

Algorithms and Data Structures

Fast Fourier Transform

Multiplying two polynomials

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The product is a polynomial $C(x)$ of degree $2n - 2$ where the coefficient of the term x^k is

$$c_k = \sum_{(i,j):i+j=k} a_i b_j$$

Equivalently: the coefficient vector c of $C(x)$ is the *convolution* $a * b$ of the coefficient vectors of $A(x)$ and $B(x)$.

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Fast Fourier Transform (FFT)

**Key idea: How to represent
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Representation 1: via their coefficient vectors

$$a = (a_0, a_1, \dots, a_{n-1}), b = (b_0, b_1, \dots, b_{n-1})$$

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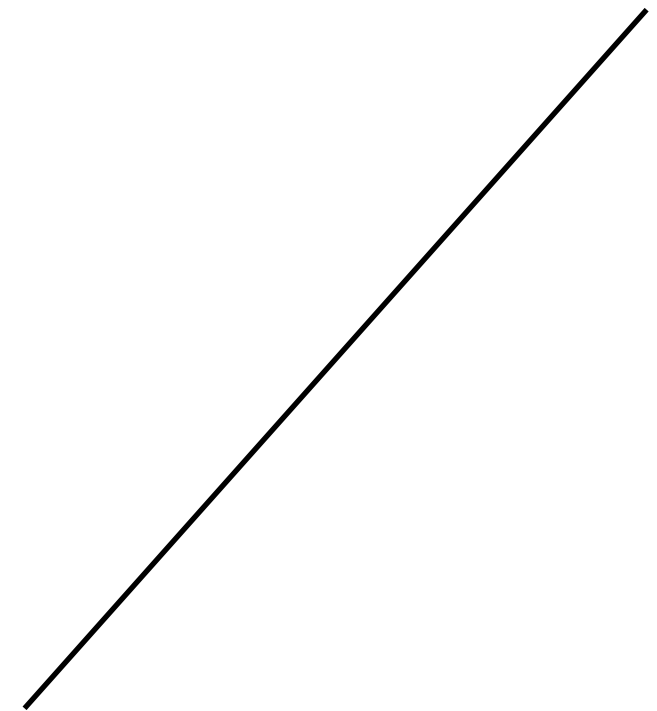
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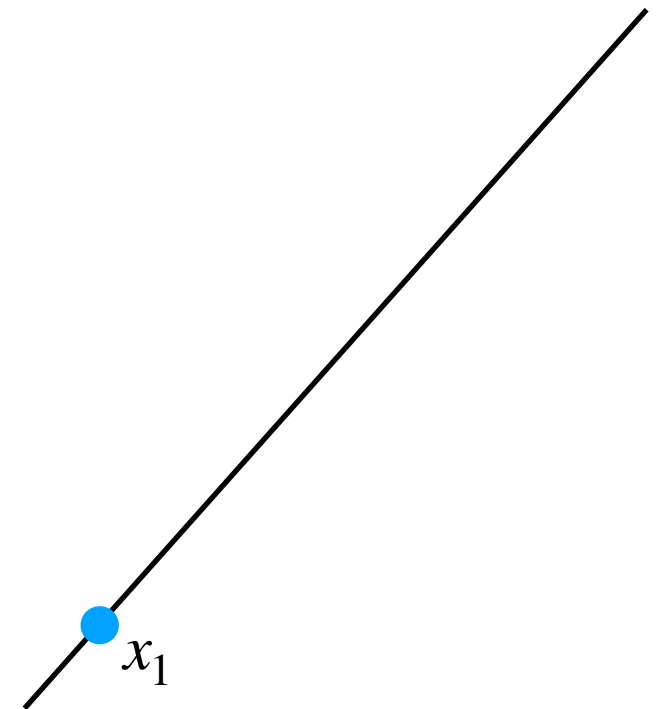
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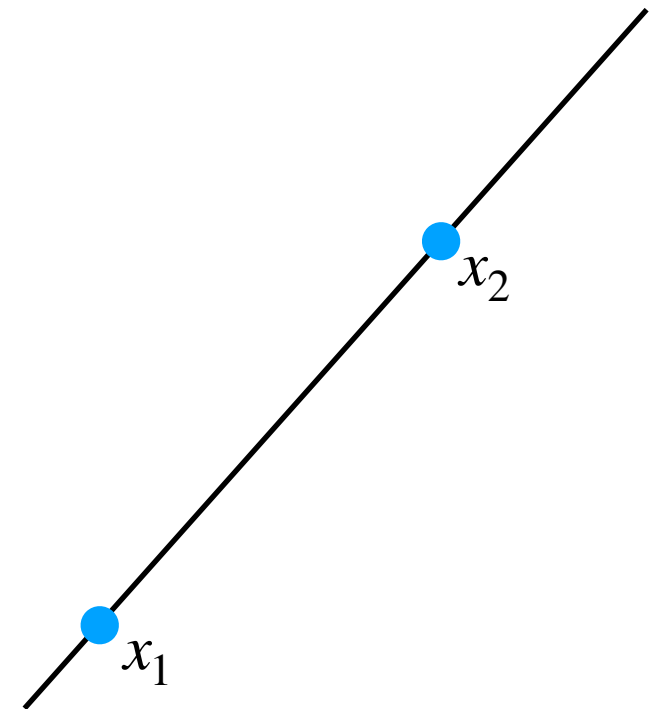
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Fact: Any polynomial of degree d can be represented by its values on at least $d + 1$ points.

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Representation 2: via their values on at least n points

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Step 1: Choose $2n$ values $x_1, x_2, \dots, x_{2n}$ and *evaluate* $A(x_j)$ and $B(x_j)$ for each $j = 1, 2, \dots, 2n$.

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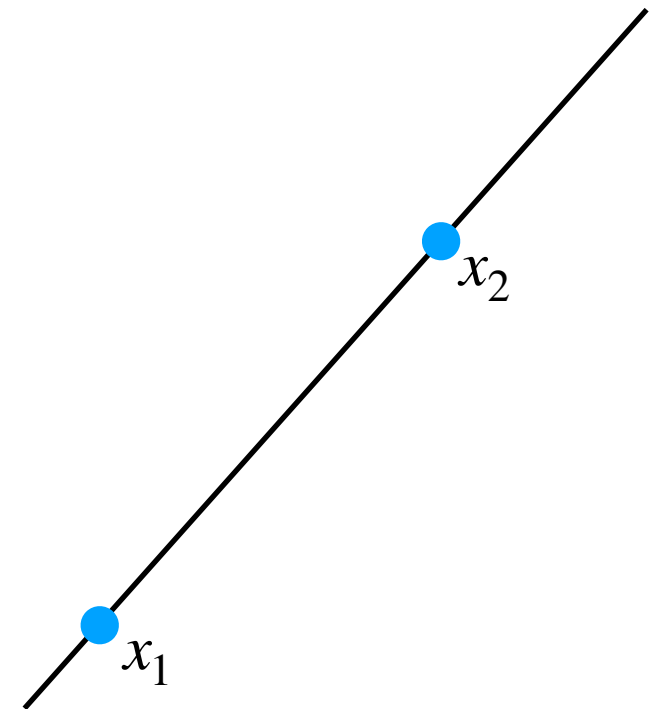
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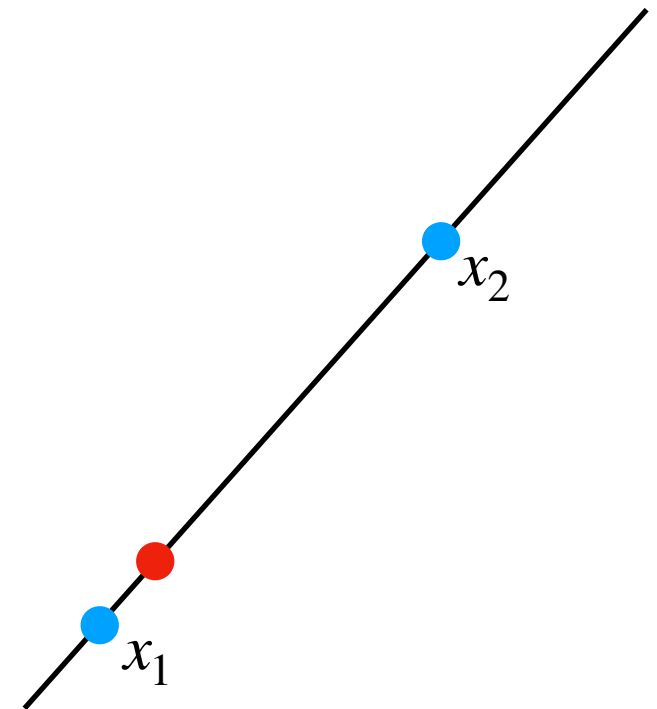
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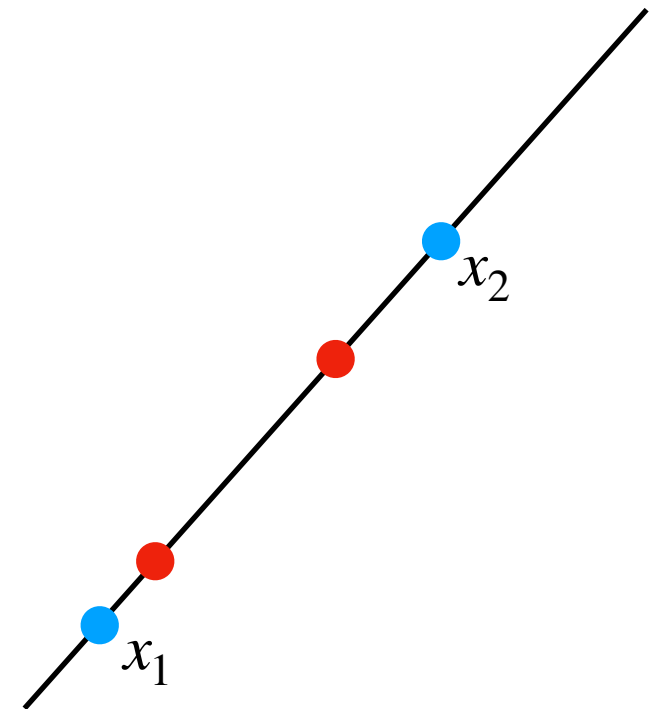
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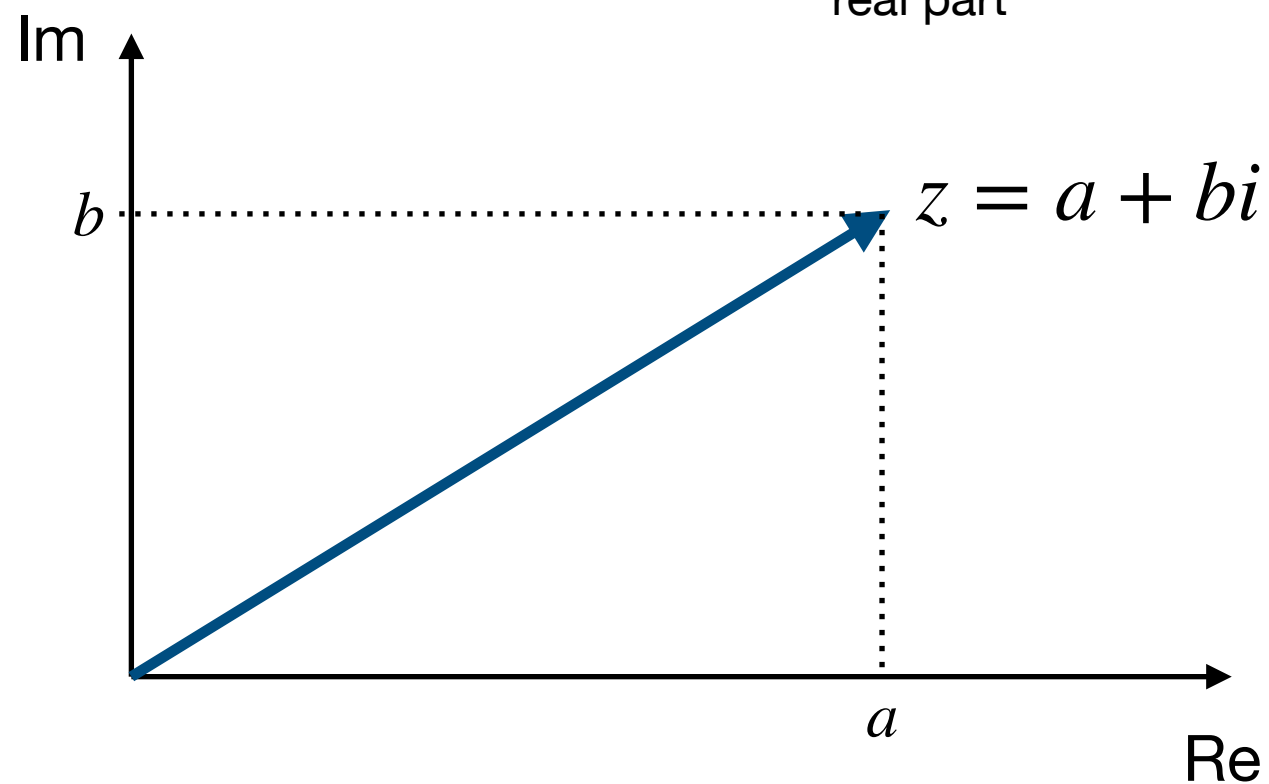
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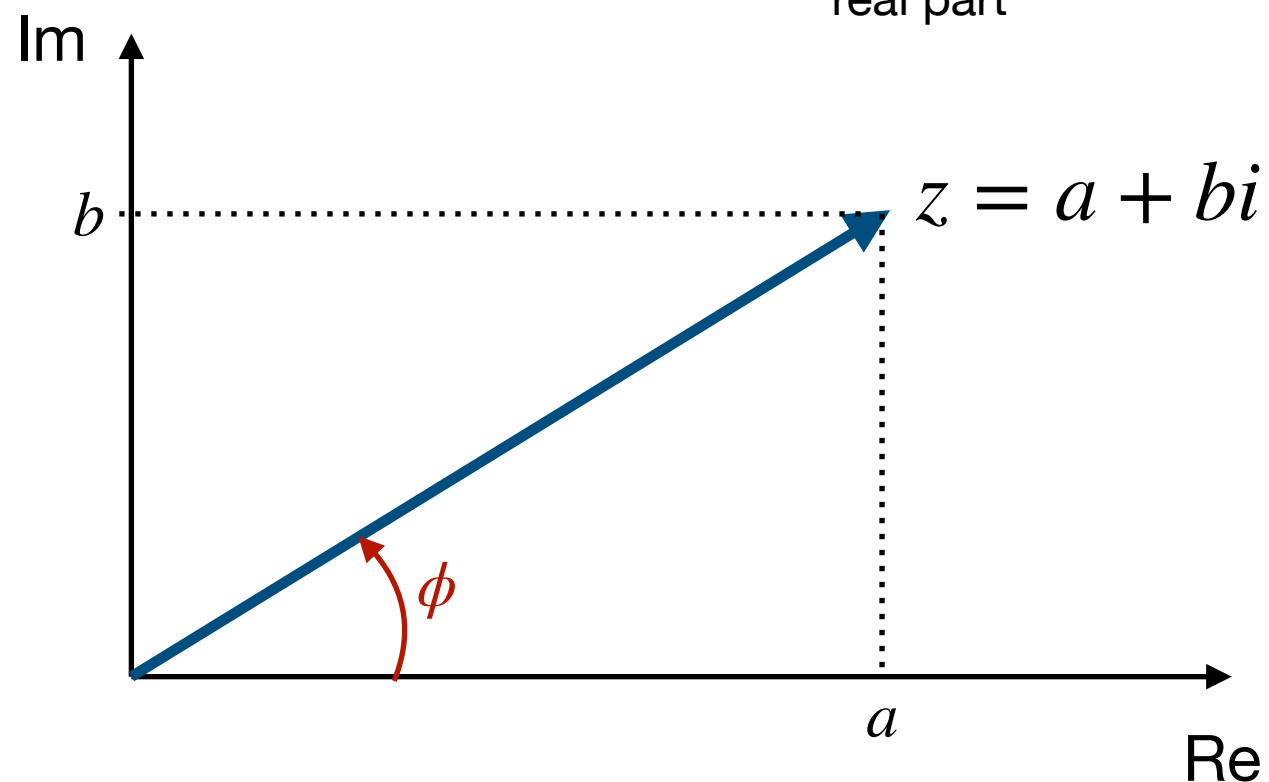
Quick Detour: Complex Numbers $\mathbb{C}$

Complex number $z = \underbrace{a}_{\text{real part}} + \underbrace{bi}_{\text{imaginary part}}$ $i^2 = -1$



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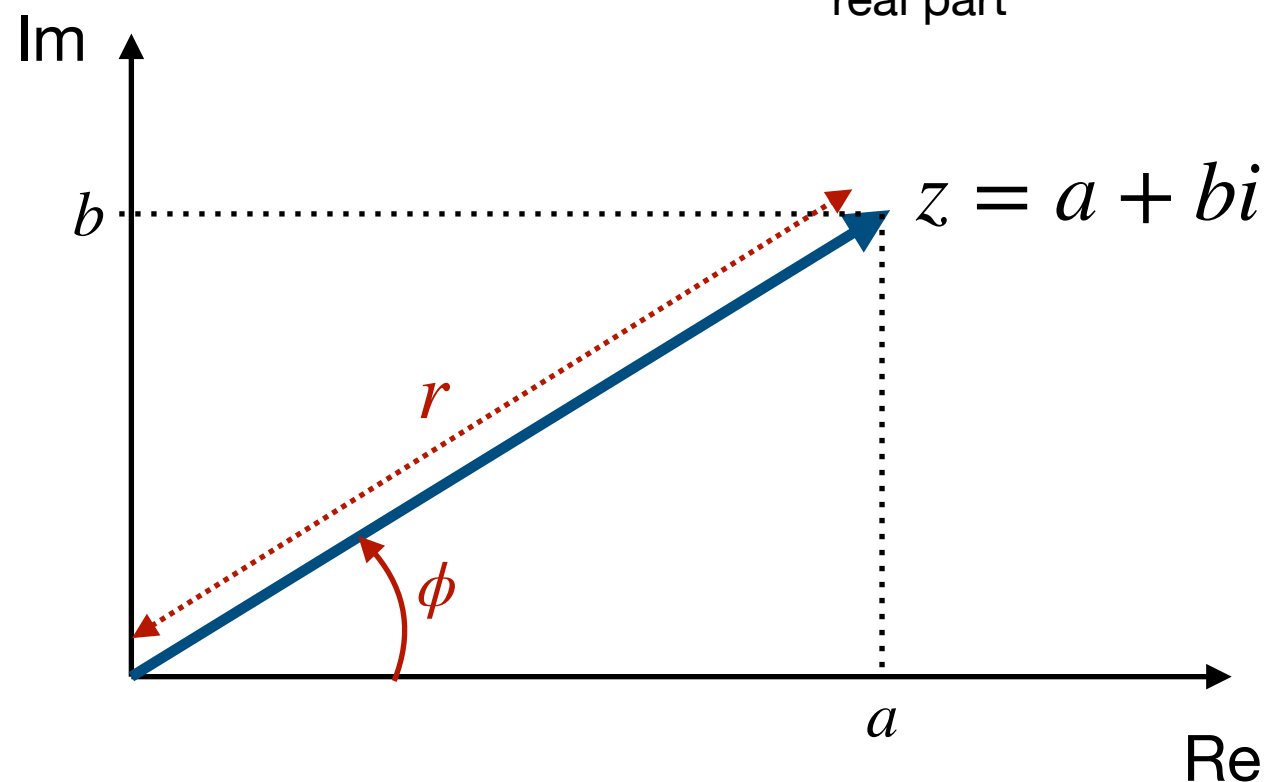
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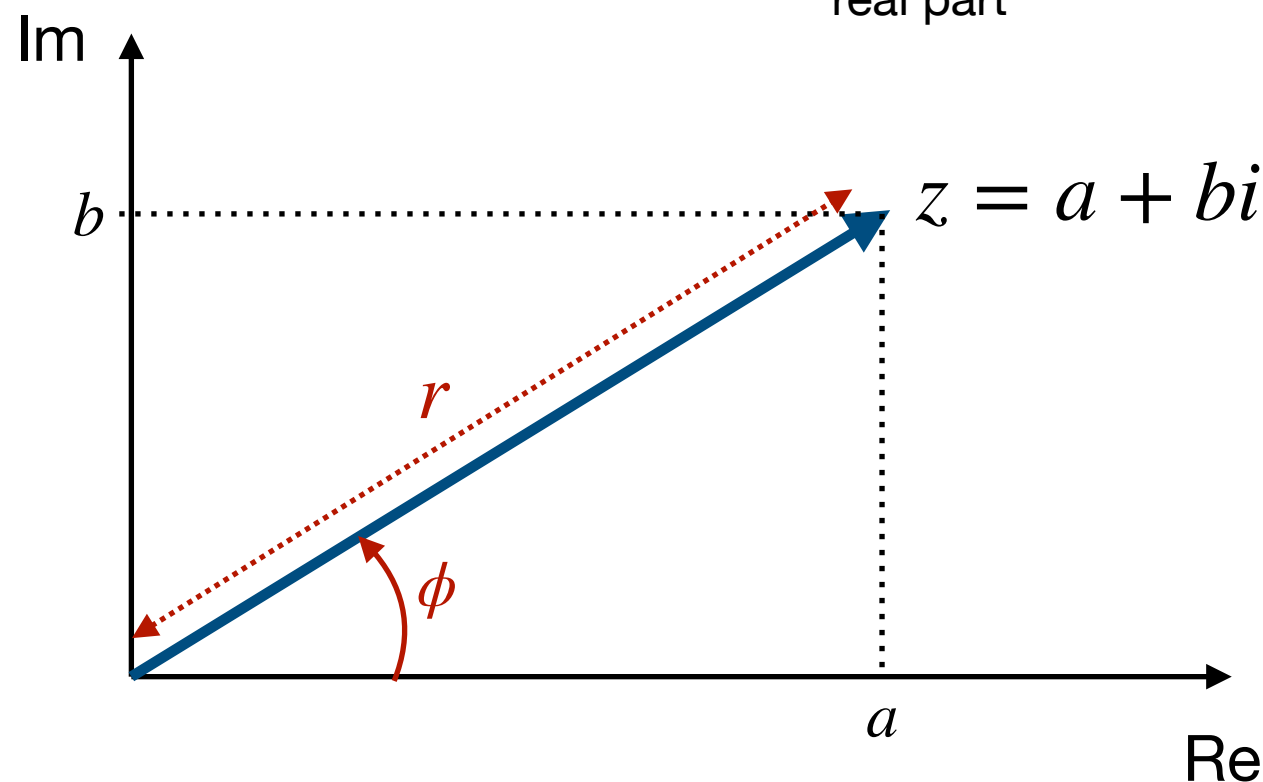
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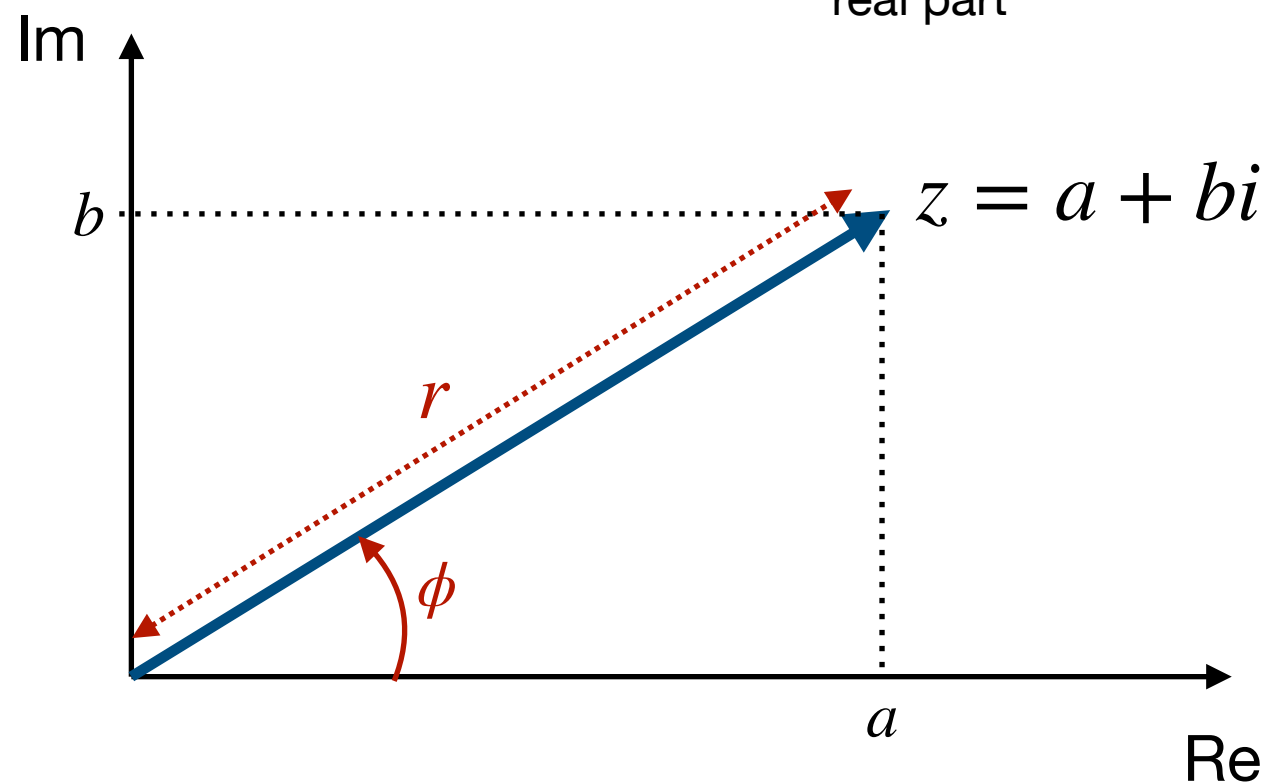
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Euler's formula: $e^{ix} = \cos x + i \sin x$

$$z = r \cdot e^{i\phi}$$



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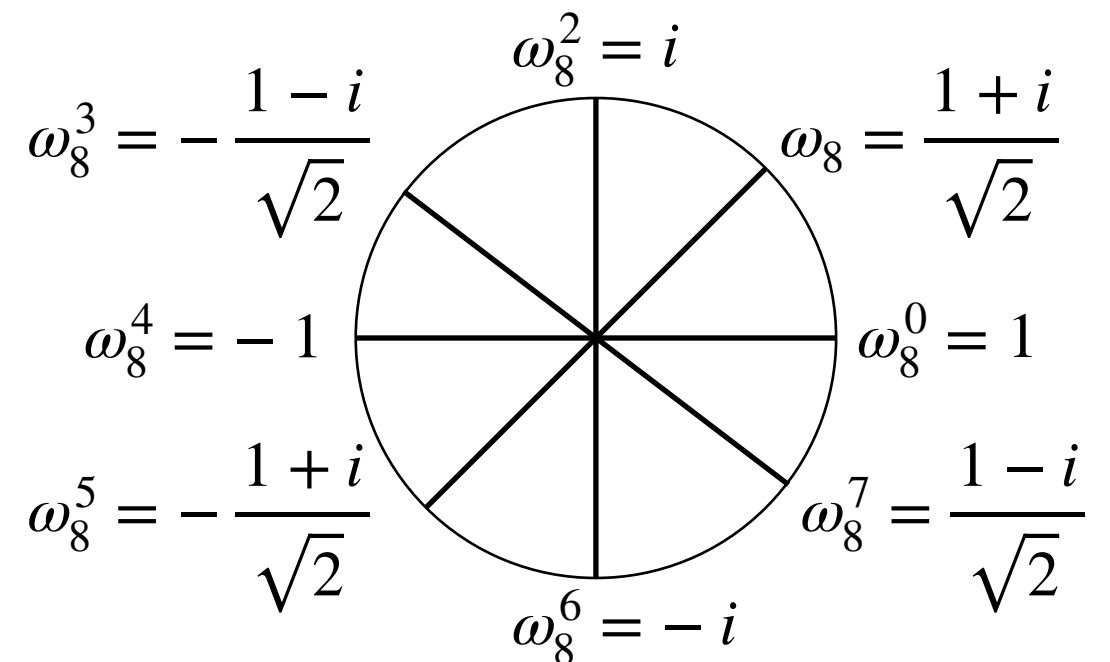
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Proof:
$$\left(\omega_n^k \right)^2 = \omega_n^{2k} = \omega_{n/2}^k, \text{ also}$$
$$\left(\omega_n^{k+n/2} \right)^2 = \omega_n^{2k+n} = \omega_n^{2k} \cdot \omega_n^n = \omega_{n/2}^k$$

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1	ω_n^2	...	ω_n^{n-2}	ω_n^n	ω_n^{n+2}	...	$\omega_n^{2(n-1)}$
$\parallel$	$\parallel$	...	$\parallel$	$\parallel$	$\parallel$	...	$\parallel$
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sum of geometric series

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We will choose the $2n$ th (complex) roots of unity.

Discrete Fourier Transform

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The *Discrete Fourier Transform (DFT)* of a sequence of m complex numbers $p_0, p_1, \dots, p_{m-1}$ is defined to be the sequence of complex numbers

$$P(1), P(\omega_m), P(\omega_m^2), \dots, P(\omega_m^{m-1})$$

obtained by evaluating the polynomial

$$P(x) = p_0 + p_1x + p_2x^2 + \dots, p_{m-1}x^{m-1}$$

on each of the m th roots of unity.

Divide and Conquer

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$$P_{\text{even}}(x) = p_0 + p_2x + p_4x^2 + \dots + p_{m-2}x^{m/2-1}$$

$$P_{\text{odd}}(x) = p_1 + p_3x + p_5x^2 + \dots + p_{m-1}x^{m/2-1}$$

Divide and Conquer

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Observe that: $P(x) = P_{\text{even}}(x^2) + x \cdot P_{\text{odd}}(x^2)$

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Observe that: $P(x) = P_{\text{even}}(x^2) + x \cdot P_{\text{odd}}(x^2)$

So to evaluate $P(x)$ at $1, \omega_m, \omega_m^2, \dots, \omega_m^{m-1}$, we can

1. Evaluate the two polynomials of degree $m/2 - 1$ at

$$1^2, (\omega_m)^2, (\omega_m^2)^2, \dots, (\omega_m^{m-1})^2$$

2. Combine the results to obtain $P(x)$

Divide and Conquer

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It seems we are still evaluating on $m - 1$ points

2. Combine the results to obtain $P(x)$

Properties of the Roots of Unity

Cancellation: Let $n \geq 0, k \geq 0, d > 0$. It holds that $\omega_n^{dk} = \omega_n^k$

Proof: $\omega_{dn}^{dk} = \left(e^{\frac{2\pi i}{dn}} \right)^{dk}$

1	ω_n^2	...	ω_n^{n-2}	ω_n^n	ω_n^{n+2}	...	$\omega_n^{2(n-1)}$
$\parallel$	$\parallel$	...	$\parallel$	$\parallel$	$\parallel$	...	$\parallel$
1	$\omega_{n/2}$	...	$\omega_{n/2}^{n/2-1}$	1	$\omega_{n/2}$	...	$\omega_{n/2}^{n/2-1}$

Halving: Let $n > 0$ be even. Then if we square all the n n th roots of unity, we get all $n/2$ ($n/2$)th roots of unity, each one twice.

Proof: $(\omega_n^k)^2 = \omega_n^{2k} = \omega_{n/2}^k$, also

$$(\omega_n^{k+n/2})^2 = \omega_n^{2k+n} = \omega_n^{2k} \cdot \omega_n^n = \omega_{n/2}^k$$

Divide and Conquer

So to evaluate $P(x)$ at $1, \omega_m, \omega_m^2, \dots, \omega_m^{m-1}$, we can

1. Evaluate the two polynomials of degree $m/2 - 1$ at

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It seems we are still evaluating on m points

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Divide and Conquer

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1. Evaluate the two polynomials of degree $m/2 - 1$ at

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We successfully halved the degree

It seems we are still evaluating on m points

2. Combine the results to obtain $P(x)$

This is a list of the $m/2$ ($m/2$)th roots of unity, each appearing twice

Divide and Conquer

So to evaluate $P(x)$ at $1, \omega_m, \omega_m^2, \dots, \omega_m^{m-1}$, we can

1. Evaluate the two polynomials of degree $m/2 - 1$ at

$1^2, (\omega_m)^2, (\omega_m^2)^2, \dots, (\omega_m^{m-1})^2$

We successfully halved the degree

It seems we are still evaluating on m points

2. Combine the results to obtain $P(x)$



This is a list of the $m/2$ ($m/2$)th roots of unity, each appearing twice

So we only need to evaluate at $m/2$ points

Divide and Conquer

So to evaluate $P(x)$ at $1, \omega_m, \omega_m^2, \dots, \omega_m^{m-1}$, we can

1. Evaluate the two polynomials of degree $m/2 - 1$ at $1^2, (\omega_m)^2, (\omega_m^2)^2, \dots, (\omega_m^{m-1})^2$
2. Combine the results to obtain $P(x)$

Divide

$$\begin{array}{cccccccc}
 1 & \omega_n^2 & \dots & \omega_n^{n-2} & \omega_n^n & \omega_n^{n+2} & \dots & \omega_n^{2(n-1)} \\
 \parallel & \parallel & \dots & \parallel & \parallel & \parallel & \dots & \parallel \\
 1 & \omega_{n/2} & \dots & \omega_{n/2}^{n/2-1} & 1 & \omega_{n/2} & \dots & \omega_{n/2}^{n/2-1}
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2. Combine the results to obtain $P(x)$

$$P(1) = P_{\text{even}}(1) + 1 \cdot P_{\text{odd}}(1)$$

Divide

$$\begin{array}{cccccccc}
 1 & \omega_n^2 & \dots & \omega_n^{n-2} & \omega_n^n & \omega_n^{n+2} & \dots & \omega_n^{2(n-1)} \\
 \parallel & \parallel & \dots & \parallel & \parallel & \parallel & \dots & \parallel \\
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Divide

$$\begin{array}{cccccccc}
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 \parallel & \parallel & \dots & \parallel & \parallel & \parallel & \dots & \parallel \\
 1 & \omega_{n/2} & \dots & \omega_{n/2}^{n/2-1} & 1 & \omega_{n/2} & \dots & \omega_{n/2}^{n/2-1}
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2. Combine the results to obtain $P(x)$

$$P(1) = P_{\text{even}}(1) + 1 \cdot P_{\text{odd}}(1)$$

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$$P(\omega_m^2) = P_{\text{even}}(\omega_{m/2}^2) + \omega_m^2 \cdot P_{\text{odd}}(\omega_{m/2}^2)$$

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$$\vdots$$

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$$P(\omega_m^{m/2-1}) = P_{\text{even}}(\omega_{m/2}^{m/2-1}) + \omega_m^2 \cdot P_{\text{odd}}(\omega_{m/2}^{m/2-1})$$

$$P(\omega_m^{m-1}) = P(\omega_m^{m/2-1})$$

Pseudocode (CLRS pp. 890)

FFT(a, n)

```
1  if  $n == 1$ 
2      return  $a$                                 // DFT of 1 element is the element itself
3   $\omega_n = e^{2\pi i/n}$ 
4   $\omega = 1$ 
5   $a^{\text{even}} = (a_0, a_2, \dots, a_{n-2})$ 
6   $a^{\text{odd}} = (a_1, a_3, \dots, a_{n-1})$ 
7   $y^{\text{even}} = \text{FFT}(a^{\text{even}}, n/2)$ 
8   $y^{\text{odd}} = \text{FFT}(a^{\text{odd}}, n/2)$ 
9  for  $k = 0$  to  $n/2 - 1$                         // at this point,  $\omega = \omega_n^k$ 
10      $y_k = y_k^{\text{even}} + \omega y_k^{\text{odd}}$ 
11      $y_{k+(n/2)} = y_k^{\text{even}} - \omega y_k^{\text{odd}}$ 
12      $\omega = \omega \omega_n$ 
13 return  $y$ 
```

Running time

Step 1: Choose $2n$ values $x_1, x_2, \dots, x_{2n}$ and *evaluate* $A(x_j)$ and $B(x_j)$ for each $j = 1, 2, \dots, 2n$.

Step 2: Compute $C(x_j) = A(x_j) \cdot B(x_j)$ for all j
(*these are now just numbers*).

Step 3: Recover C from $C(x_1), C(x_2), \dots, C(x_{2n})$.

Running time

Step 1: Choose the $2n$ $2n$ th roots of unity $1, \omega_{2n}, \omega_{2n}^2, \dots, \omega_{2n}^{2n-1}$ and *evaluate* $A(\omega_{2n}^j)$ and $B(\omega_{2n}^j)$ for each $j = 0, 1, \dots, 2n - 1$.

Running time

Step 1: Choose the $2n$ $2n$ th roots of unity $1, \omega_{2n}, \omega_{2n}^2, \dots, \omega_{2n}^{2n-1}$ and *evaluate* $A(\omega_{2n}^j)$ and $B(\omega_{2n}^j)$ for each $j = 0, 1, \dots, 2n - 1$.

How much time do we need for each of the evaluations?

Running time

Step 1: Choose the $2n$ $2n$ th roots of unity $1, \omega_{2n}, \omega_{2n}^2, \dots, \omega_{2n}^{2n-1}$ and *evaluate* $A(\omega_{2n}^j)$ and $B(\omega_{2n}^j)$ for each $j = 0, 1, \dots, 2n - 1$.

How much time do we need for each of the evaluations?

Let $T(n)$ be the time required to evaluate a polynomial of degree $n - 1$ on all of the $2n$ $2n$ th roots of unity.

Running time

Step 1: Choose the $2n$ $2n$ th roots of unity $1, \omega_{2n}, \omega_{2n}^2, \dots, \omega_{2n}^{2n-1}$ and *evaluate* $A(\omega_{2n}^j)$ and $B(\omega_{2n}^j)$ for each $j = 0, 1, \dots, 2n - 1$.

How much time do we need for each of the evaluations?

Let $T(n)$ be the time required to evaluate a polynomial of degree $n - 1$ on all of the $2n$ $2n$ th roots of unity.

We need to evaluate $P(x) = P_{\text{even}}(x^2) + x \cdot P_{\text{odd}}(x^2)$ at $1, \omega_{2n}, \omega_{2n}^2, \dots, \omega_{2n}^{2n-1}$

Running time

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Running time: $T(n) \leq 2T(n/2) + cn$

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Running time: $T(n) \leq 2T(n/2) + cn$

Asymptotic running time: $O(n \log n)$

What if we divided like this?

Assume that $m = 2^\ell$ for some positive integer ℓ .

Let

$$P_{\text{small}}(x) = p_0 + p_1x + p_2x^2 + \dots + p_{m/2-1}x^{m/2-1}$$

$$P_{\text{big}}(x) = p_{m/2} + p_{m/2+1}x + p_{m/2+2}x^2 + \dots + p_{m-1}x^{m/2-1}$$

We would have: $P(x) = P_{\text{big}}(x) + x^{m/2} \cdot P_{\text{small}}(x)$

What is the issue with this?

Running time

$O(n \log n)$

Step 1: Choose the $2n$ $2n$ th roots of unity $1, \omega_{2n}, \omega_{2n}^2, \dots, \omega_{2n}^{2n-1}$ and *evaluate* $A(\omega_{2n}^j)$ and $B(\omega_{2n}^j)$ for each $j = 0, 1, \dots, 2n - 1$.

Step 2: Compute $C(x_j) = A(x_j) \cdot B(x_j)$ for all j
(these are now just numbers).

$O(n)$

Step 3: Recover C from $C(x_1), C(x_2), \dots, C(x_{2n})$.

What about this?

(Fast) Polynomial Interpolation

(Fast) Polynomial Interpolation

Recover C from $C(x_1), C(x_2), \dots, C(x_{2n})$

(Fast) Polynomial Interpolation

Recover C from $C(x_1), C(x_2), \dots, C(x_{2n})$

Main idea: We will reduce *polynomial interpolation* to *polynomial evaluation*, which we saw how to do using D&C earlier.

(Fast) Polynomial Interpolation

Recover C from $C(x_1), C(x_2), \dots, C(x_{2n})$

Main idea: We will reduce *polynomial interpolation* to *polynomial evaluation*, which we saw how to do using D&C earlier.

Define the polynomial $D(x) = \sum_{s=0}^{2n-1} C(\omega_{2n}^s) \cdot x^s$, and evaluate it at the $2n$ th roots of unity.

(Fast) Polynomial Interpolation

Define the polynomial $D(x) = \sum_{s=0}^{2n-1} C(\omega_{2n}^s) \cdot x^s$, and evaluate it at the $2n$ th roots of unity.

(Fast) Polynomial Interpolation

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Properties of the Roots of Unity

Summation: Suppose $n \geq 1$ and k is not divisible by n . It

holds that $\sum_{j=0}^{n-1} (\omega_n^k)^j = 0$

Proof:
$$\sum_{j=0}^{n-1} (\omega_n^k)^j = \frac{(\omega_n^k)^n - 1}{\omega_n^k - 1} = \frac{(\omega_n^n)^k - 1}{\omega_n^k - 1} = \frac{1^k - 1}{\omega_n^k - 1} = 0$$

sum of geometric series

(Fast) Polynomial Interpolation

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For all t such that $t + k$ is not divisible by $2n$, we have:

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(Fast) Polynomial Interpolation

Recover C from $C(x_1), C(x_2), \dots, C(x_{2n})$

Define the polynomial $D(x) = \sum_{s=0}^{2n-1} C(\omega_{2n}^s) \cdot x^s$, and evaluate it at the $2n$ th roots of unity.

We get: $c_s = \frac{1}{2n} \cdot D(\omega_{2n}^{2n-s})$

Alternative viewpoint

The *Discrete Fourier Transform (DFT)* of a sequence of m complex numbers $p_0, p_1, \dots, p_{m-1}$ is defined to be the sequence of complex numbers

$$P(1), P(\omega_m), P(\omega_m^2), \dots, P(\omega_m^{m-1})$$

obtained by evaluating the polynomial

$$P(x) = p_0 + p_1x + p_2x^2 + \dots, p_{m-1}x^{m-1}$$

on each of the m th roots of unity.

Alternative viewpoint

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_m & \omega_m^2 & \cdots & \omega_m^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_m^{m-1} & \omega_m^{2(m-1)} & \cdots & \omega_m^{(m-1)^2} \end{bmatrix}}_M \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{m-1} \end{bmatrix} = \begin{bmatrix} \hat{p}_0 \\ \hat{p}_1 \\ \vdots \\ \hat{p}_{m-1} \end{bmatrix}$$

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We can compute $\overrightarrow{p} = M^{-1} \overrightarrow{\hat{p}}$

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Is M invertible?

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Is M invertible?

How can we compute M^{-1} ?

M is invertible

$$\begin{bmatrix} 1 & z_0 & z_0^2 & \cdots & z_0^m \\ 1 & z_1 & z_1^2 & \cdots & z_1^m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_\ell & z_\ell^2 & \cdots & z_\ell^m \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{m-1} \end{bmatrix} = \begin{bmatrix} \hat{p}_0 \\ \hat{p}_1 \\ \vdots \\ \hat{p}_{m-1} \end{bmatrix}$$



Vandermonde matrix

M is invertible

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Vandermonde matrix

$$\mathbf{det}(M) = \prod_{0 \leq i < j \leq \ell} (x_j - x_i)$$

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Vandermonde matrix

$$\det(M) = \prod_{0 \leq i < j \leq \ell} (x_j - x_i)$$

When $m = \ell$ (i.e., M is square) and $z_i \neq z_j$ for all $i \neq j$ (i.e., all z_i 's are distinct) and thus $\det(M) \neq 0$, then M is invertible.

How to compute M

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_m & \omega_m^2 & \cdots & \omega_m^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_m^{m-1} & \omega_m^{2(m-1)} & \cdots & \omega_m^{(m-1)^2} \end{bmatrix}}_{M(\omega_m)} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{m-1} \end{bmatrix} = \begin{bmatrix} \hat{p}_0 \\ \hat{p}_1 \\ \vdots \\ \hat{p}_{m-1} \end{bmatrix}$$

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Proof: $M(\omega_m)(j, j') = \omega_m^{jj'}$, and $\frac{1}{m} M(\omega_m^{-1})(j, j') = \frac{1}{m} \omega_m^{-jj'}$

Consider the matrix $\frac{1}{m} M(\omega_m^{-1}) \cdot M(\omega_m)$

How to compute M

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Then we have:

$$\frac{1}{m} M(\omega_m^{-1}) \cdot M(\omega_m)(j, j') = \frac{1}{m} \sum_{k=0}^{m-1} \omega_m^{-kj} \cdot \omega_m^{kj'} = \frac{1}{m} \sum_{k=0}^{m-1} \omega_m^{k(j'-j)}$$

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Why? Because $-(m-1) \leq j' - j \leq m-1$

How to compute M

Lemma: $M(\omega_m)^{-1} = \frac{1}{m} M(\omega_n^{-1})$

Hence $\frac{1}{m} M(\omega_m^{-1}) \cdot M_m(\omega_m) = I_m$ (the identify matrix).

Running time

$O(n \log n)$

Step 1: Choose the $2n$ $2n$ th roots of unity $1, \omega_{2n}, \omega_{2n}^2, \dots, \omega_{2n}^{2n-1}$ and *evaluate* $A(\omega_{2n}^j)$ and $B(\omega_{2n}^j)$ for each $j = 0, 1, \dots, 2n - 1$.

Step 2: Compute $C(x_j) = A(x_j) \cdot B(x_j)$ for all j
(these are now just numbers).

$O(n)$

Step 3: Recover C from $C(x_1), C(x_2), \dots, C(x_{2n})$.

What about this?

Running time

$O(n \log n)$

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$O(n \log n)$

Convolution Theorem

For any two vectors a and b of length n where n is a power of 2, the convolution $a * b$ of a and b can be computed as:

$$a * b = \text{DFT}_{2n}^{-1} \left(\text{DFT}_{2n}(a) + \text{DFT}_{2n}(b) \right)$$