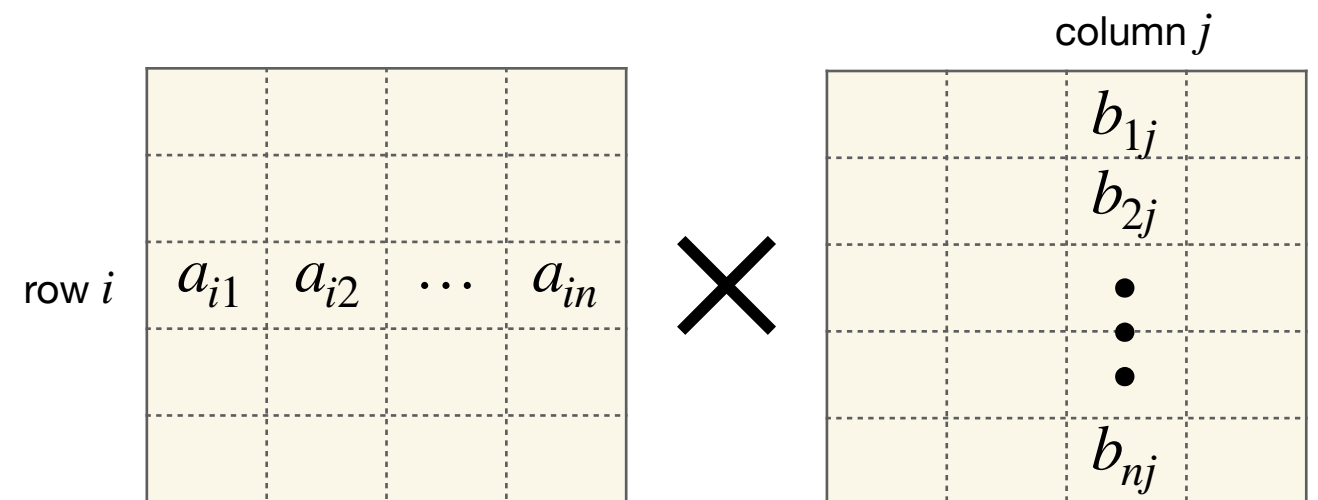
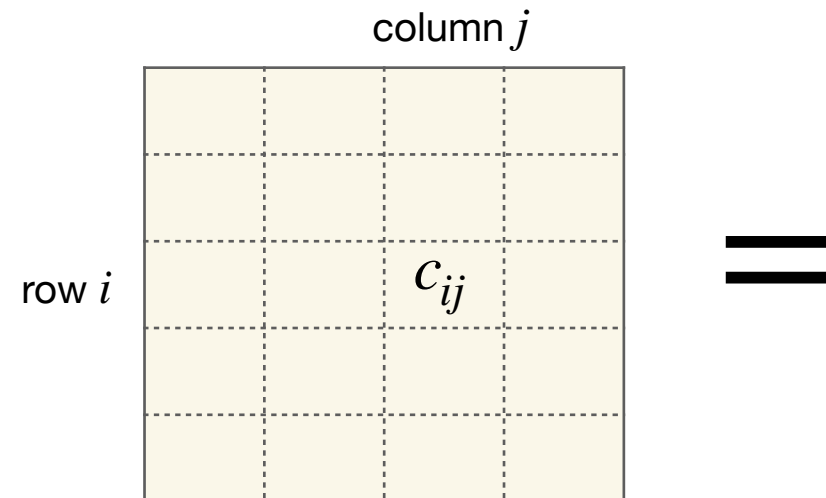
$$A = (a_{ij})_{1 \leq i, j \leq n} \text{ and}$$

$$B = (b_{ij})_{1 \leq i, j \leq n}$$

The product of A and B is the $(n \times n)$ -matrix

$$C = (c_{ij})_{1 \leq i, j \leq n} \text{ with entries}$$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$



A straightforward approach

Compute the sum of pairwise products $a_{ik} \cdot b_{kj}$ for each entry c_{ij} of C .

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Running time: $\Theta(n^3)$

A naive D&C approach

Suppose we divide our matrices A and B as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

We can write C as:

$$\begin{aligned} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ A_{21} \cdot B_{11} + A_{22} \cdot B_{21} & A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{pmatrix} \end{aligned}$$

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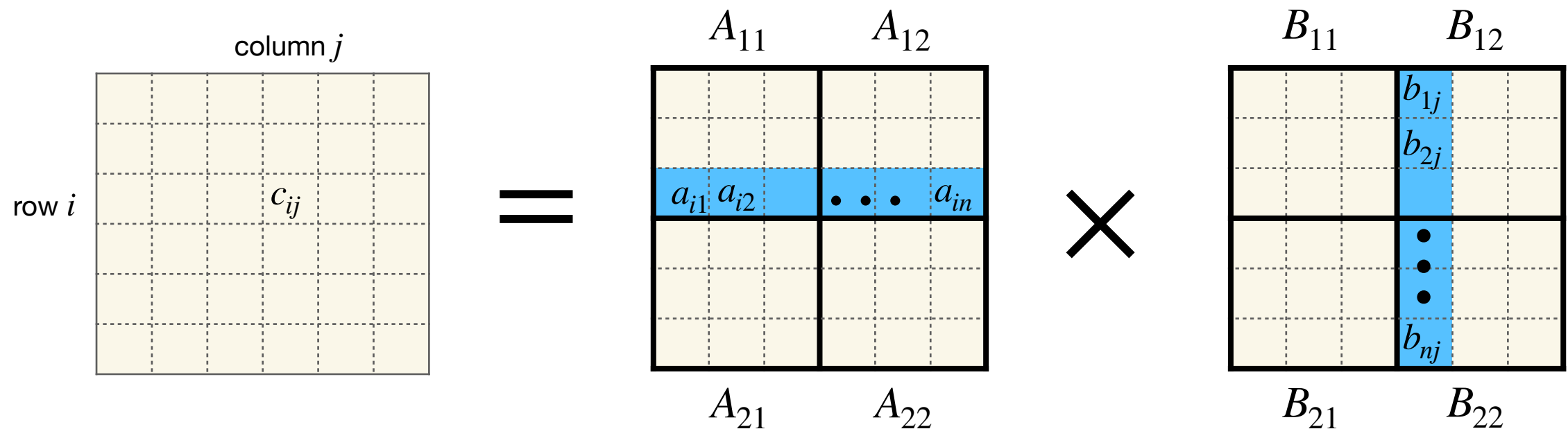
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We will assume from now on
that $n = 2^k$ for some k .

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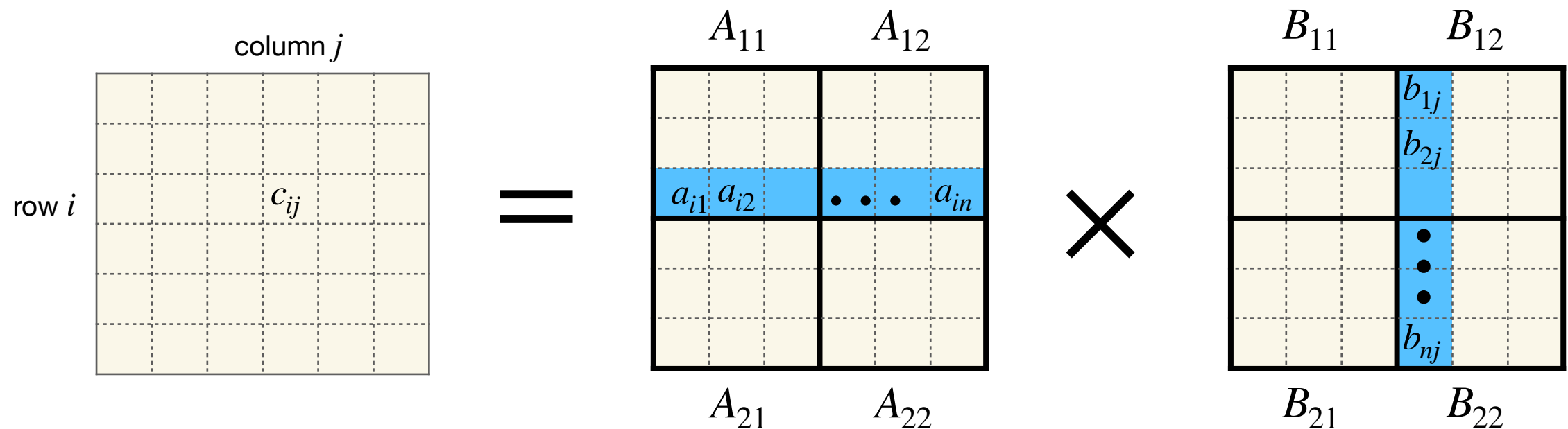


Suppose $i \leq n/2$ and $j > n/2$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} = \sum_{k=1}^{n/2} a_{ik} b_{kj} + \sum_{k=n/2+1}^n a_{ik} b_{kj}$$

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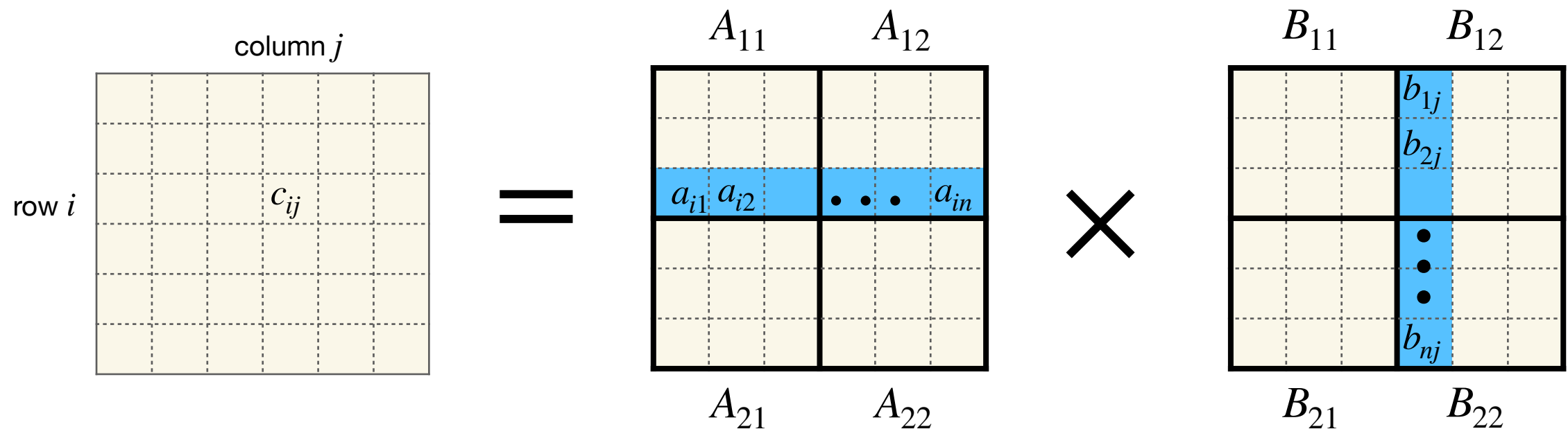


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The D&C algorithm

Matrix-Multiply-DC (A, B)

if $n = 1$, **do**

$$c_{11} = a_{11} \cdot b_{11}$$

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Partition $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ **and** $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$

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Running Time

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$8T(n/2)$

Running Time

Recurrence relation: For **some constant c**,

$$T(n) = \begin{cases} 8T(n/2) + cn^2, & \text{when } n > 1. \\ c, & \text{when } n = 1. \end{cases}$$

The Master Theorem

The Master Theorem is a very general theorem for solving recurrence relations.

Suppose $T(n) \leq \alpha T(\lceil n/b \rceil) + O(n^d)$
for some constants $\alpha > 0$, $b > 1$ and $d \geq 0$.

$$\text{Then, } T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b \alpha \\ O(n^d \log_b n), & \text{if } d = \log_b \alpha \\ O(n^{\log_b \alpha}), & \text{if } d < \log_b \alpha \end{cases}$$



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Strassen's remarkable algorithm
to the rescue (next lecture)

