Introduction to Algorithms and Data Structures

Lecture 9: Balanced trees

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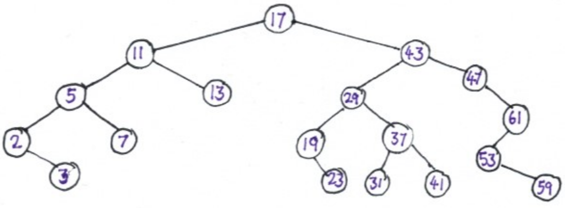
Ordered binary trees



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(Implementation detail: should use doubly linked tree structures.)

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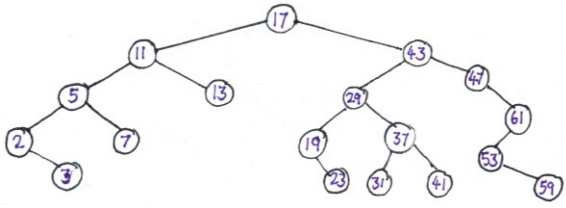

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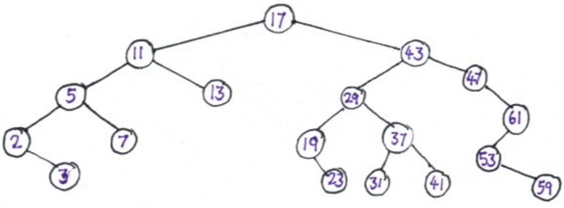

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Label nodes with keys (e.g. integers or strings) in such a way that for every node x we have

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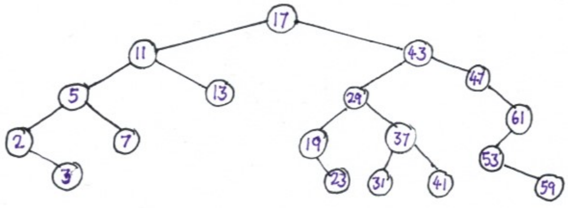

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Can use such trees to represent sets of keys.
(For dictionaries, just add value component to each node.)

Implementing contains/lookup

This is easy. Let a node x stand for the tree rooted at x.

```
contains'(x,k):
    if x = null then return False
    else if x.key = k then return True
    else if k < x.key then return contains'(x.left,k)
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contains(k):
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Then $d = \lfloor \lg n \rfloor$, so **contains** will take time $O(\lg n)$.

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However, worst case is still $\Theta(n)$!

Insert on binary trees

This too is easy: walk down tree to find where *k* wants to go, and create a new leaf node for it.

```
insert'(x,k):
    if x.key = k then return KeyAlreadyPresent
    else if k < x.key then
        if x.left = null then x.left = new Node(k)
        else insert'(x.left,k)
    else
        if x.right = null then x.right = new Node(k)
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    if root = null then root = new Node(k)
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NB. Nothing here to guard against tree *becoming* unbalanced!

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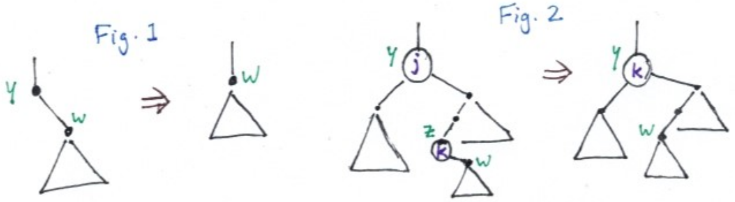

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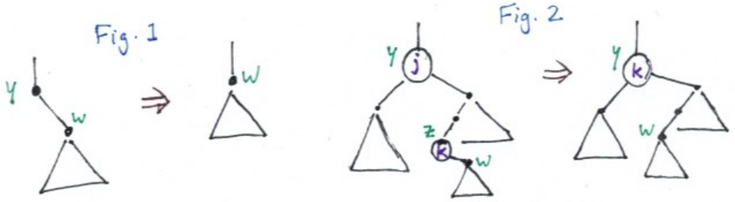

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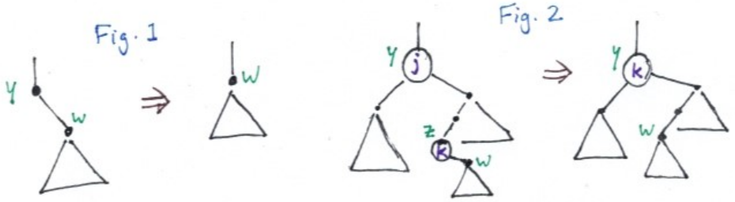

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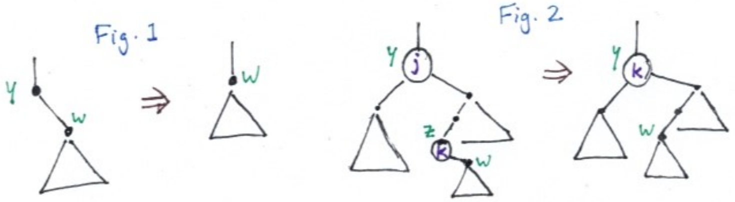
- If y has one child, easy to elide the node y (Fig. 1).
- If y has two children:
  - Locate leftmost node in $R(y)$, i.e. starting at y, turn right, then left as often as possible. This finds the node z bearing the smallest key in $R(y)$ (call it k).
  - Copy z.key to y.key.
  - If z has a right child, elide z, otherwise just delete z. (Fig. 2).



Same runtime characteristics.

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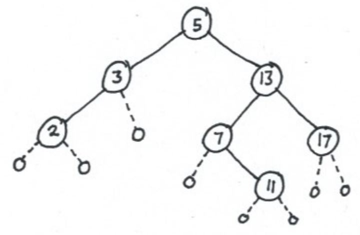
We choose **red-black** trees as the most entertaining of these.
Covered in detail in Sedgewick+Wayne and in CLRS.

Small preliminary: adding trivial nodes

For mathematical convenience, extend our trees so that original *null* branches now point to trivial nodes, with no children and bearing no key. Original nodes are proper nodes.



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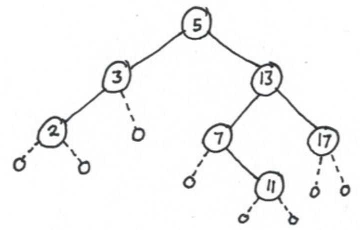
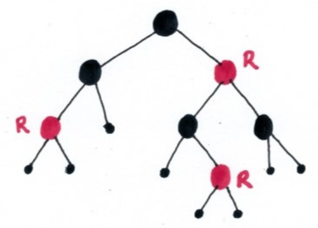


Call this an extended tree.

Just makes rules easier to state.
Wouldn't need these trivial nodes in an implementation.

Red-black trees



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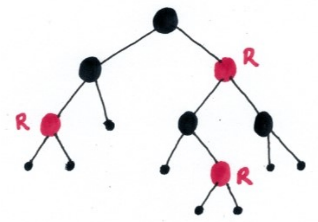

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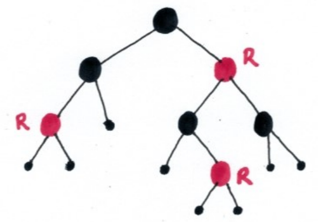

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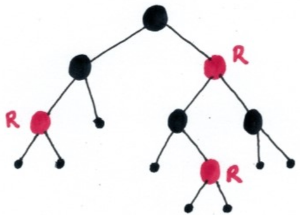

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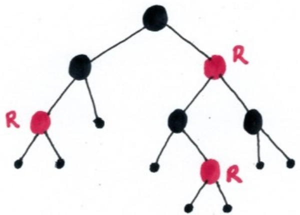

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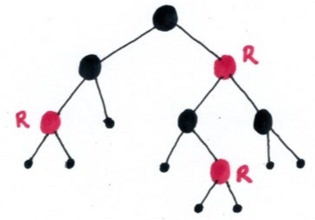

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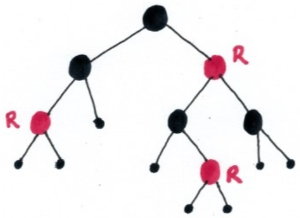

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There are $b - 1$ 'complete levels' of proper nodes, so $n \geq 2^{b-1} - 1$.
Hence $b \leq \lg(n+1) + 1$, so all path lengths $\leq 2\lg(n+1) + 1$.

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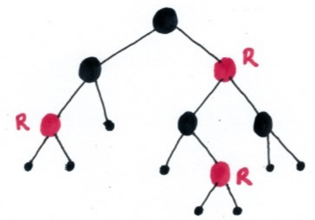

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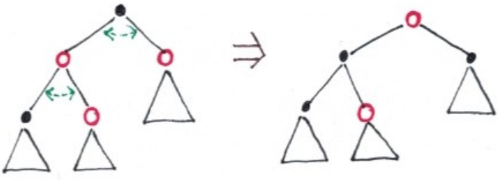
Main ingredient is the red-uncle rule:



(Just colour-flipping: fast. No rewiring involved!)

Insert fix-up, continued

Applying the red-uncle rule pushes a red upward, so may result in another double-red higher up.

So we apply the red-uncle rule as often as possible (will be at most $O(\lg n)$ times). We'll then be in one of three endgame scenarios:

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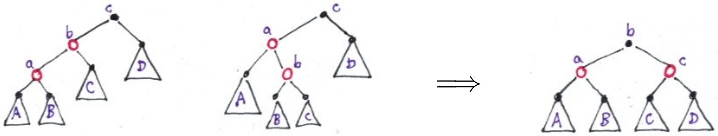
3. Have some configuration involving a black with 4 'nearest black descendants'. Replace by obvious 'balanced' version:



   $O(1)$ amount of rewiring.
   Note order of constituents is preserved: AaBbCcD.
   (Subtrees A,B,C,D may be empty.)

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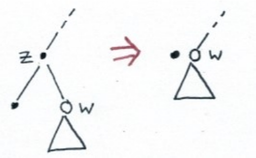

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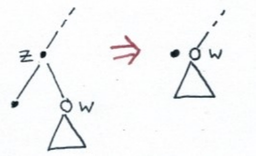

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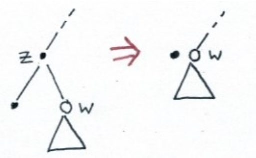
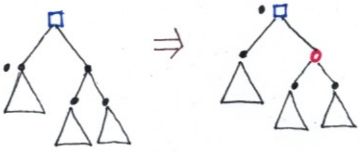


Easy case: Node it haunts is now red: can just turn it black.

Wandering black rule: apply this as often as possible (will be $O(\lg n)$ times).



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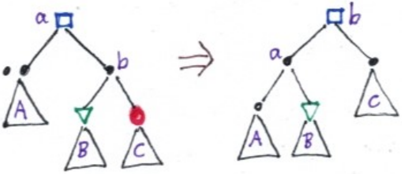
- Floating black haunts a red node: **turns it black**.
- Floating black reaches root: just remove it.
- We're in some other fixable scenario, e.g.



Blue square and green triangle are colour variables.

- 4 other scenarios like this: see CLRS 13 for full details. (For an exam, you're not expected to know these fix-up rules — not even the one above.)

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Reading:

Sedgewick+Wayne 3.2 (first half) and 3.3 (second half)
CLRS 12.1-12.3, 13.1-13.3

Visualization tool for red-black trees:
https://www.cs.usfca.edu/∼galles/visualization/RedBlack.html