Introduction to Algorithms and Data Structures

Lecture 8: Sets, dictionaries and hashing

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12 October 2023

Sets and dictionaries

Two important datatypes ...

- (Finite) sets of items of a given type X. E.g. $\{3, 5\}$

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\textbf{contains} & : X \to \texttt{bool} \\
\textbf{insert} & : X \to \texttt{void} \\
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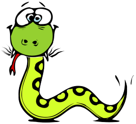
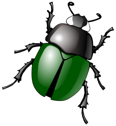
- Dictionaries (i.e. lookup tables) mapping keys of type X to values of type Y.

$$\begin{array}{rl} \textbf{lookup} & : X \to Y \\ \textbf{insert} & : X * Y \to \texttt{void} \\ \textbf{delete} & : X \to \texttt{void} \\ \textbf{isEmpty} & : \texttt{void} \to \texttt{bool} \end{array}$$

Sets and dictionaries in Python

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'George' in Beatles                    # returns True
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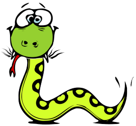

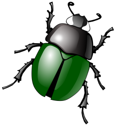

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Beatles = {'John', 'Paul', 'George', 'Ringo'}
'George' in Beatles                # returns True

BeatlesYearsOfBirth =
   {'John':1940, 'Paul':1942, 'George':1943, 'Ringo':1940}
BeatlesYearsOfBirth['George']      # returns 1943
```

Sets and dictionaries via sorted arrays

Could implement sets/dictionaries via (any impl of) lists:

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Beatles_Rep = ['John', 'Paul', 'George', 'Ringo']
BeatlesYearsOfBirth_Rep = [('John',1940), ('Paul',1942), ....]
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binarySearch(A,key,i,j):      # searches A[i], ..., A[j−1]
    if j−1 = i
        if A[i].key = key then return A[i].value else FAIL
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        k = ⌊ i+j/2 ⌋
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But **insert/delete** still costly. Can we do better?

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Then try to use an array A of size m, storing the entry for key s at position $\#(s)$ in A.

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2	3	4	5	6	7
$> (100 - 10^{-123})\%$	$> 99.9999\%$	$> 99.8\%$	66%	15%	2%

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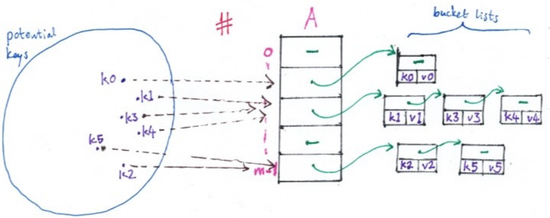

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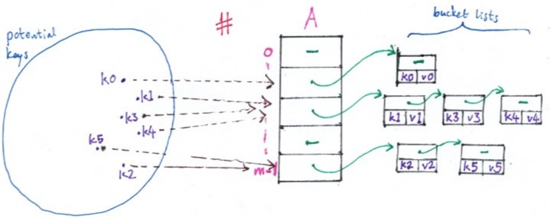

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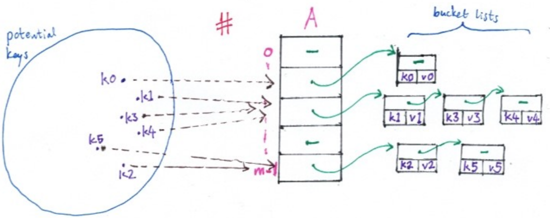

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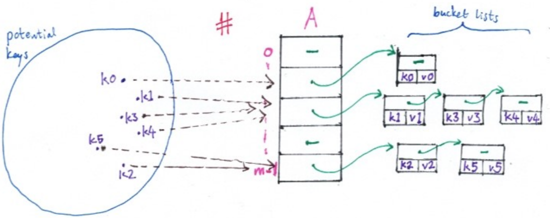

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If we've decided on a desired load α, can 'expand-and-rehash' any time n gets too large (amortized cost is reasonable).

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Can also show the same for successful lookup, assuming all keys present in table are equally likely. See CLRS 11.2.

Making a *proper* hash of it

Rarely true that all *potential* keys (e.g. strings) 'equally probable'. But in the interests of 'balancing' our hash table, we'd like the hash codes $0, \ldots, m-1$ to be all equally likely.

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But whatever we do, worst case (all keys hashing to same code) is always terrible. A malicious user who knew your hash function could force this to happen ...

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Solution 2: Rather than keeping bucket lists outside the hash table, store all items within the table itself (open addressing).

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To deal with clashes, we use not just a simple hash function $\#(k)$, but a function $\#(k, i)$ where $0 \leq i < m$. For a key k:

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To **insert** an item e with key k, probe $A[\#(k, 0)], A[\#(k, 1)], \ldots$ until we find a free slot $A[\#(k, i)]$, then put e there.

To **lookup** an item with key k, probe $A[\#(k, 0)], A[\#(k, 1)], \ldots$ until we find either an item with key k, or free cell (lookup failed).

Probing: example

Let's use an array A of size $m = 10$ to store a set of integers.

0	1	2	3	4	5	6	7	8	9

Probe function: $\#(k, i) = (k + i) \bmod 10$.

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insert(58). $\#(58, 0) = 8$: taken. $\#(58, 1) = 9$: taken.
$\#(58, 2) = 0$: free.

Probing: example

Let's use an array A of size $m = 10$ to store a set of integers.

0	1	2	3	4	5	6	7	8	9
58								28	49

Probe function: $\#(k, i) = (k + i) \bmod 10$.

insert(49). $\#(49, 0) = 9$: free.

insert(28). $\#(28, 0) = 8$: free.

insert(58). $\#(58, 0) = 8$: taken. $\#(58, 1) = 9$: taken.
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contains(28).

Probing: example

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insert(58). $\#(58, 0) = 8$: taken. $\#(58, 1) = 9$: taken.
$\#(58, 2) = 0$: free.

contains(28). $\#(28, 0) = 8$, $A[8] = 28$.

Probing: example

Let's use an array A of size $m = 10$ to store a set of integers.

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insert(49). $\#(49, 0) = 9$: free.

insert(28). $\#(28, 0) = 8$: free.

insert(58). $\#(58, 0) = 8$: taken. $\#(58, 1) = 9$: taken.
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contains(28). $\#(28, 0) = 8$, $A[8] = 28$. So true.

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Let's use an array A of size $m = 10$ to store a set of integers.

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contains(58).

Probing: example

Let's use an array A of size $m = 10$ to store a set of integers.

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$\#(58, 2) = 0$: free.

contains(28). $\#(28, 0) = 8$, $A[8] = 28$. So true.

contains(58). $\#(58, 0) = 8$, $A[8] = 28 \neq 58$.

Probing: example

Let's use an array A of size $m = 10$ to store a set of integers.

0	1	2	3	4	5	6	7	8	9
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insert(58). $\#(58, 0) = 8$: taken. $\#(58, 1) = 9$: taken.
$\#(58, 2) = 0$: free.

contains(28). $\#(28, 0) = 8$, $A[8] = 28$. So true.

contains(58). $\#(58, 0) = 8$, $A[8] = 28 \neq 58$.
$\#(58, 1) = 9$, $A[9] = 49 \neq 58$.

Probing: example

Let's use an array A of size $m = 10$ to store a set of integers.

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$\#(58, 2) = 0$: free.

contains(28). $\#(28, 0) = 8$, $A[8] = 28$. So true.

contains(58). $\#(58, 0) = 8$, $A[8] = 28 \neq 58$.
$\#(58, 1) = 9$, $A[9] = 49 \neq 58$.
$\#(58, 2) = 0$: $A[0] = 58$. So true.

Probing: example

Let's use an array A of size $m = 10$ to store a set of integers.

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contains(28). $\#(28, 0) = 8$, $A[8] = 28$. So true.

contains(58). $\#(58, 0) = 8$, $A[8] = 28 \neq 58$.
$\#(58, 1) = 9$, $A[9] = 49 \neq 58$.
$\#(58, 2) = 0$: $A[0] = 58$. So true.

contains(39).

Probing: example

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$\#(58, 2) = 0$: free.

contains(28). $\#(28, 0) = 8$, $A[8] = 28$. So true.

contains(58). $\#(58, 0) = 8$, $A[8] = 28 \neq 58$.
$\#(58, 1) = 9$, $A[9] = 49 \neq 58$.
$\#(58, 2) = 0$: $A[0] = 58$. So true.

contains(39). $\#(39, 0) = 9$, $A[9] = 49 \neq 39$.

Probing: example

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$\#(58, 2) = 0$: free.

contains(28). $\#(28, 0) = 8$, $A[8] = 28$. So true.

contains(58). $\#(58, 0) = 8$, $A[8] = 28 \neq 58$.
$\#(58, 1) = 9$, $A[9] = 49 \neq 58$.
$\#(58, 2) = 0$: $A[0] = 58$. So true.

contains(39). $\#(39, 0) = 9$, $A[9] = 49 \neq 39$.
$\#(39, 1) = 0$, $A[0] = 58 \neq 39$.

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Let's use an array A of size $m = 10$ to store a set of integers.

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$\#(58, 2) = 0$: free.

contains(28). $\#(28, 0) = 8$, $A[8] = 28$. So true.

contains(58). $\#(58, 0) = 8$, $A[8] = 28 \neq 58$.
$\#(58, 1) = 9$, $A[9] = 49 \neq 58$.
$\#(58, 2) = 0$: $A[0] = 58$. So true.

contains(39). $\#(39, 0) = 9$, $A[9] = 49 \neq 39$.
$\#(39, 1) = 0$, $A[0] = 58 \neq 39$.
$\#(39, 2) = 1$, $A[1]$ free. So false.

Probing: pros and cons

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See CLRS 11.4 for more details.

Radical alternative: Perfect hashing

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If set of keys is static (no **insert/delete** required), may be worth finding a perfect hash function (no clashes) for this set of keys.

As part of Coursework 1, we'll explore a modern approach to perfect hashing.

Reading:

Roughgarden 12.1-12.4 (good!)
CLRS Chapter 11, omitting theorems and their proofs, except for Theorem 11.1 which corresponds to slide 8.