Algorithms and Data Structures

Max Flow in Polynomial Time: The Edmonds-Karp Algorithm

Ford-Fulkerson analysis

Feasibility

Does the algorithm produce a flow if it terminates?

Termination

Does the algorithm always terminate?

Running Time

What is the running time of the algorithm?

Optimality / Correctness

Does the algorithm produce a maximum flow?

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Running Time

The running time of *FF* is $O(m\text{F})$, where F is the value of the maximum flow.

Since F $\leq c$, this is in fact $O(mc)$.

Is this an efficient algorithm?

The running time is *pseudopolynomial*, as it runs in time polynomial in n and the unary representation of the total capacity c.

It is fairly efficient, if in the numbers involved in the input are reasonably small.

The Ford-Fulkerson Algorithm

Max-Flow

Initially set f(e) = 0 for all e in E.

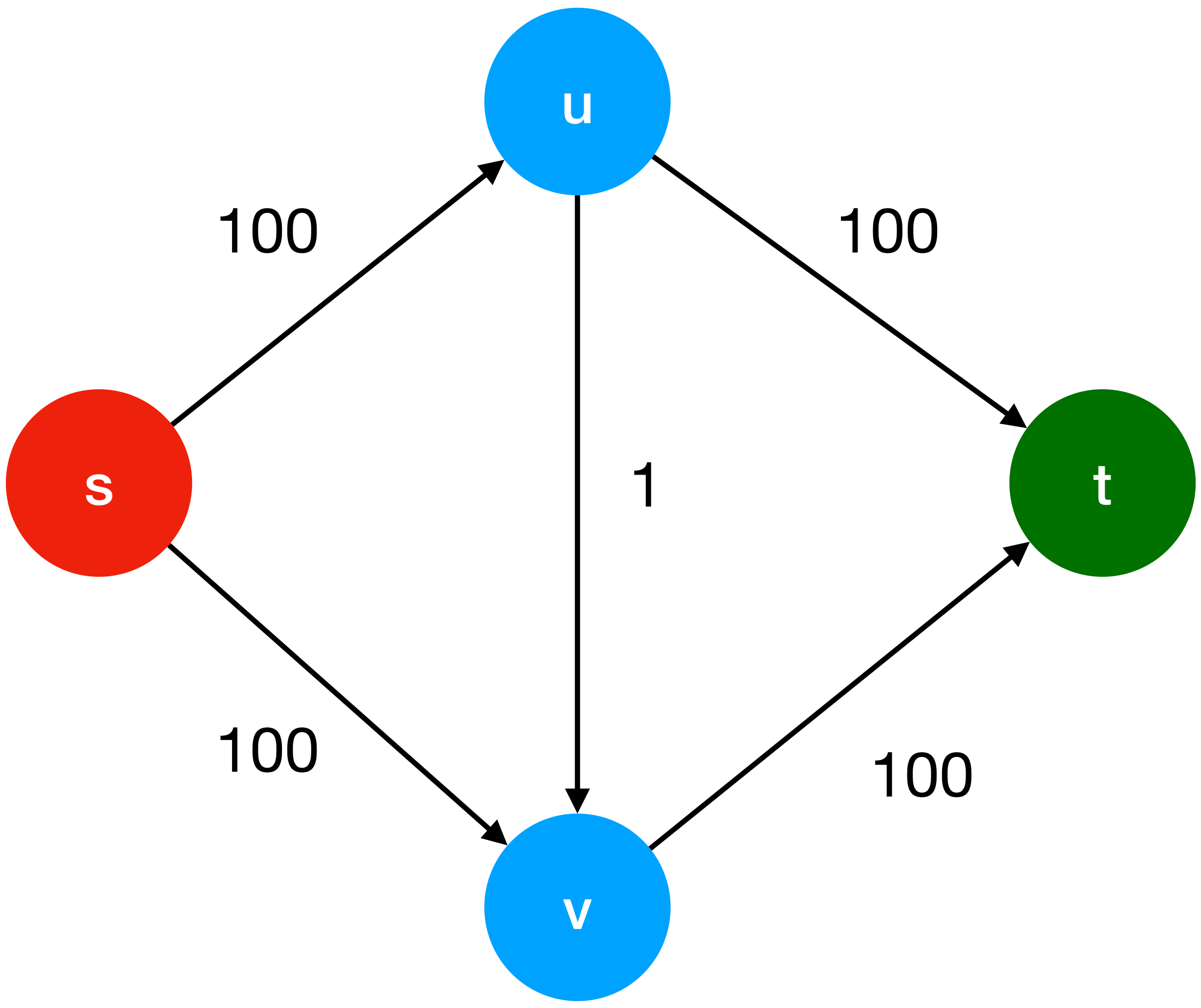
While there exists an s-t path in the residual graph Gf

Choose such a path P
f' = **augment**(f, P)
Update f to be f'
Update the residual graph to be Gf'

Endwhile

Return (f)

Example



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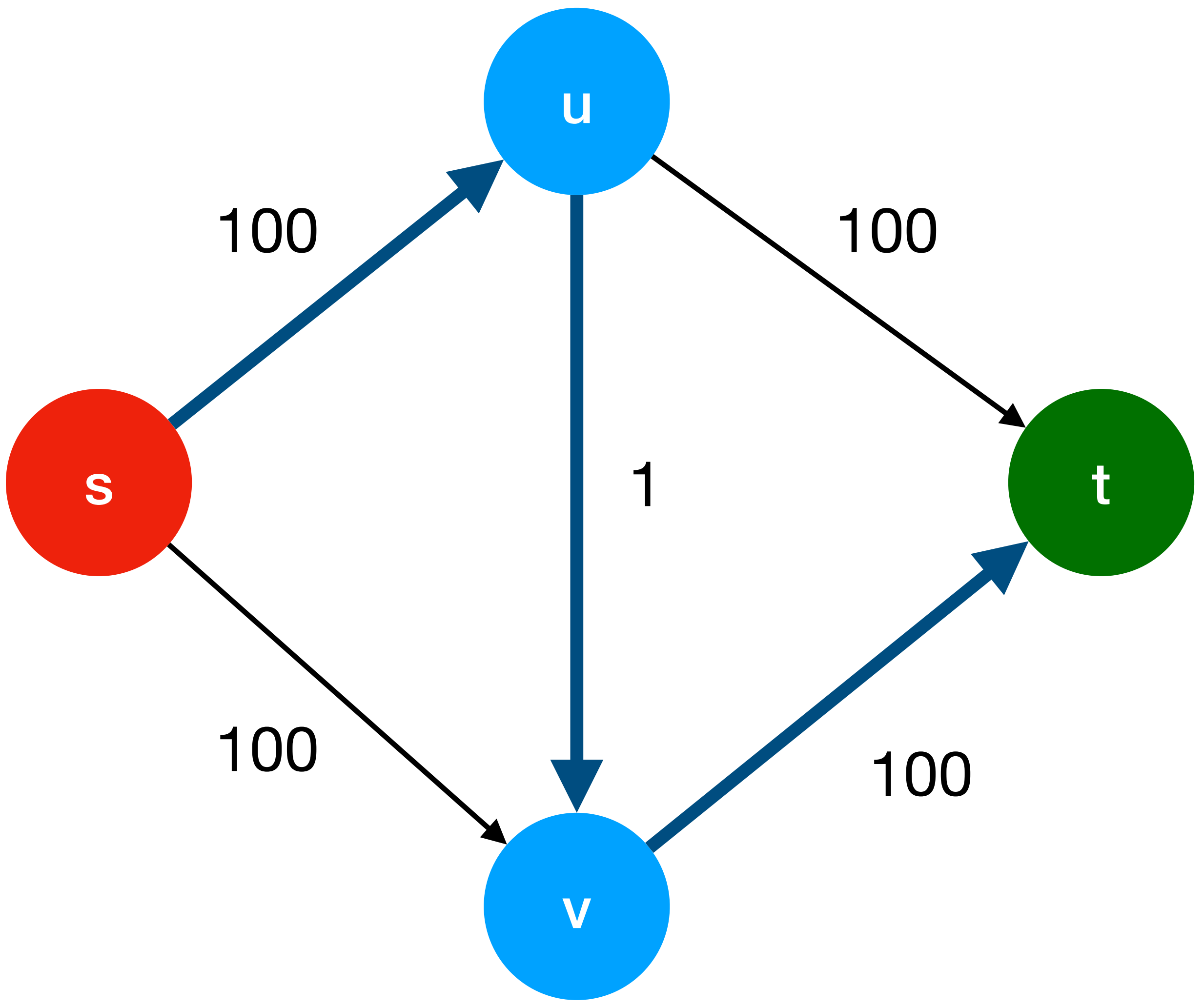

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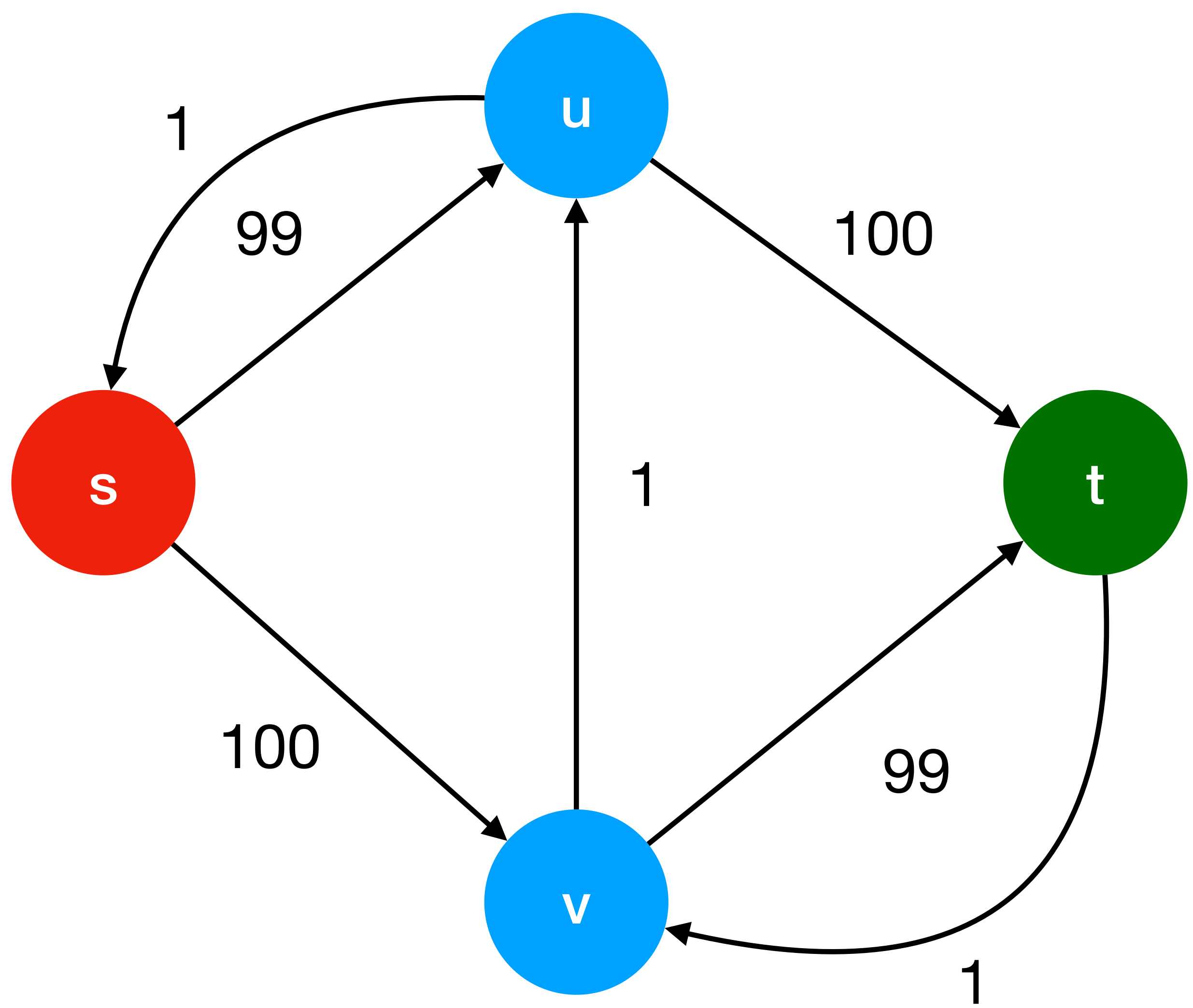

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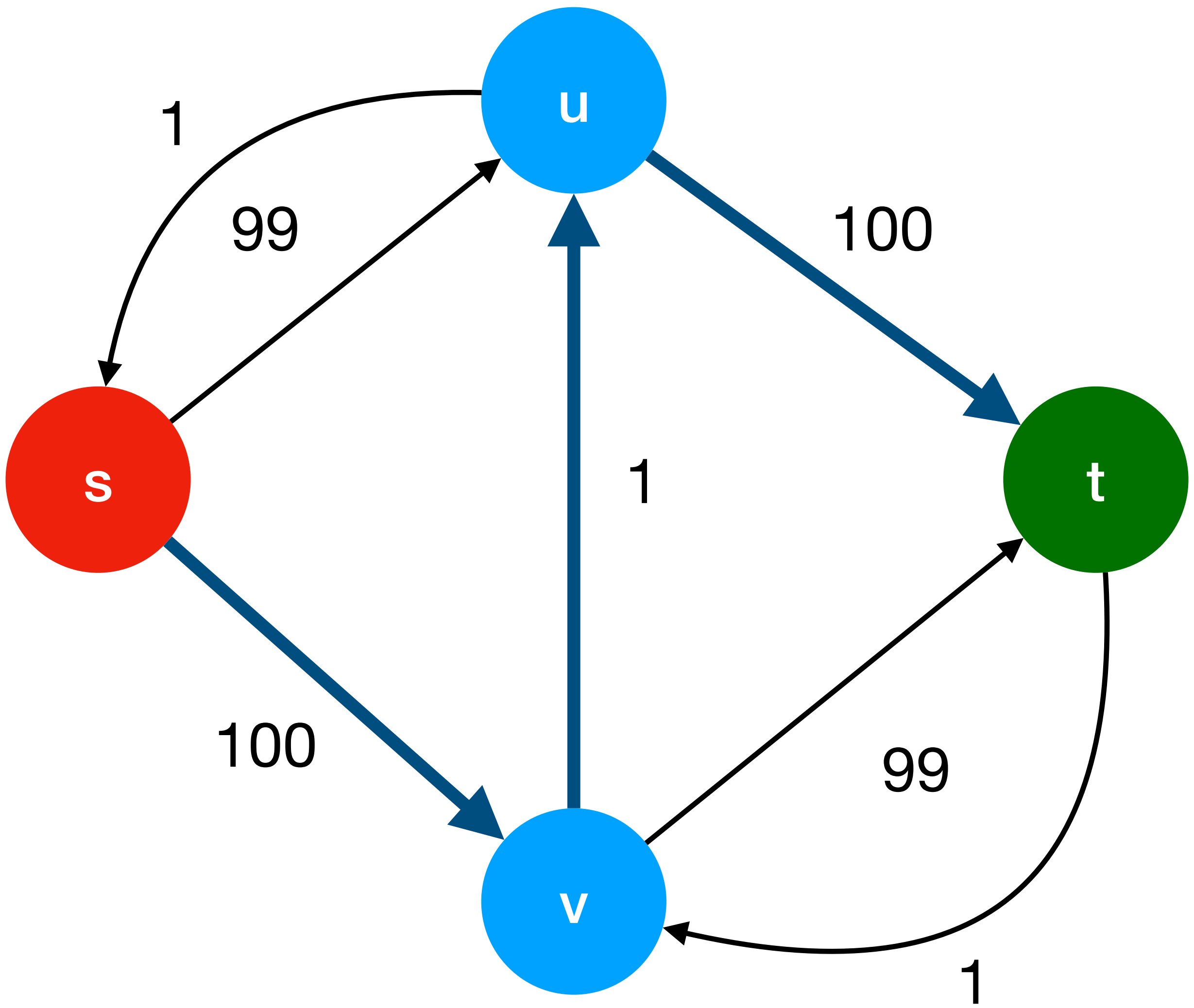

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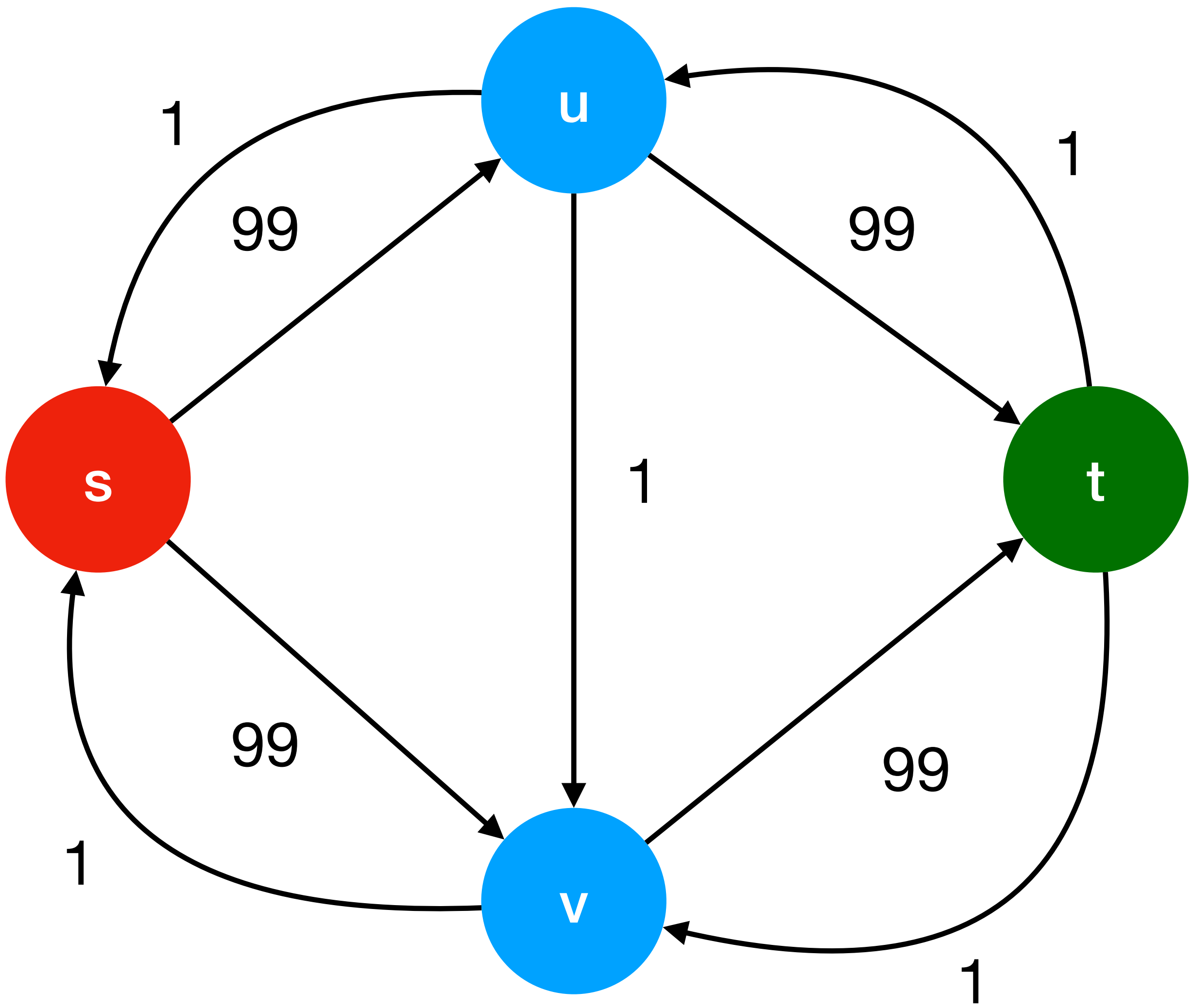

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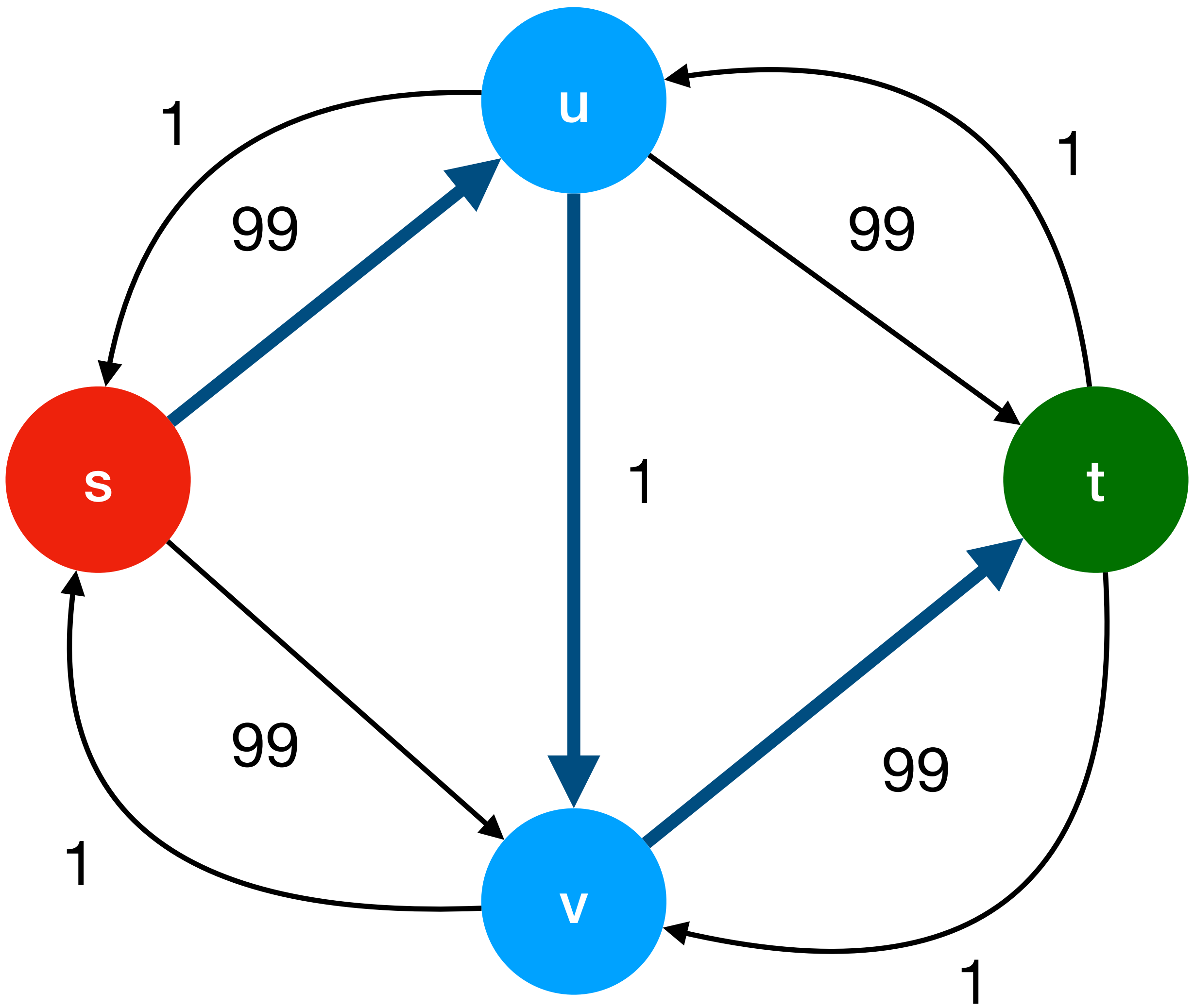

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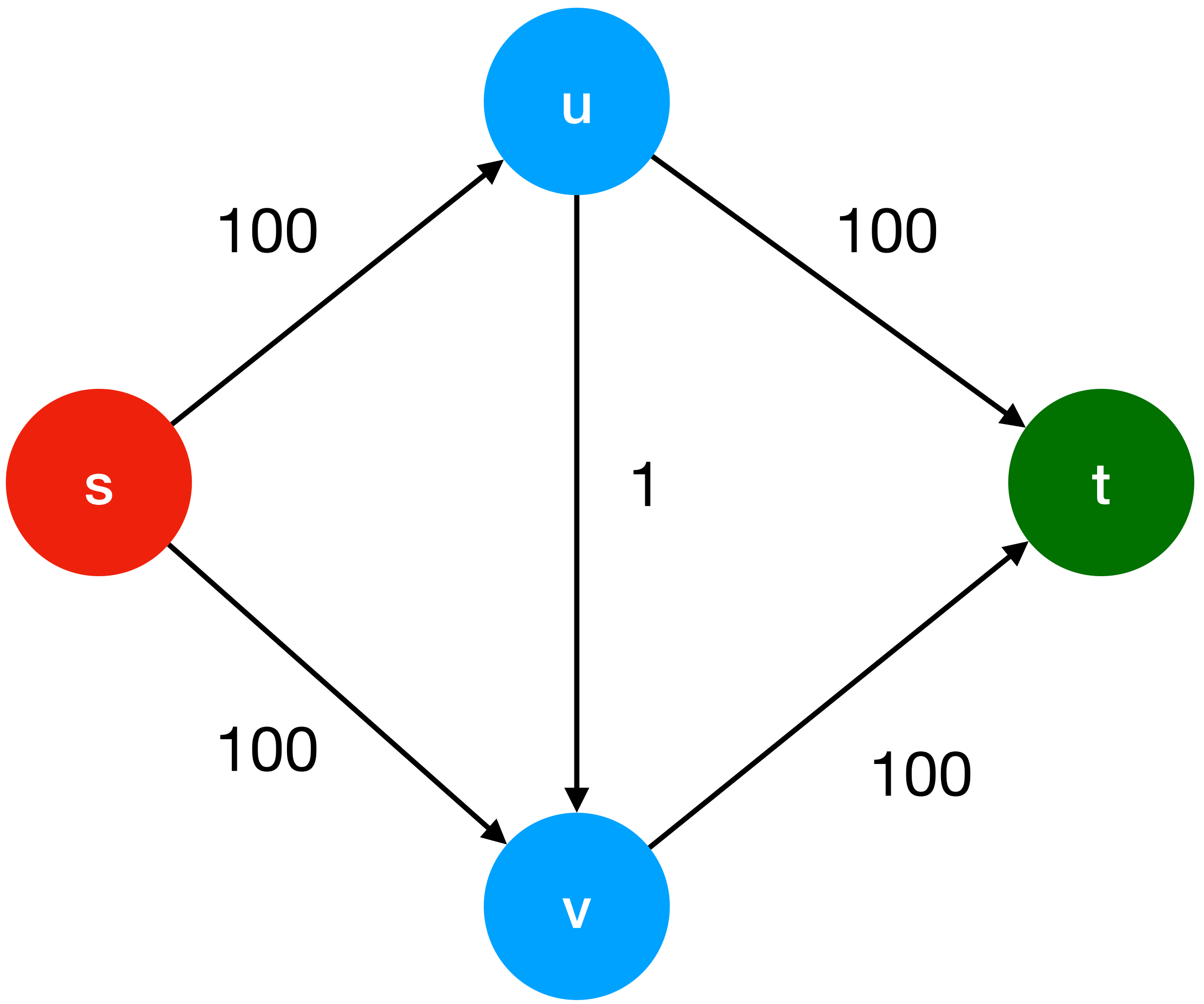

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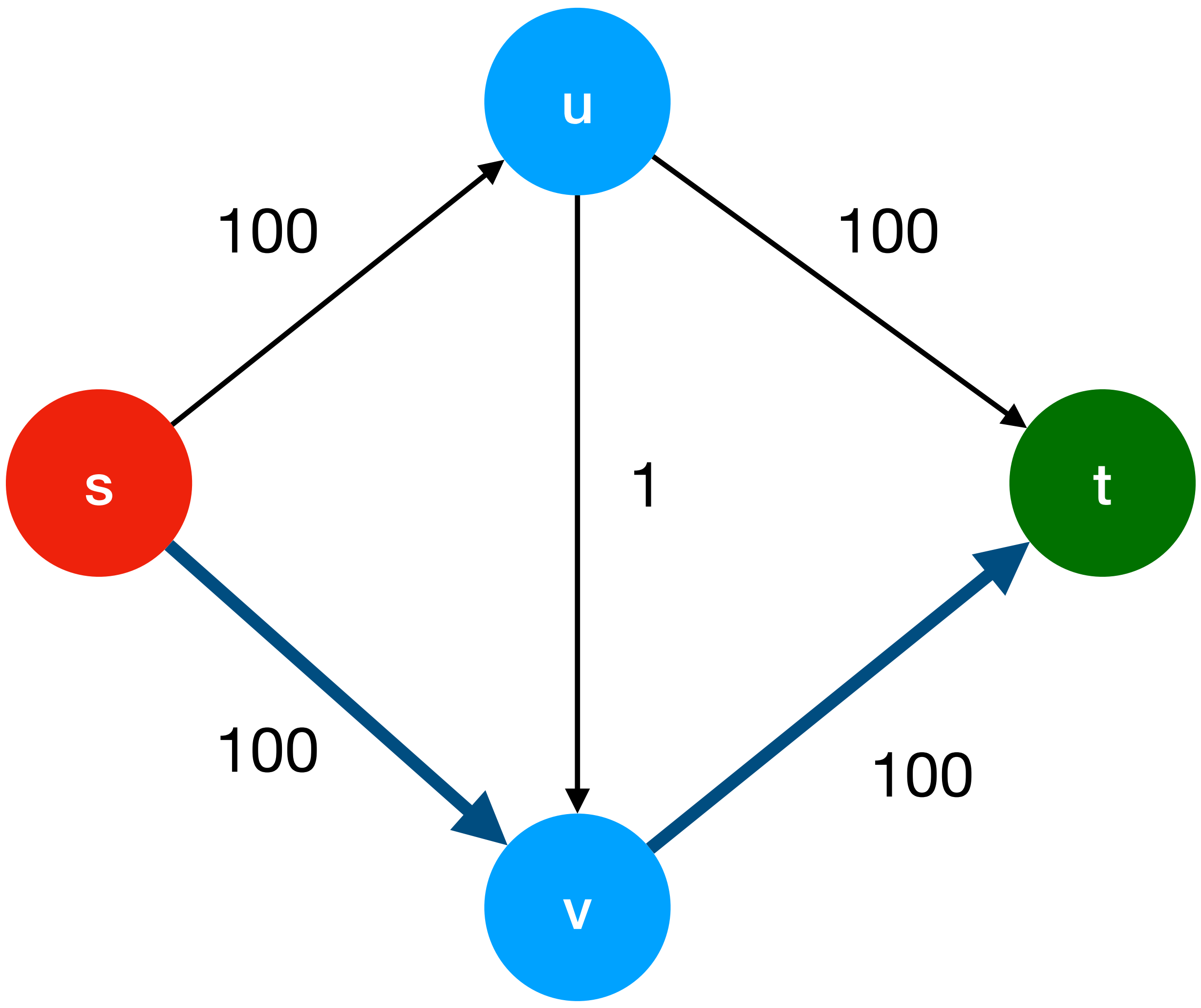

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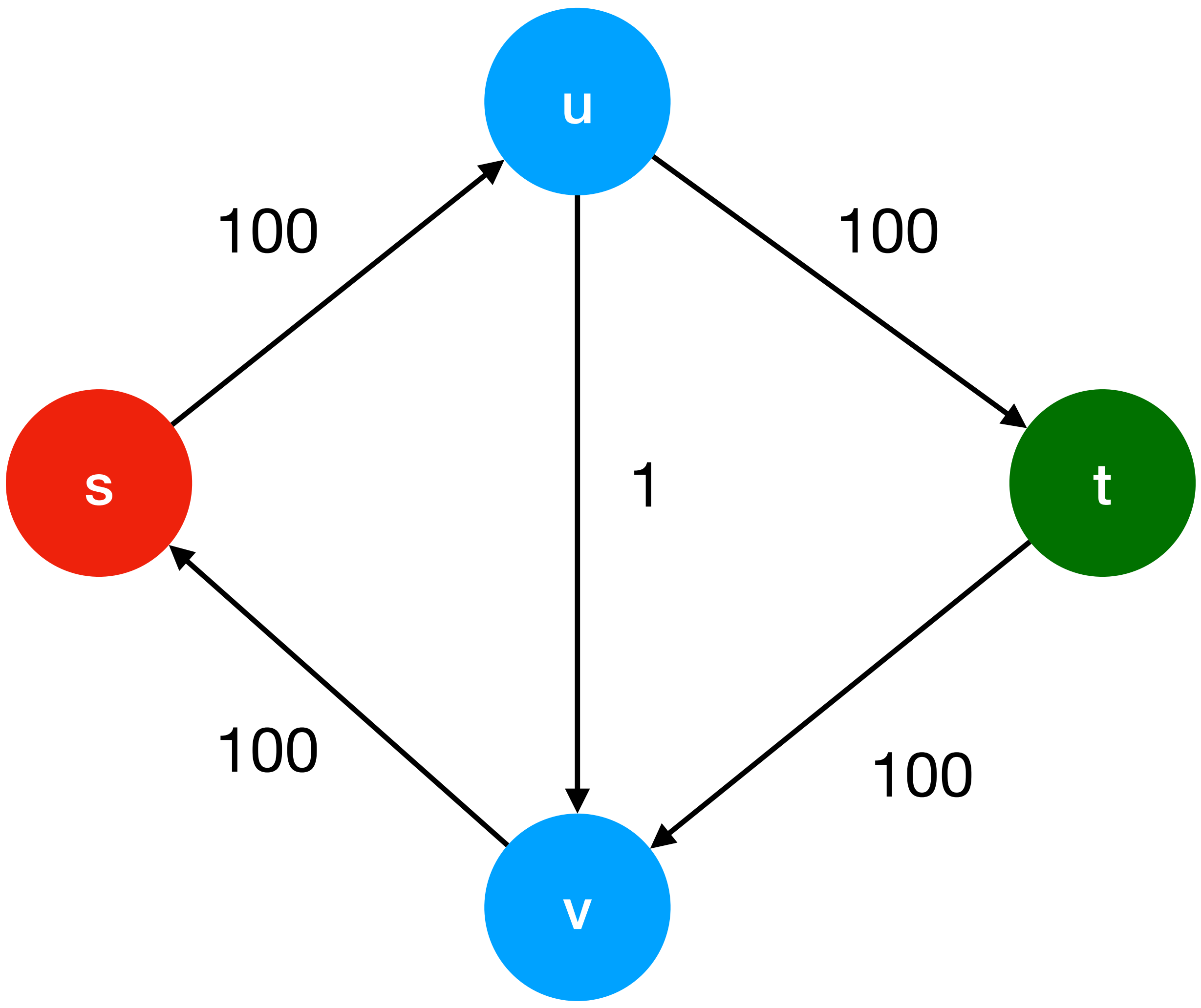

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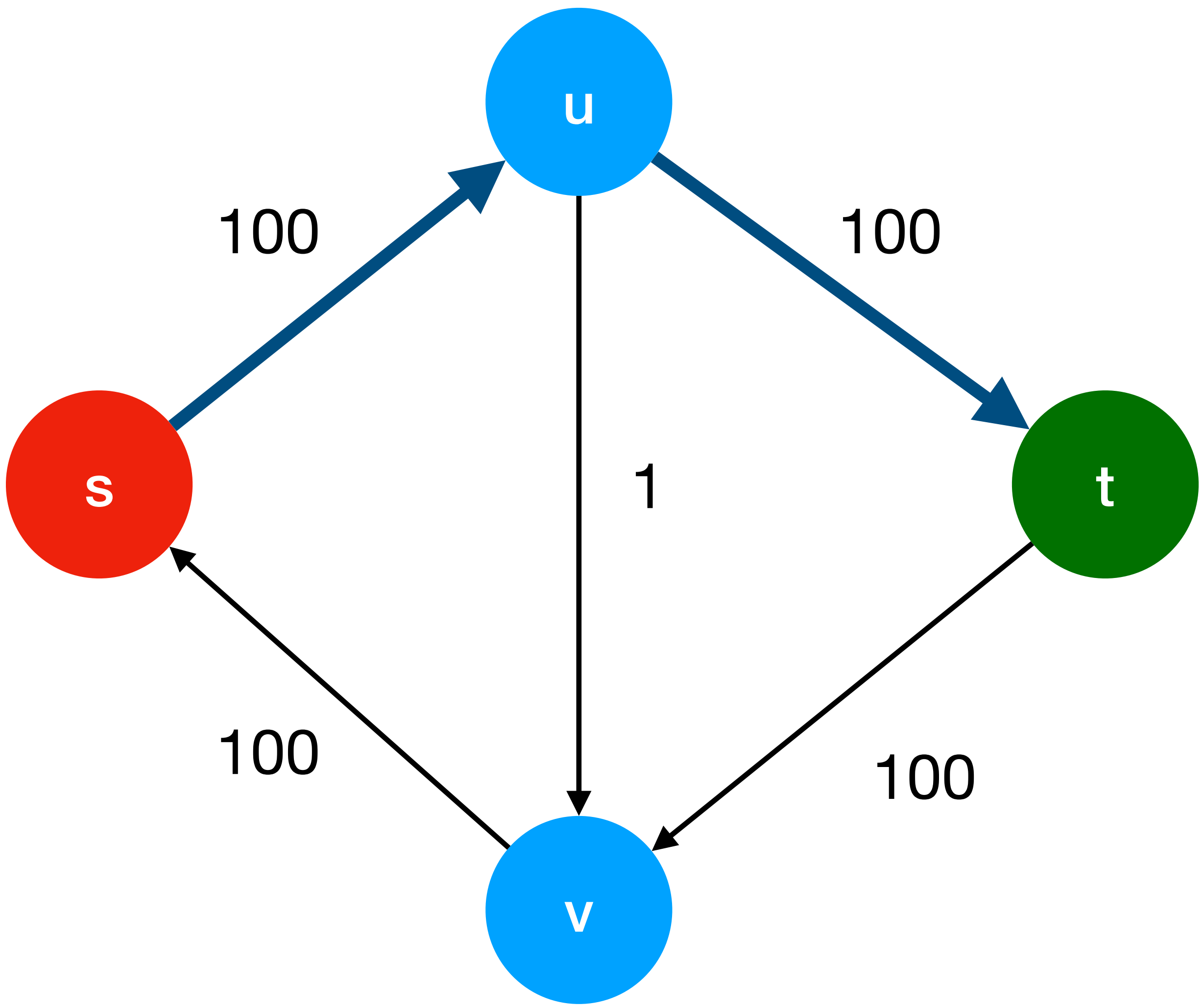

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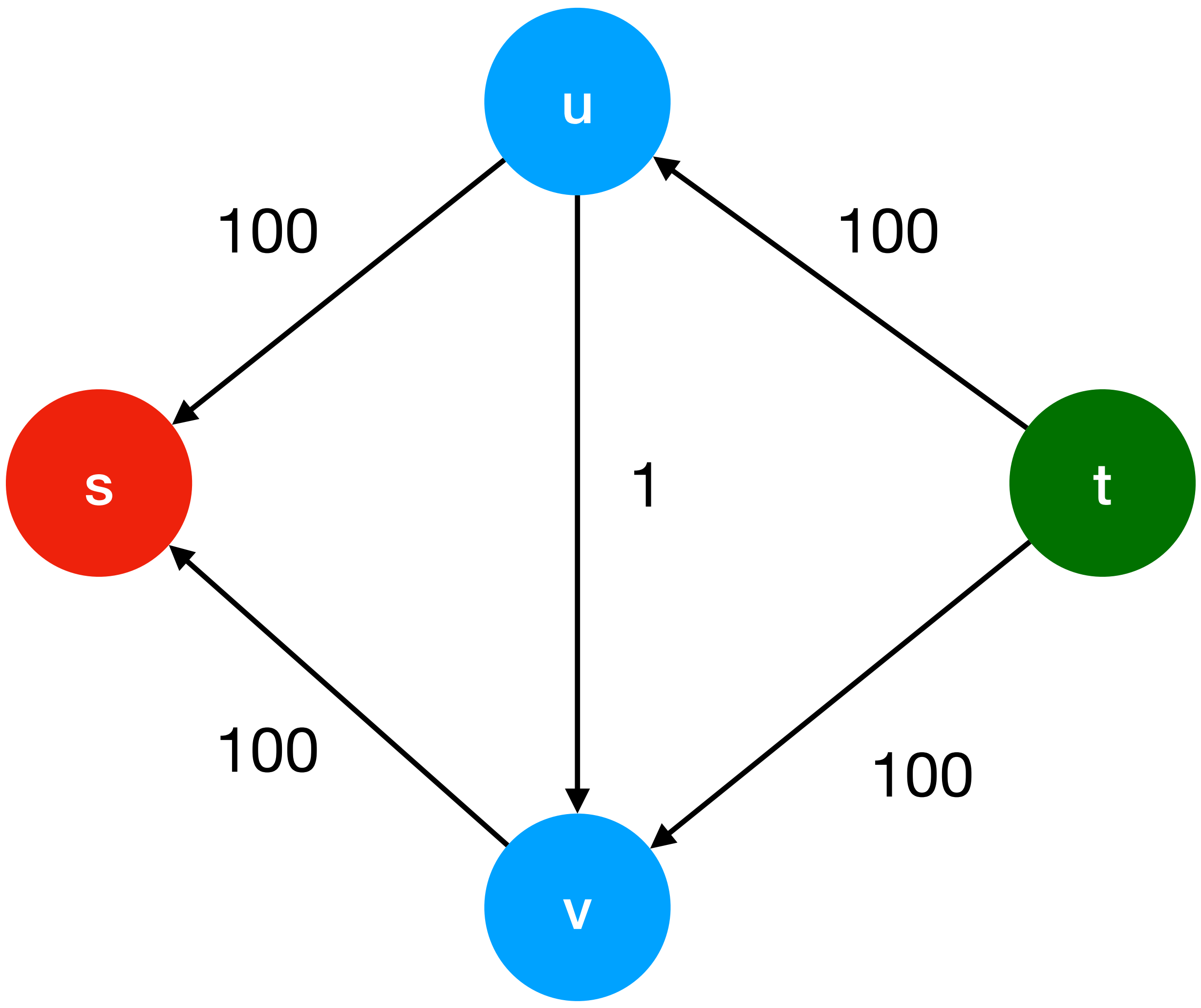

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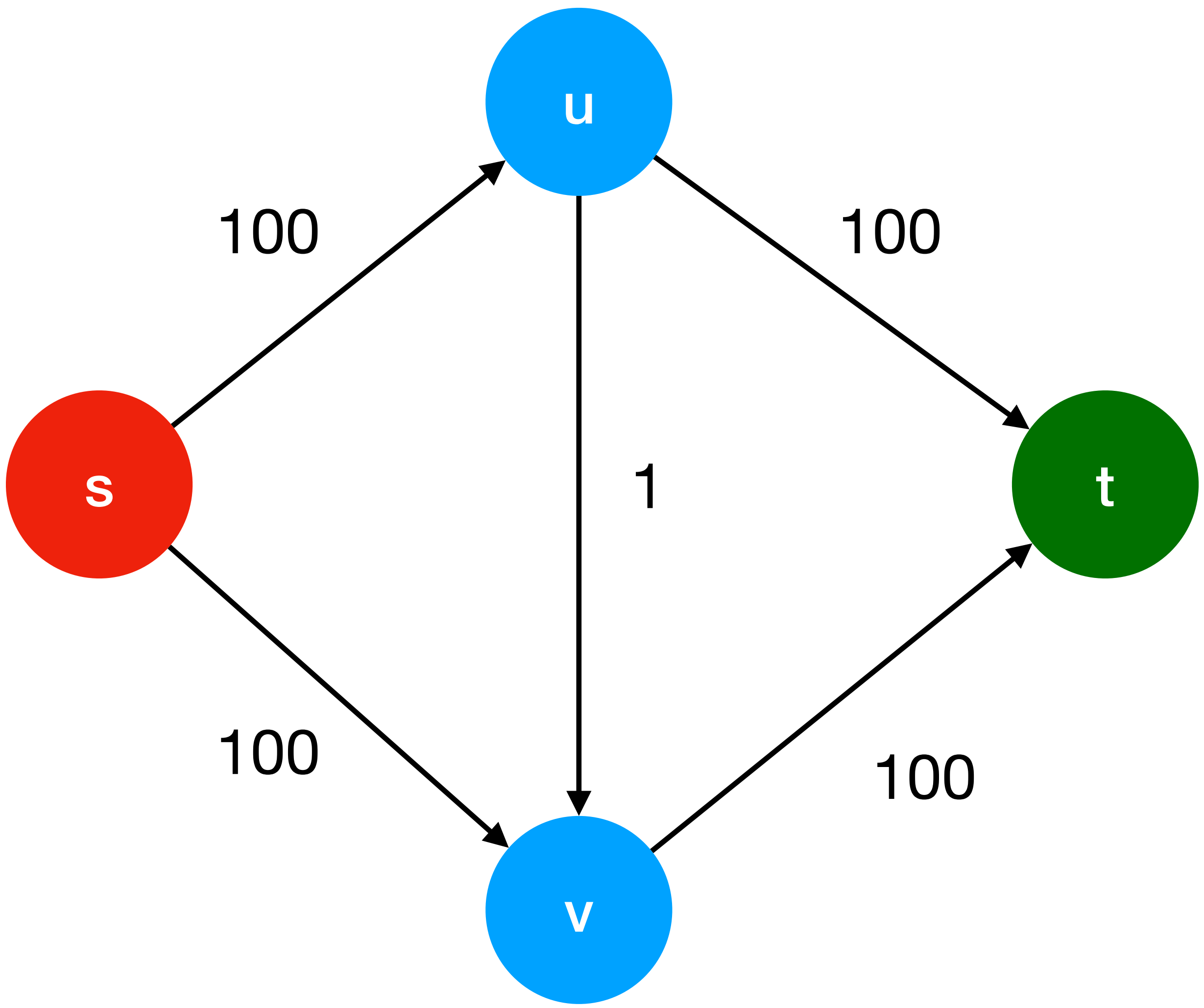

Max-Flow in polynomial time

We made the algorithm must faster by simply selecting the shortest path with available capacity.

Can we always hope to do that?

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The Edmonds-Karp Algorithm

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Useful Lemma

Lemma: Suppose that we run the Edmonds-Karp algorithm on a flow network $G = (V, E)$ and consider any flow augmentation. For every node $u \in V \setminus \{s, t\}$, the length of the shortest path $d_f(s, u)$ from s to u in the residual graph G_f given by f does not decrease.

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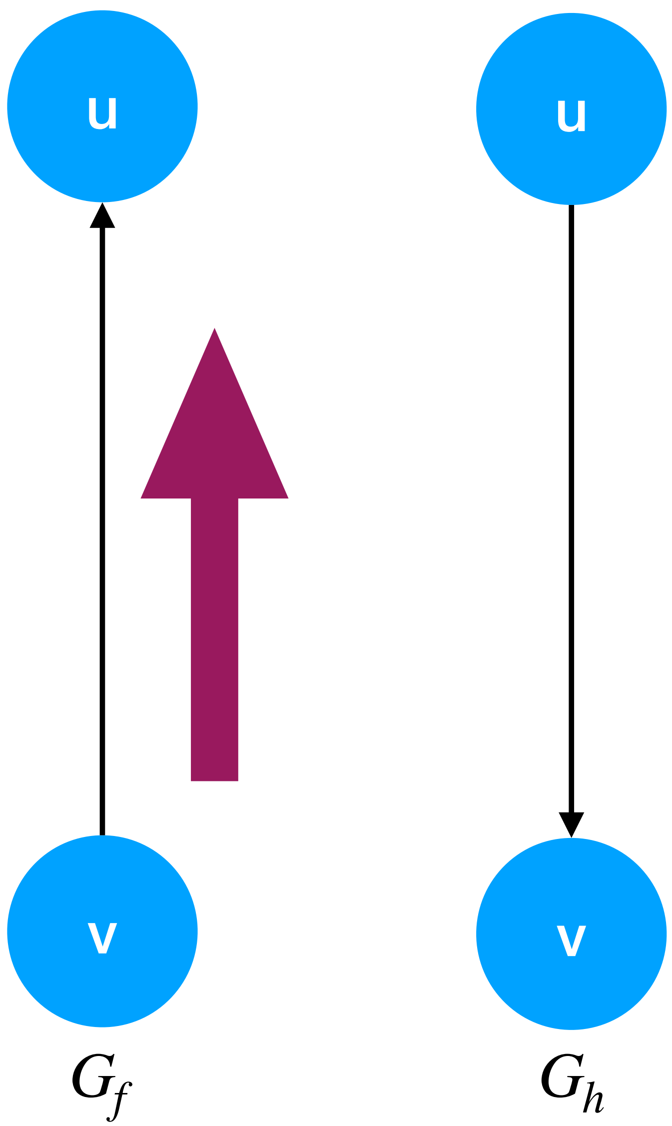

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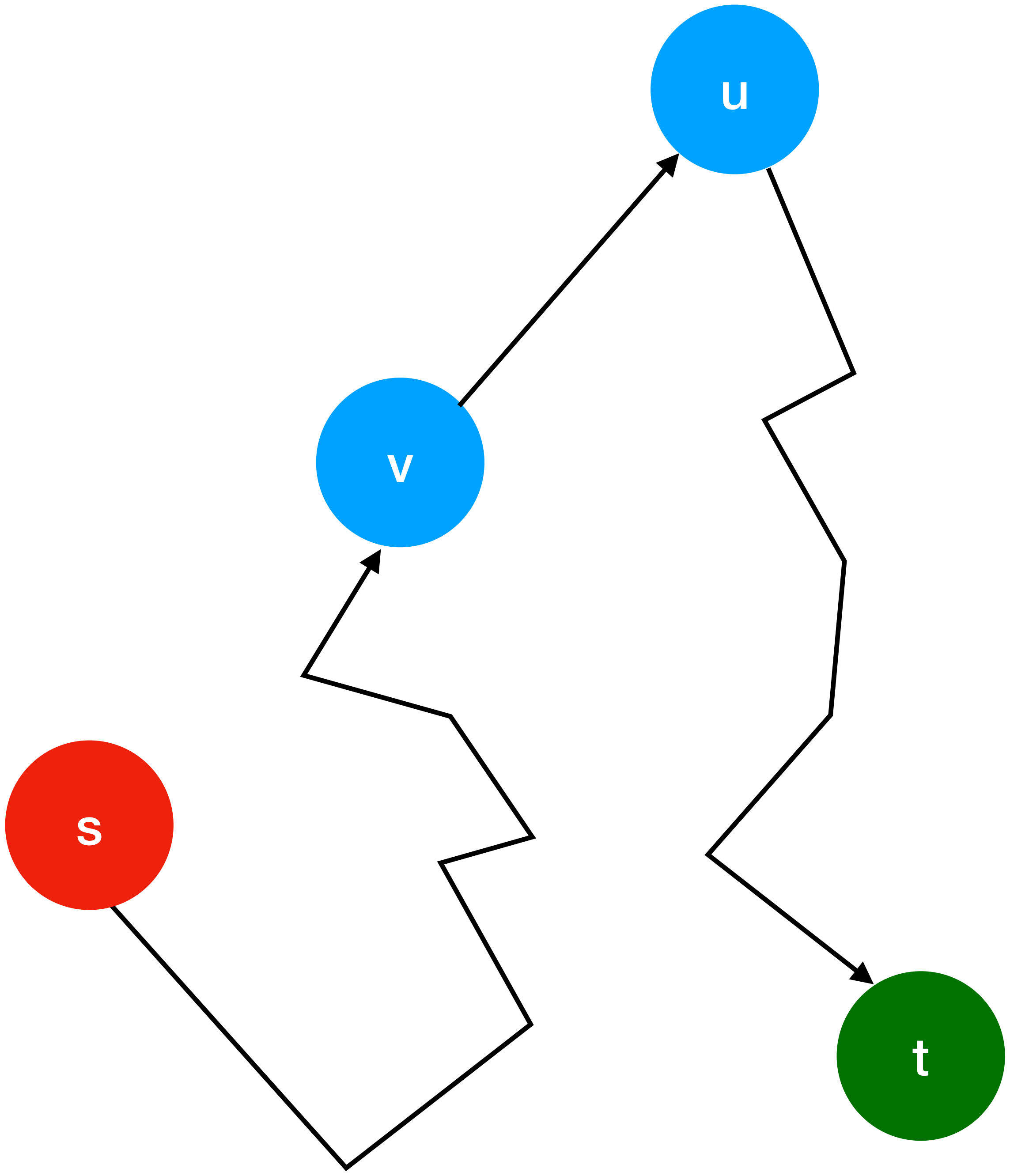

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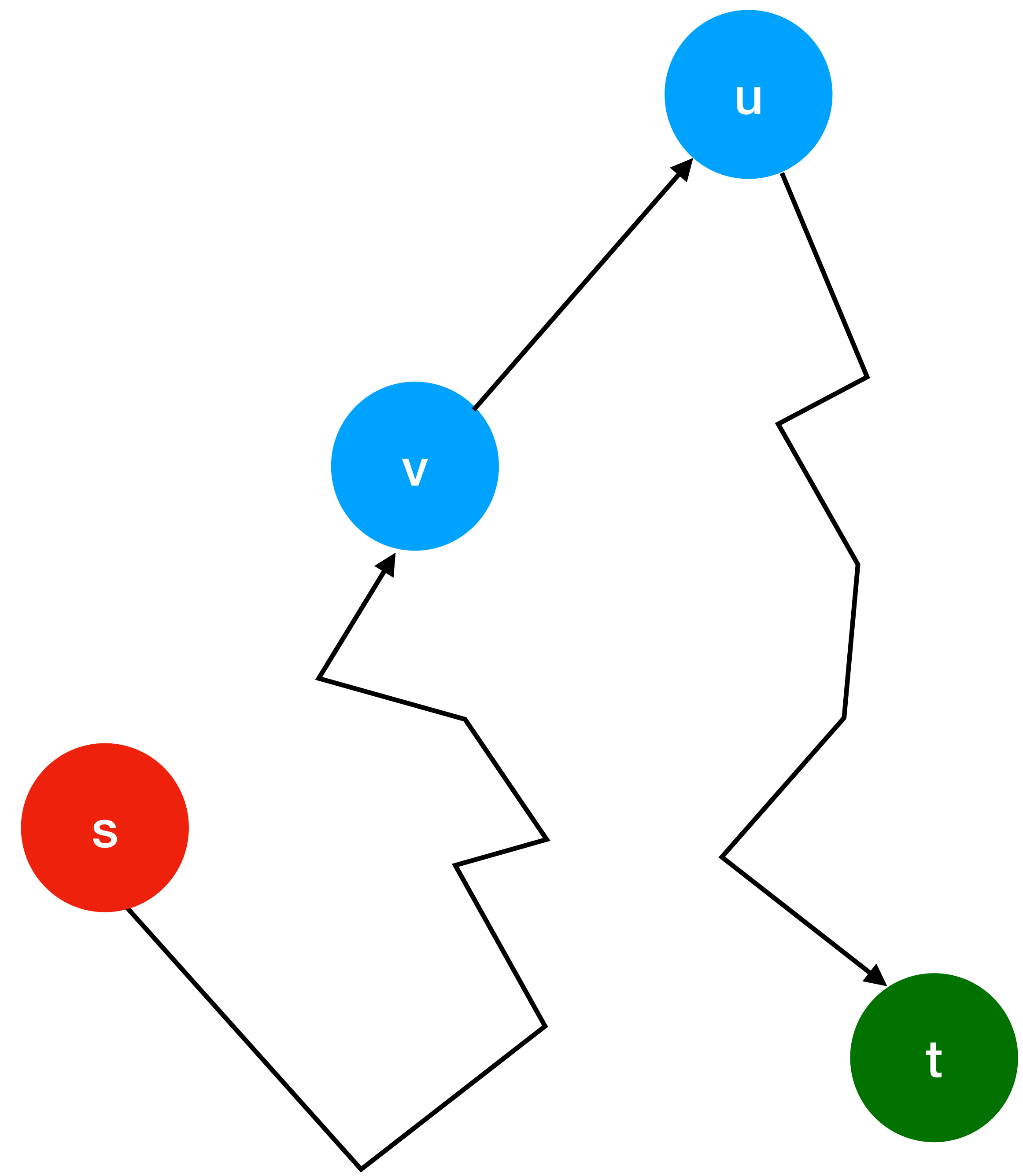

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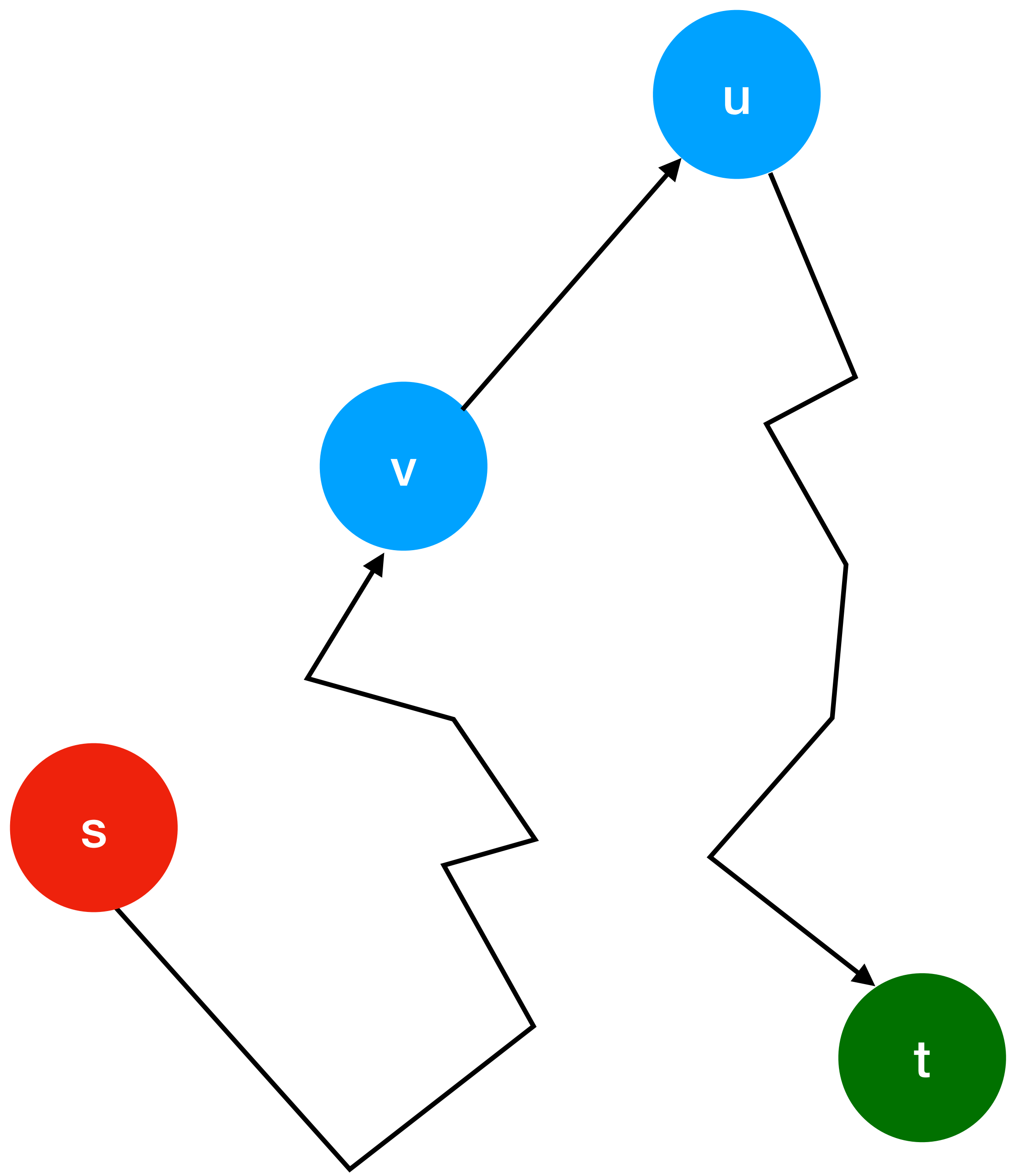

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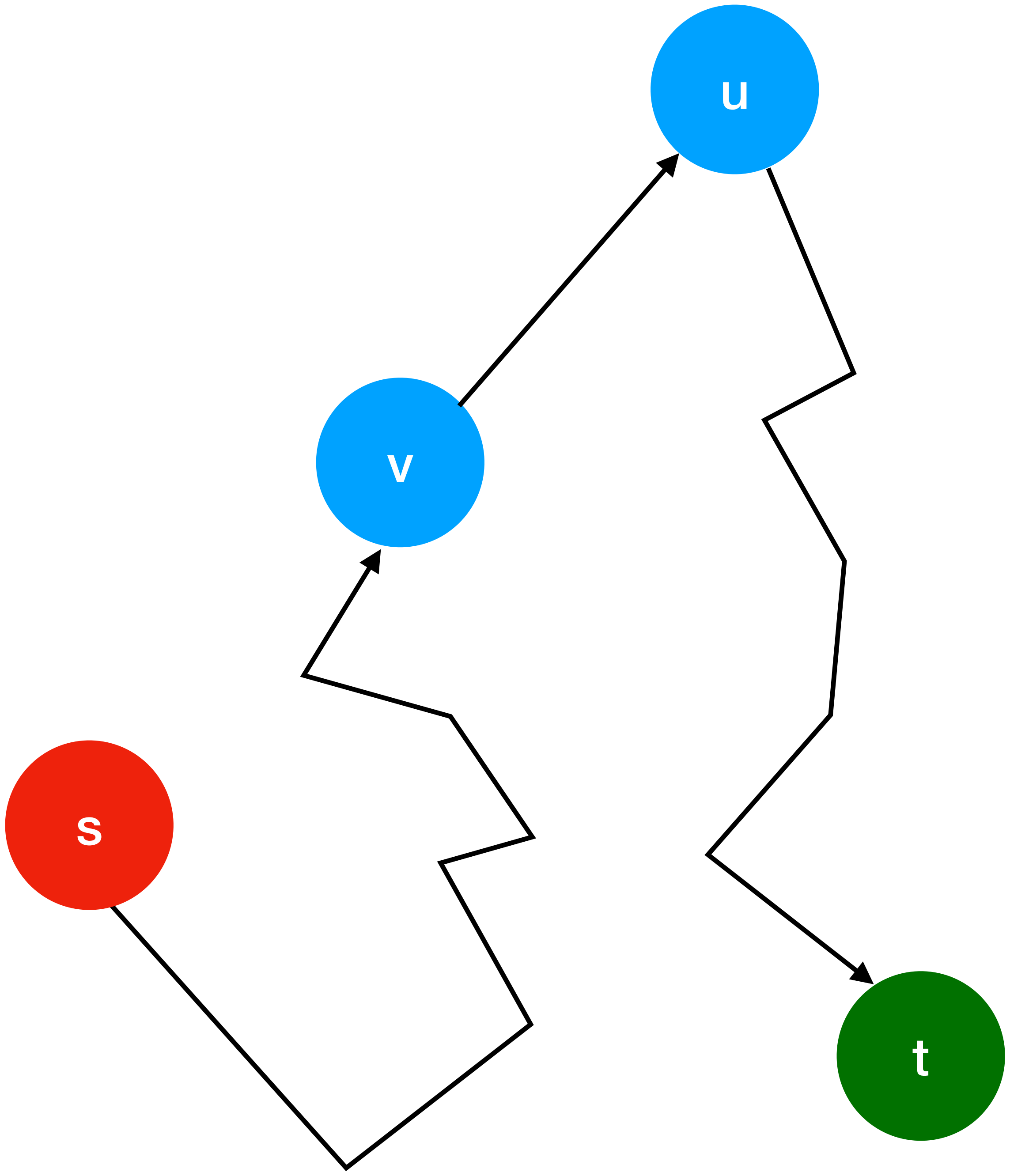

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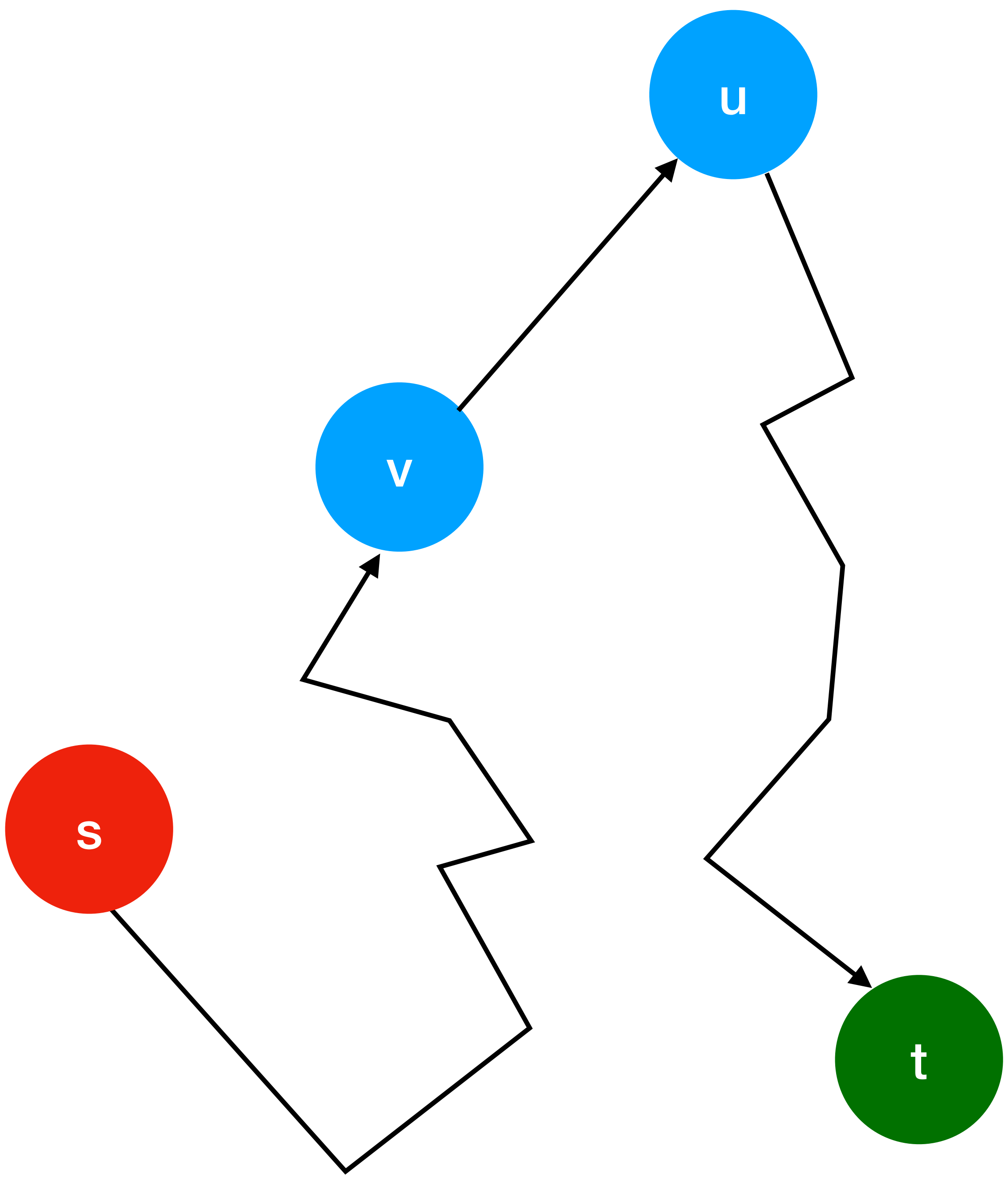

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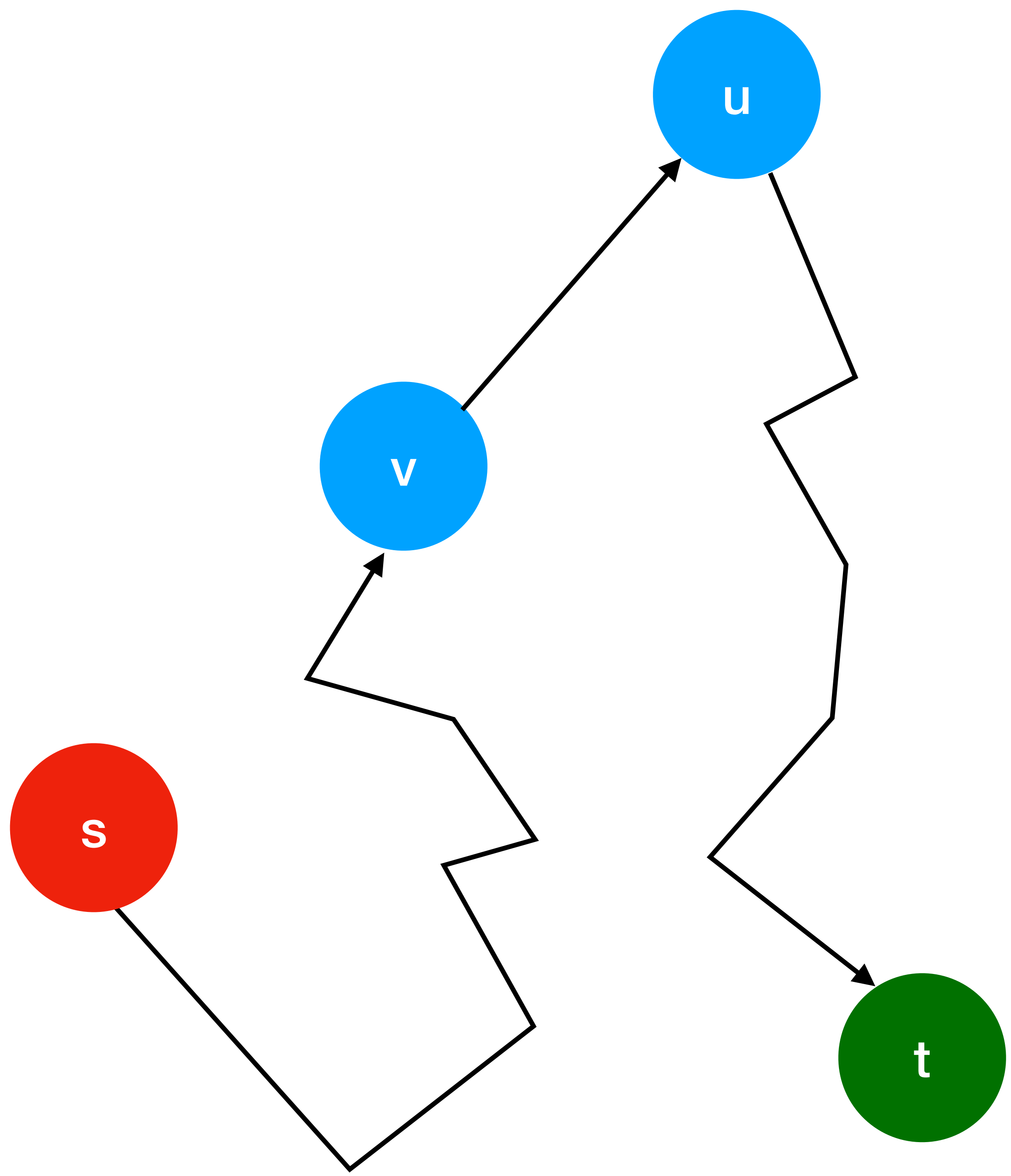

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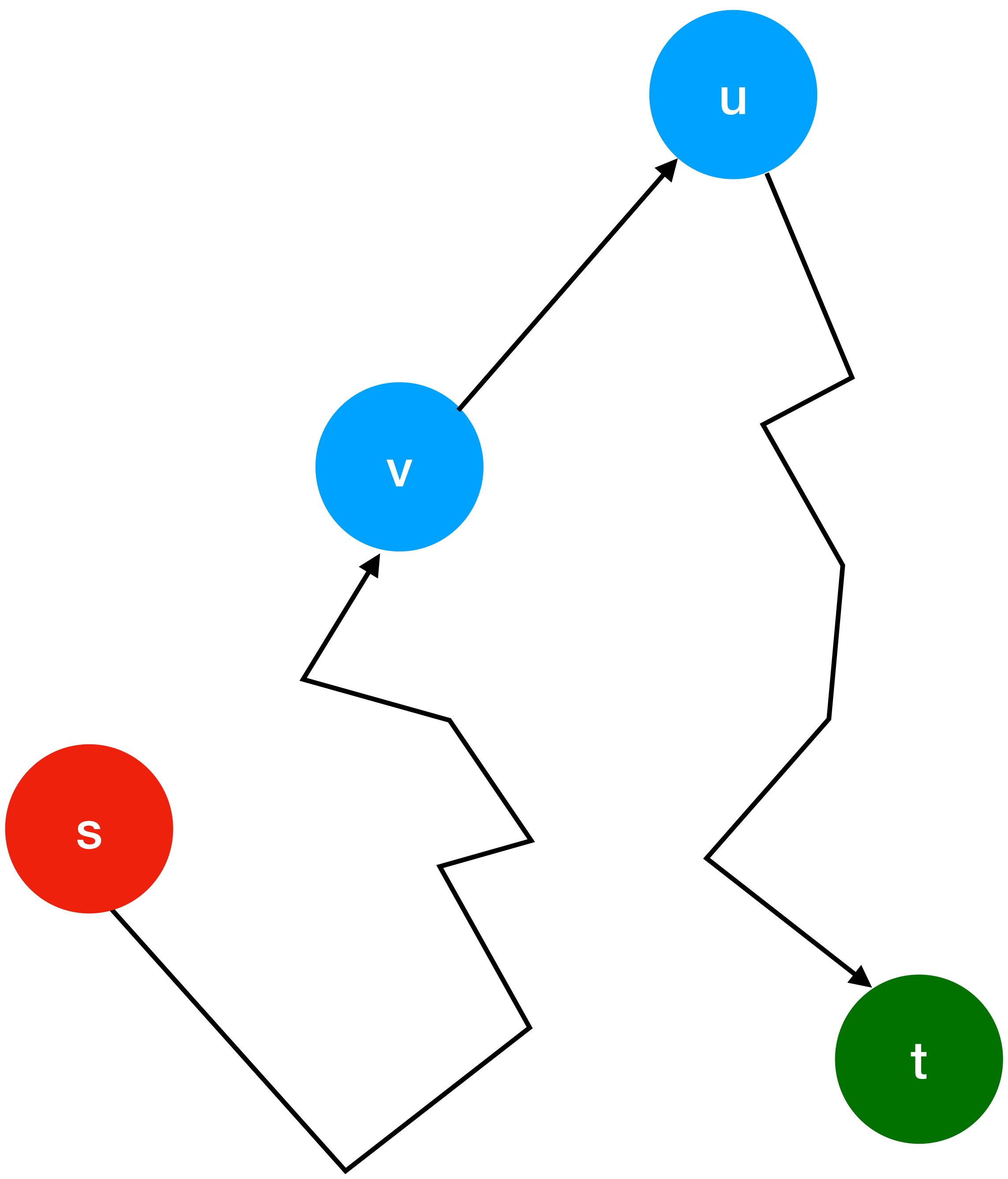

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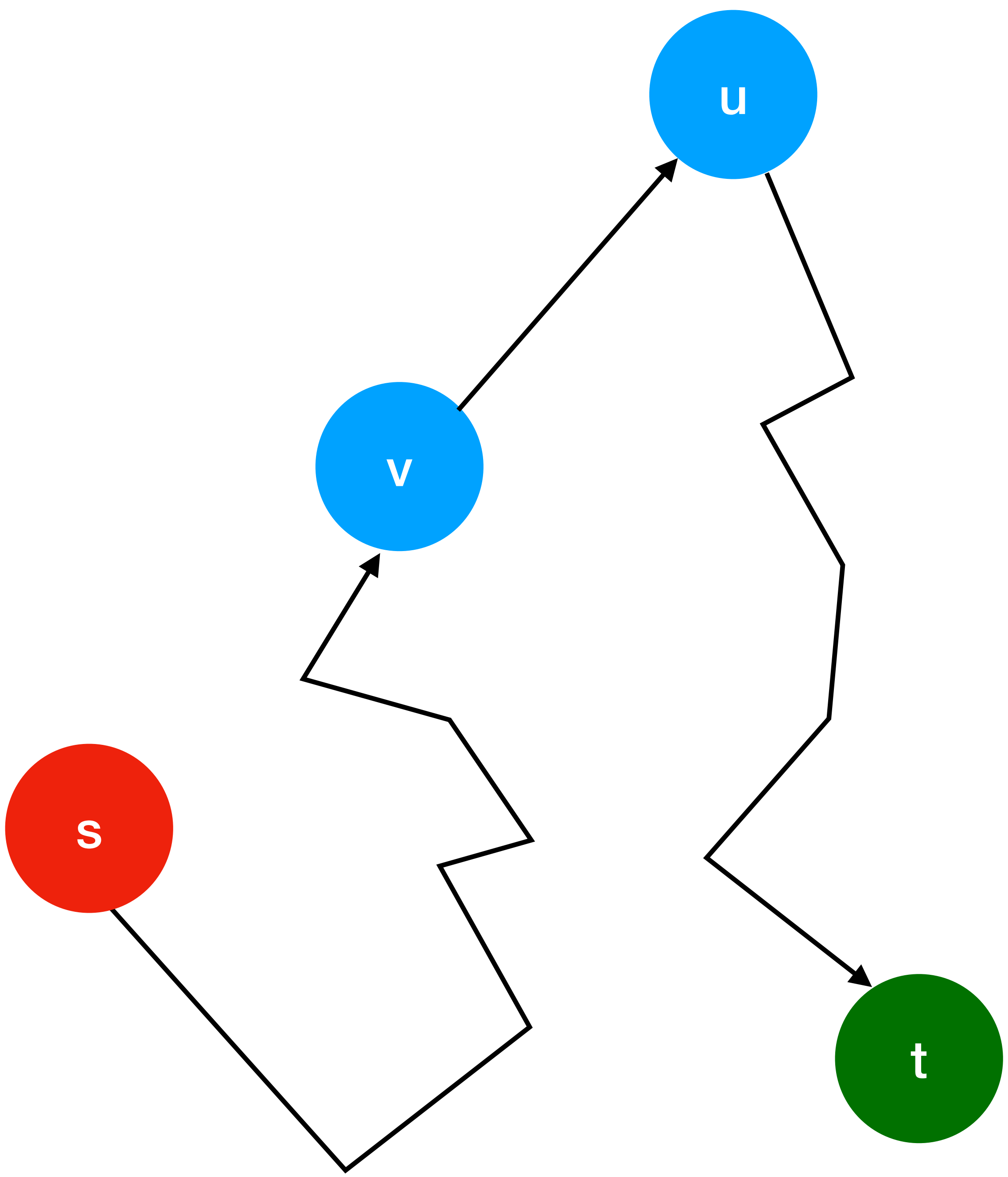

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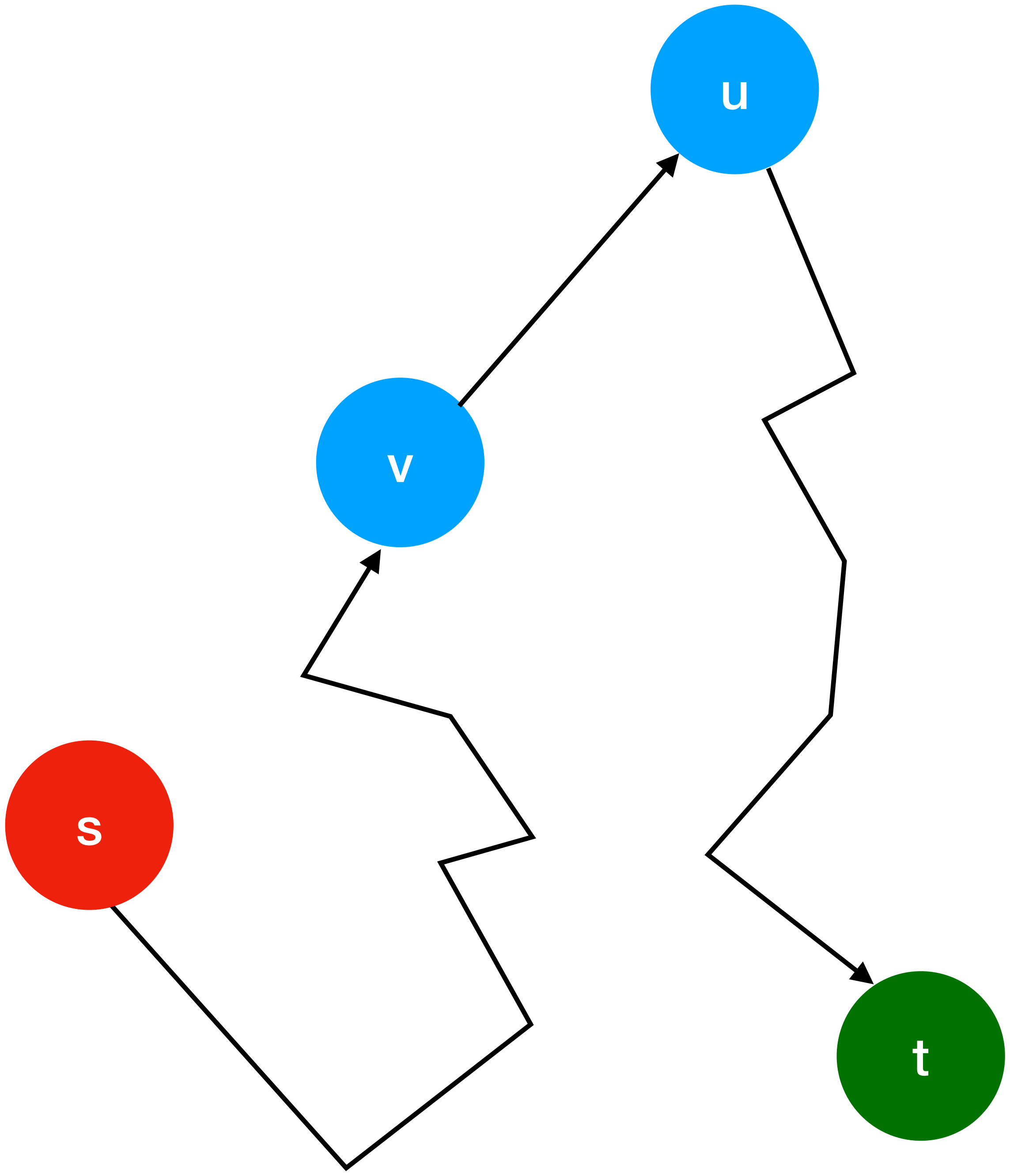

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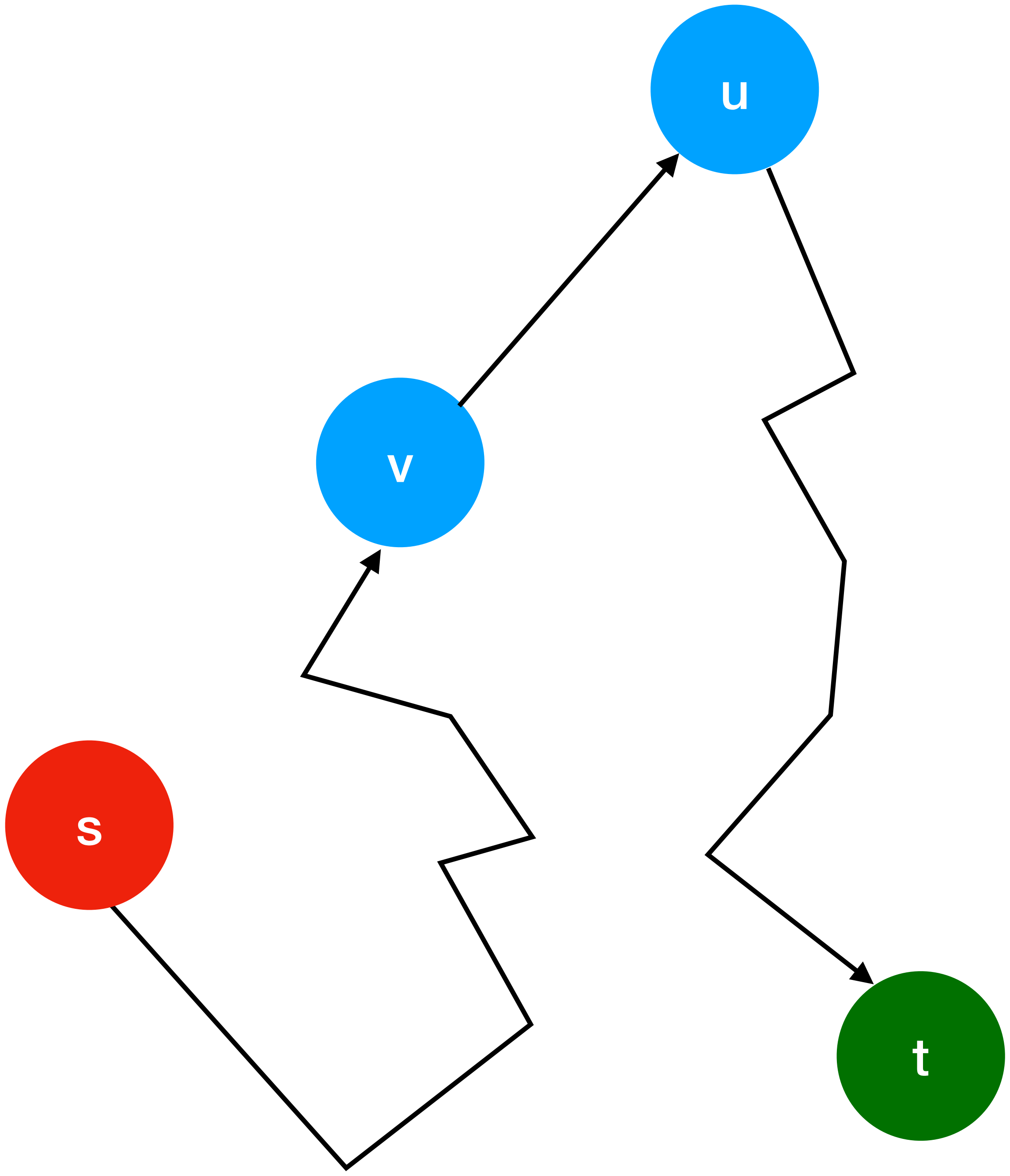

Useful Lemma

Lemma: Suppose that we run the Edmonds-Karp algorithm on a flow network $G = (V, E)$ and consider any flow augmentation. For every node $u \in V \backslash \{s, t\}$, the length of the shortest path $d_f(s, u)$ from s to u in the residual graph G_f given by f does not decrease.

Proof: Suppose that there exists some node for which it decreases with some flow augmentation; let f to h be this augmentation.

Let v be the node with the minimum $d_h(s, v)$ for which $d_f(s, v) > d_h(s, v)$.

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That means that (v, u) was part of the chosen augmenting path, which was a shortest (s, t) path in G_f.

Consider the sub-path (s, u) of the (s, t) path above. This is also a shortest (s, u) path.

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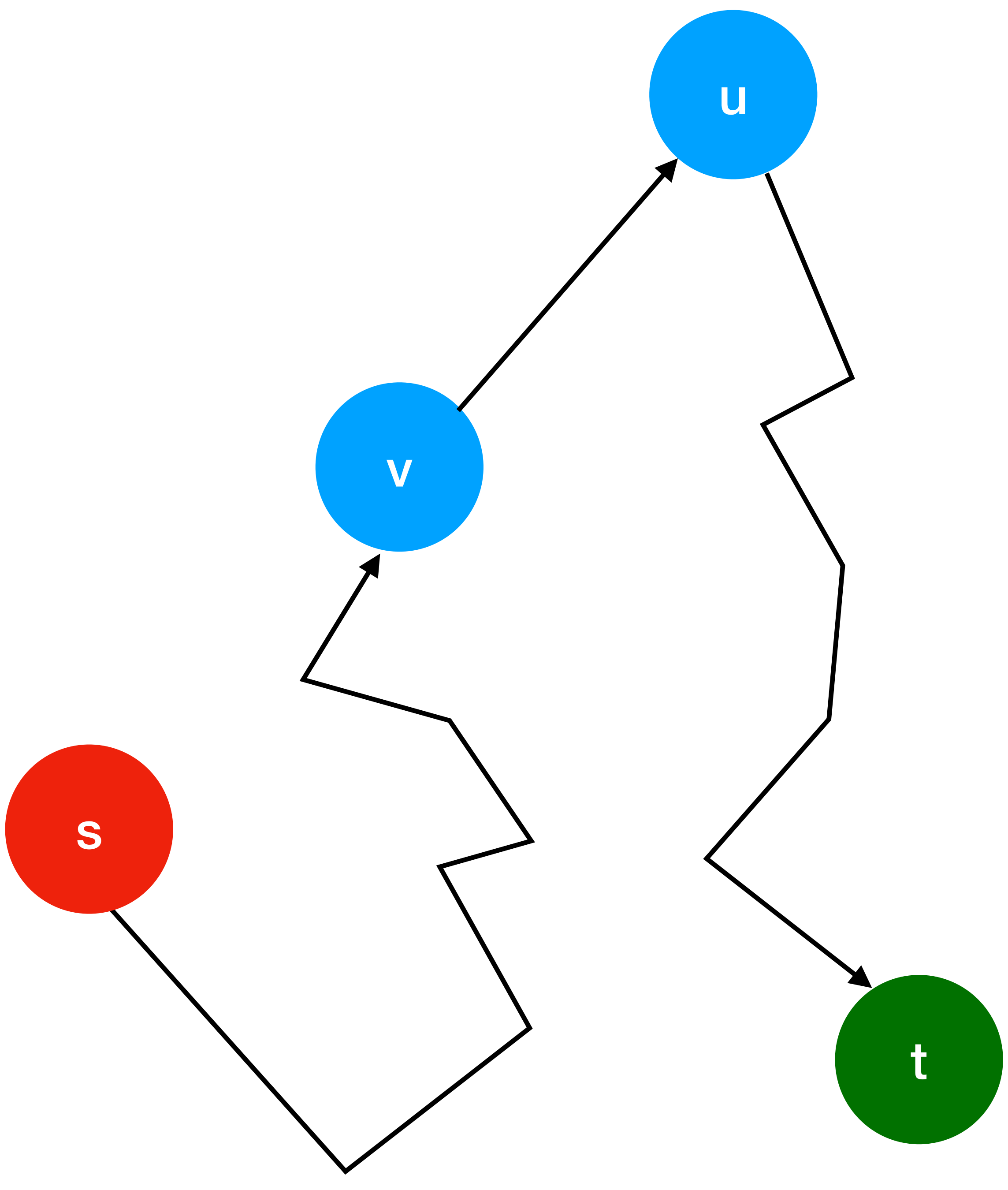

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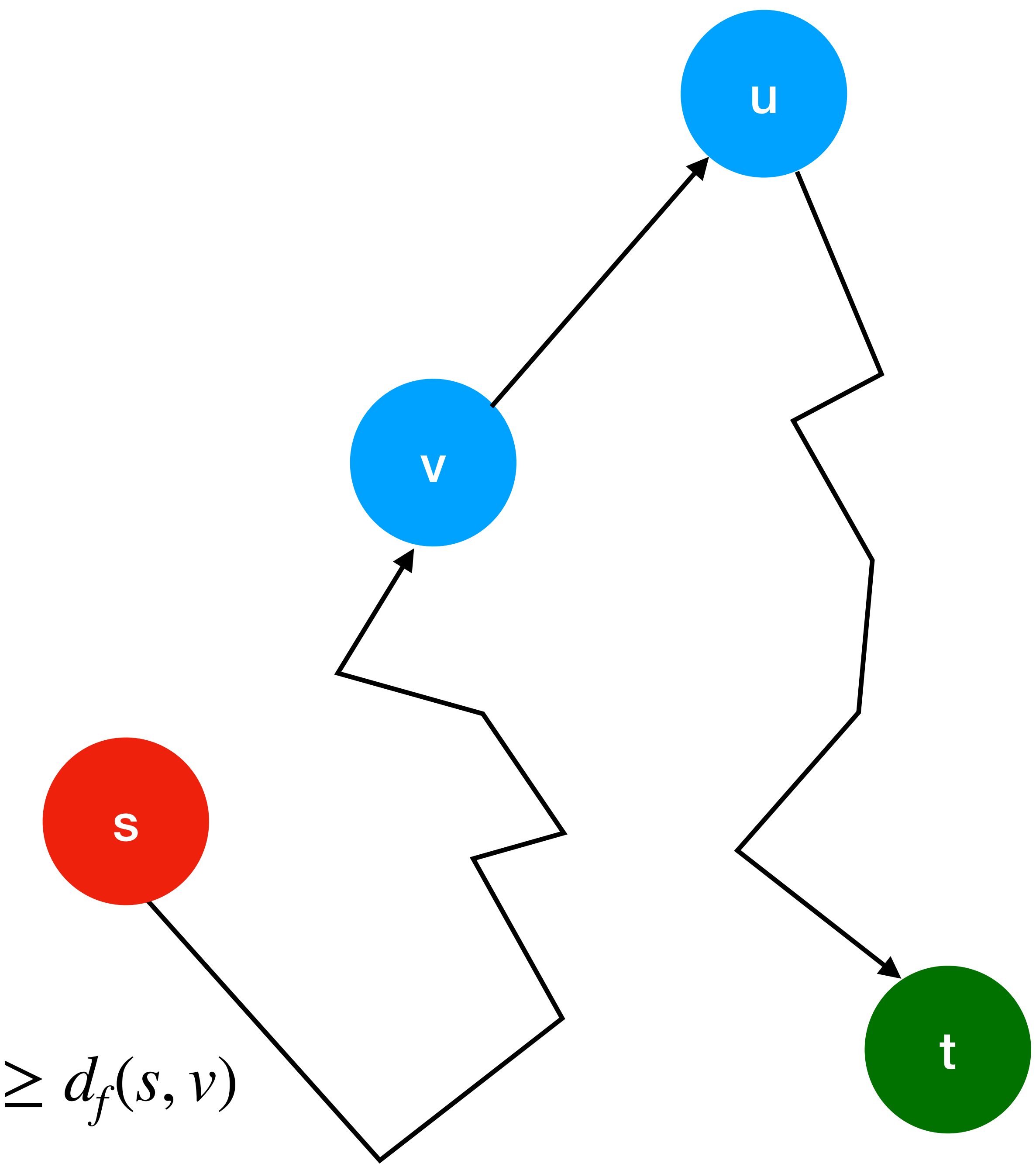

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contradiction!

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How many times can an edge e become critical during the execution of the algorithm?

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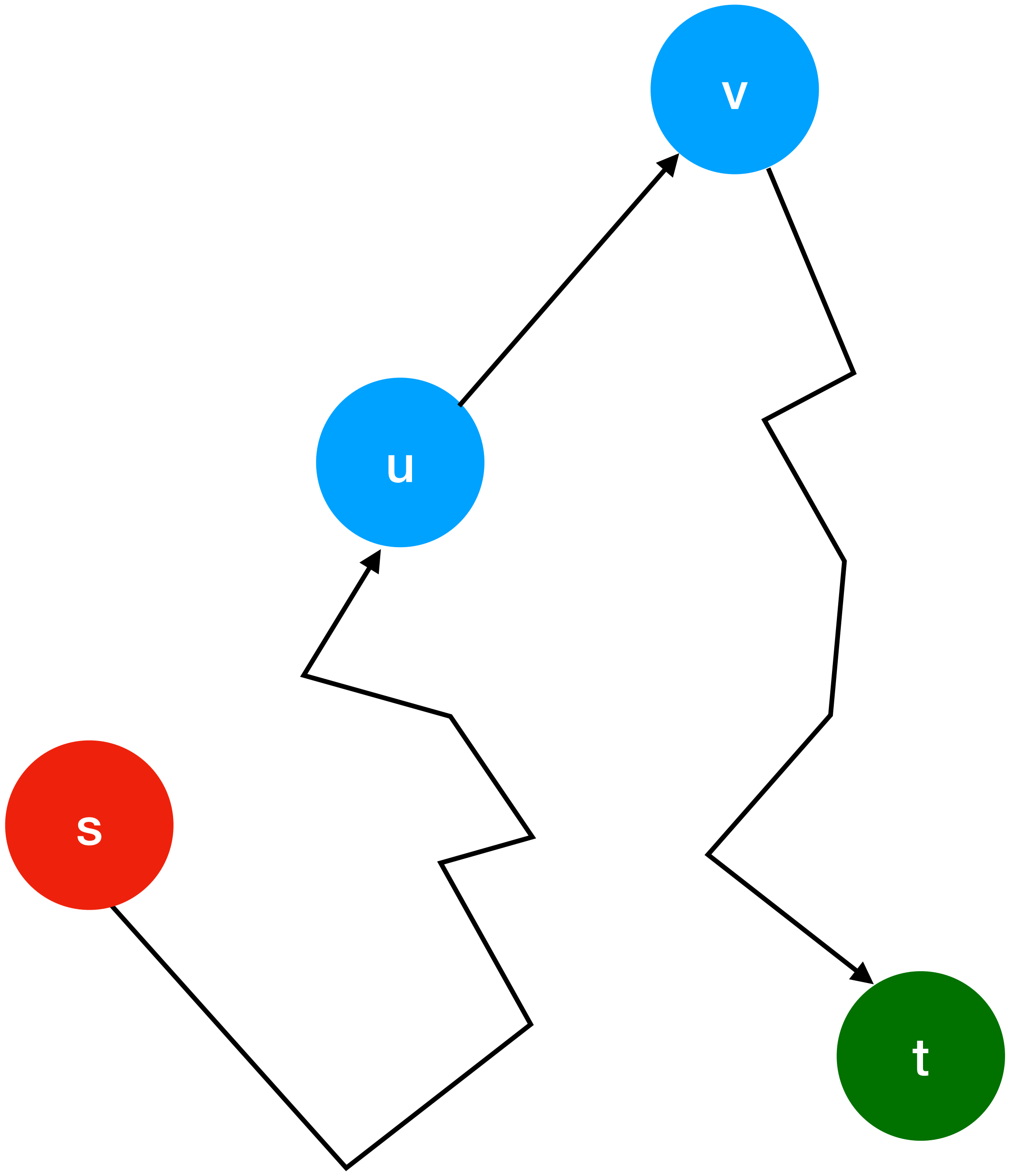
Let (u, v) be an edge. Consider the first time it becomes critical, at which point the residual graph is G_f.



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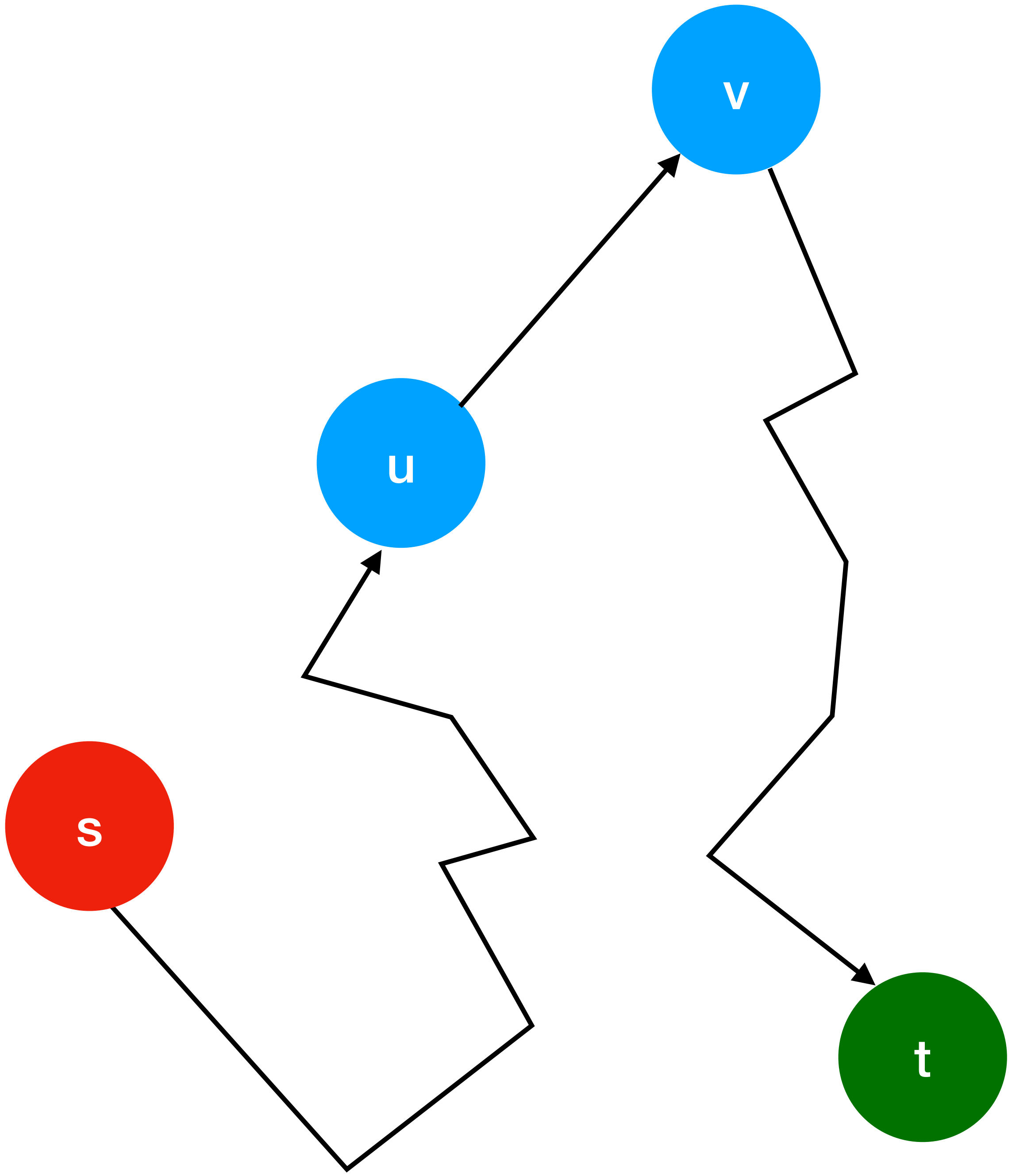

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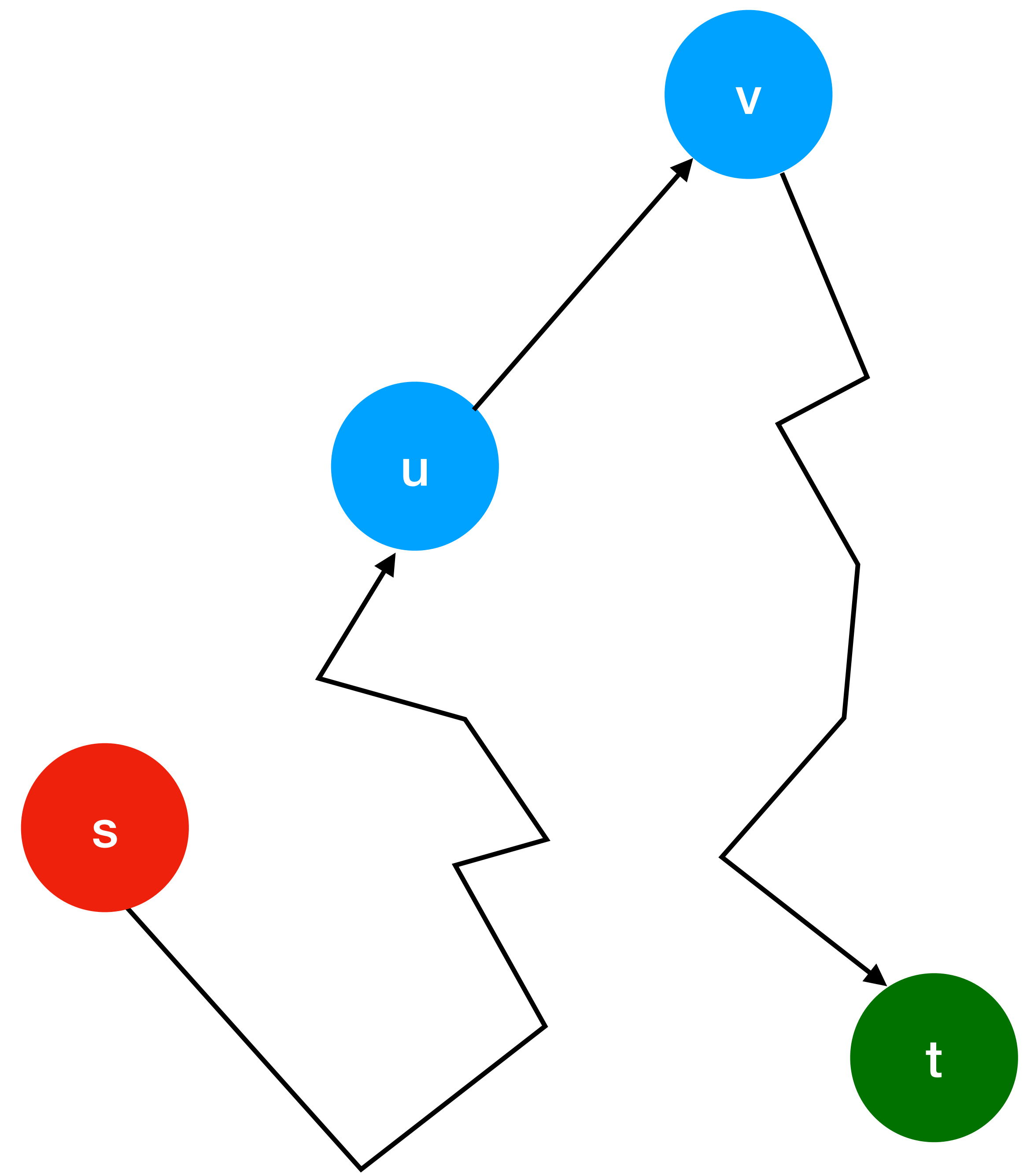

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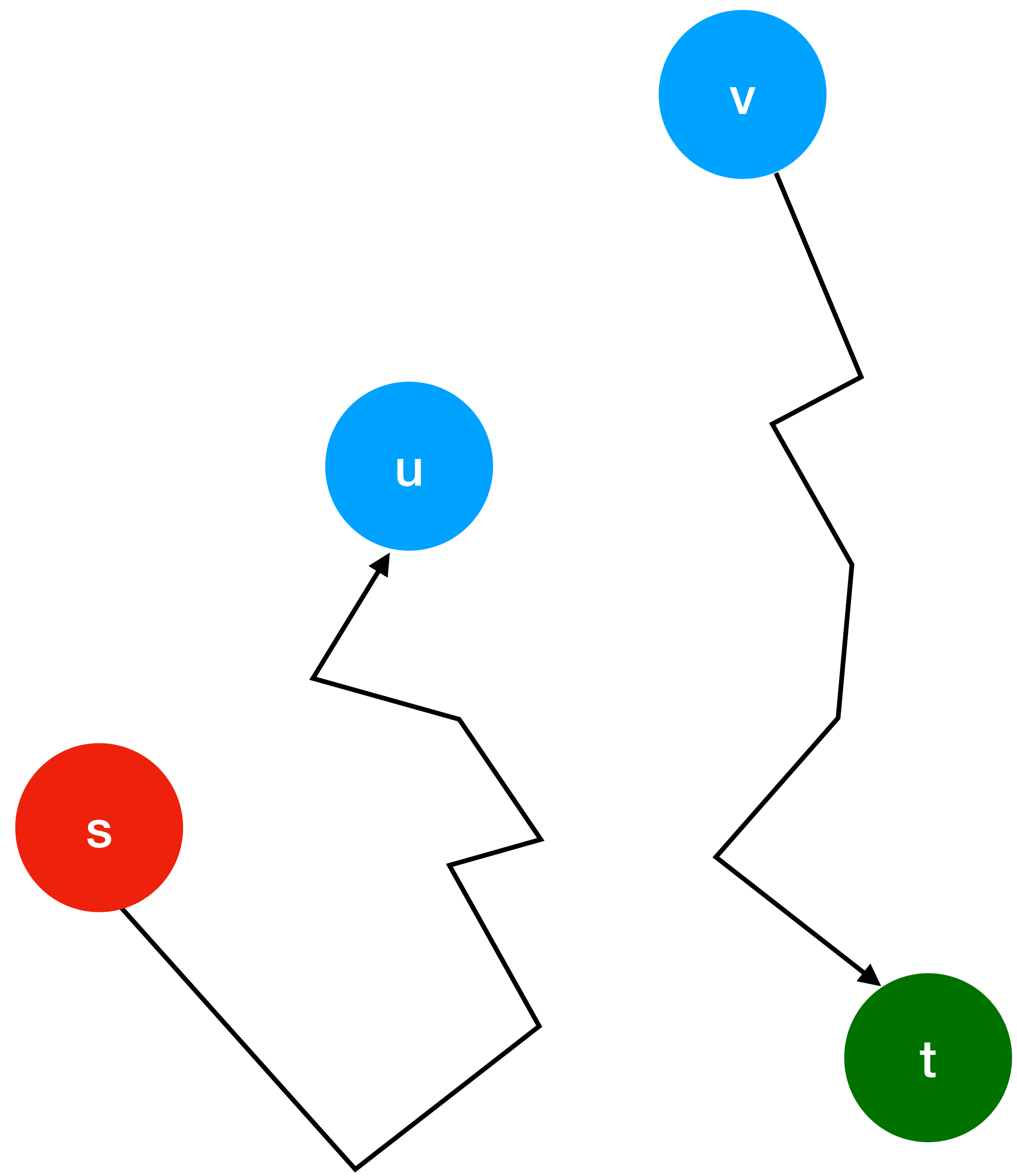

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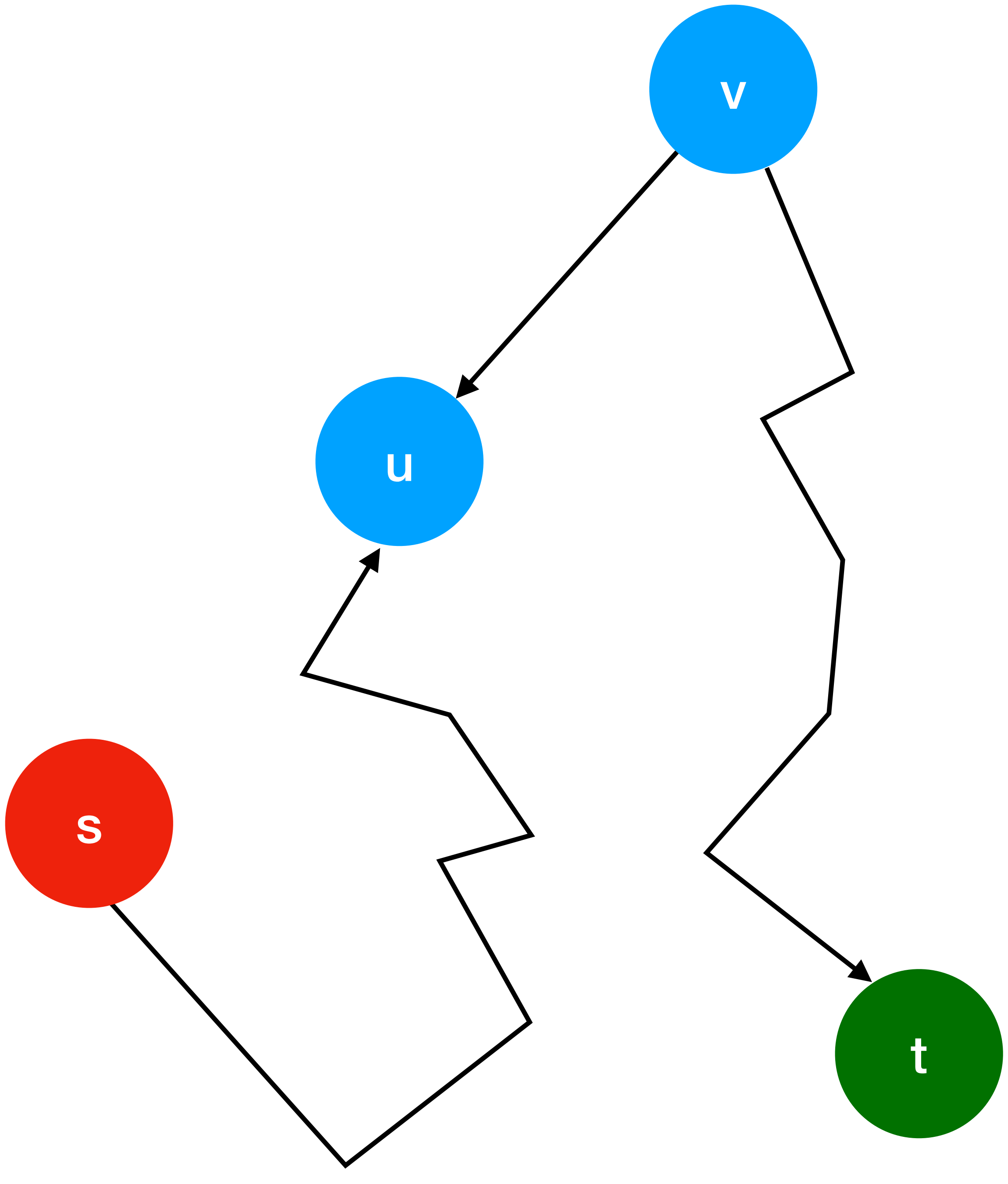

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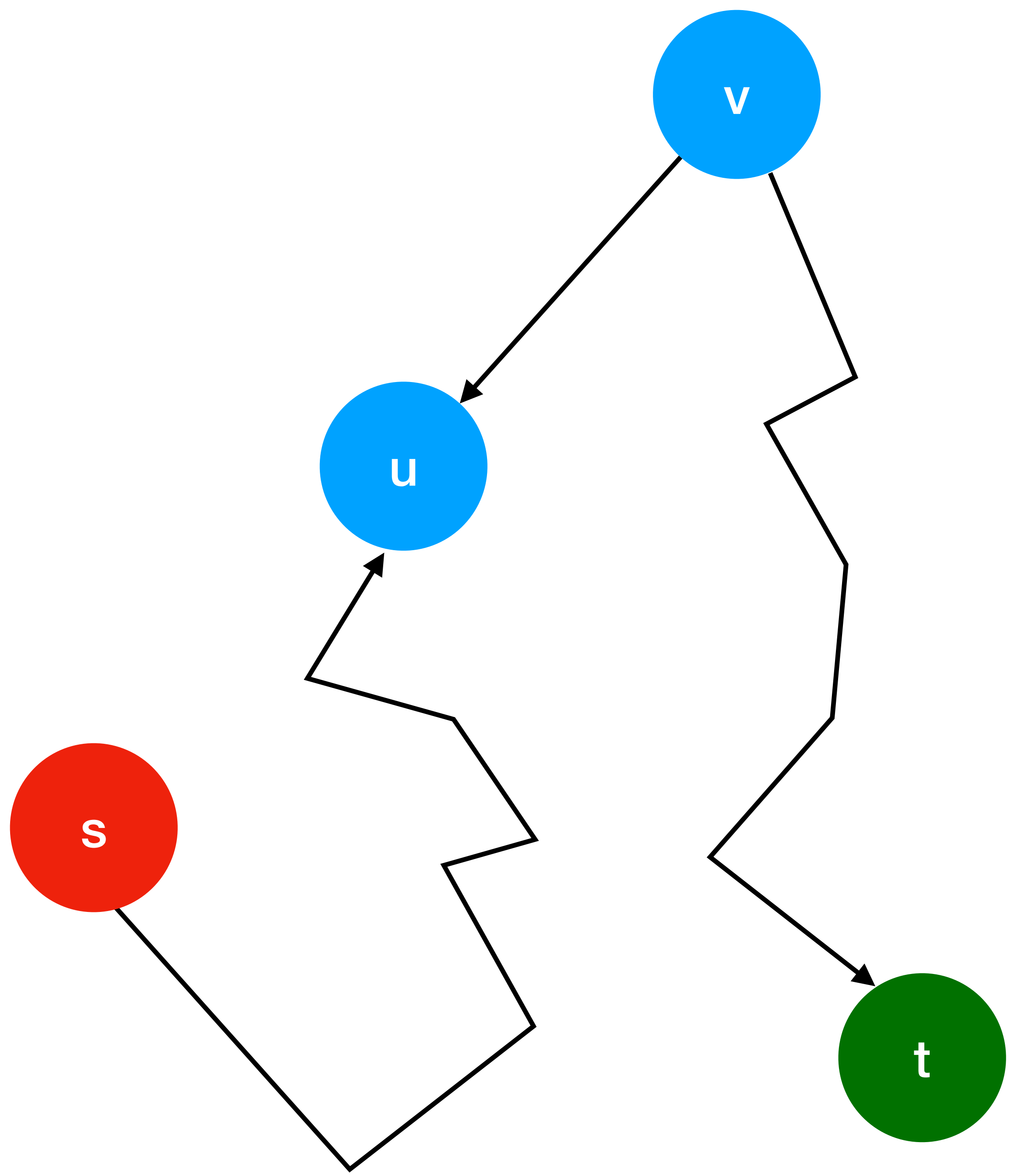

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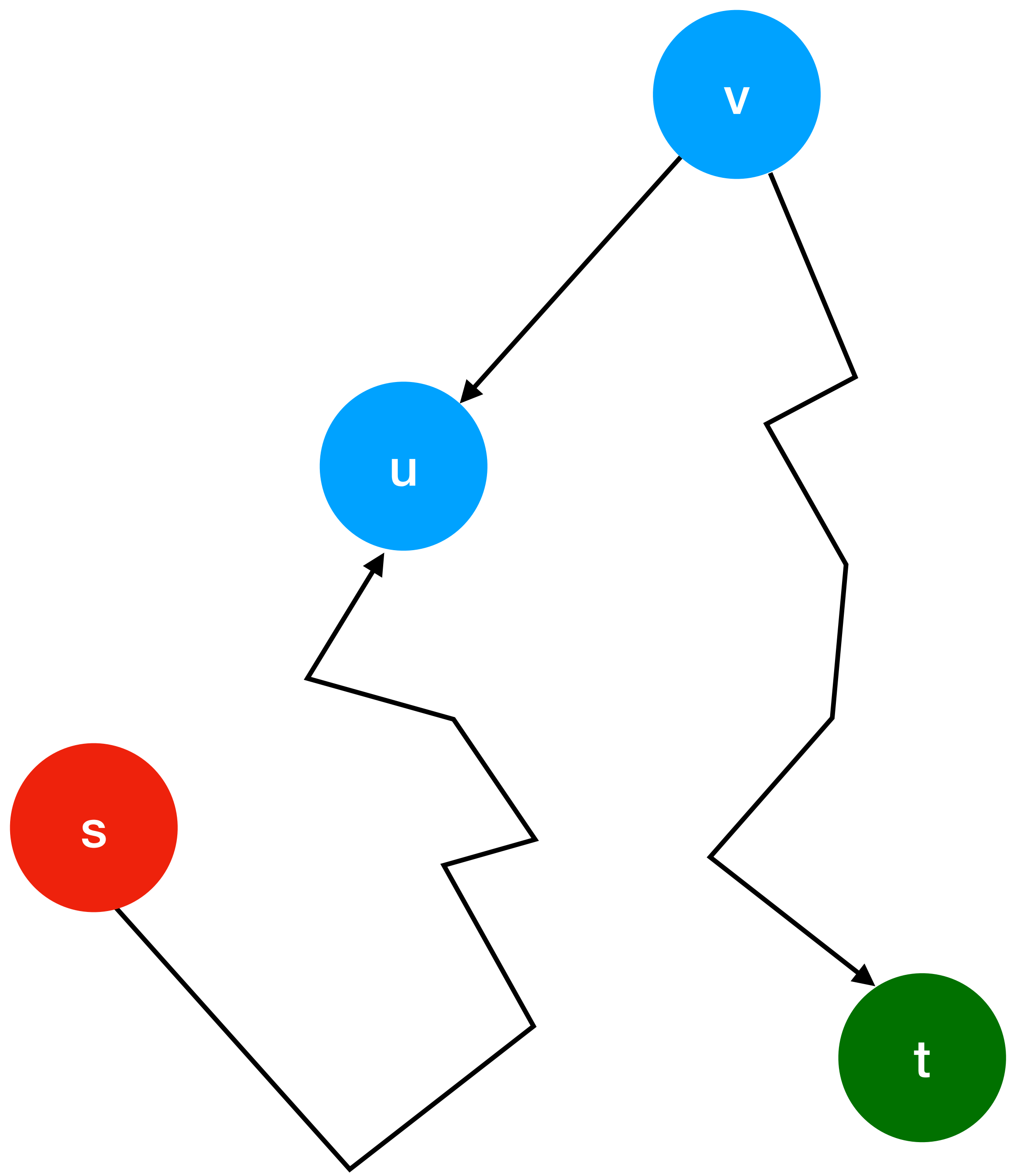

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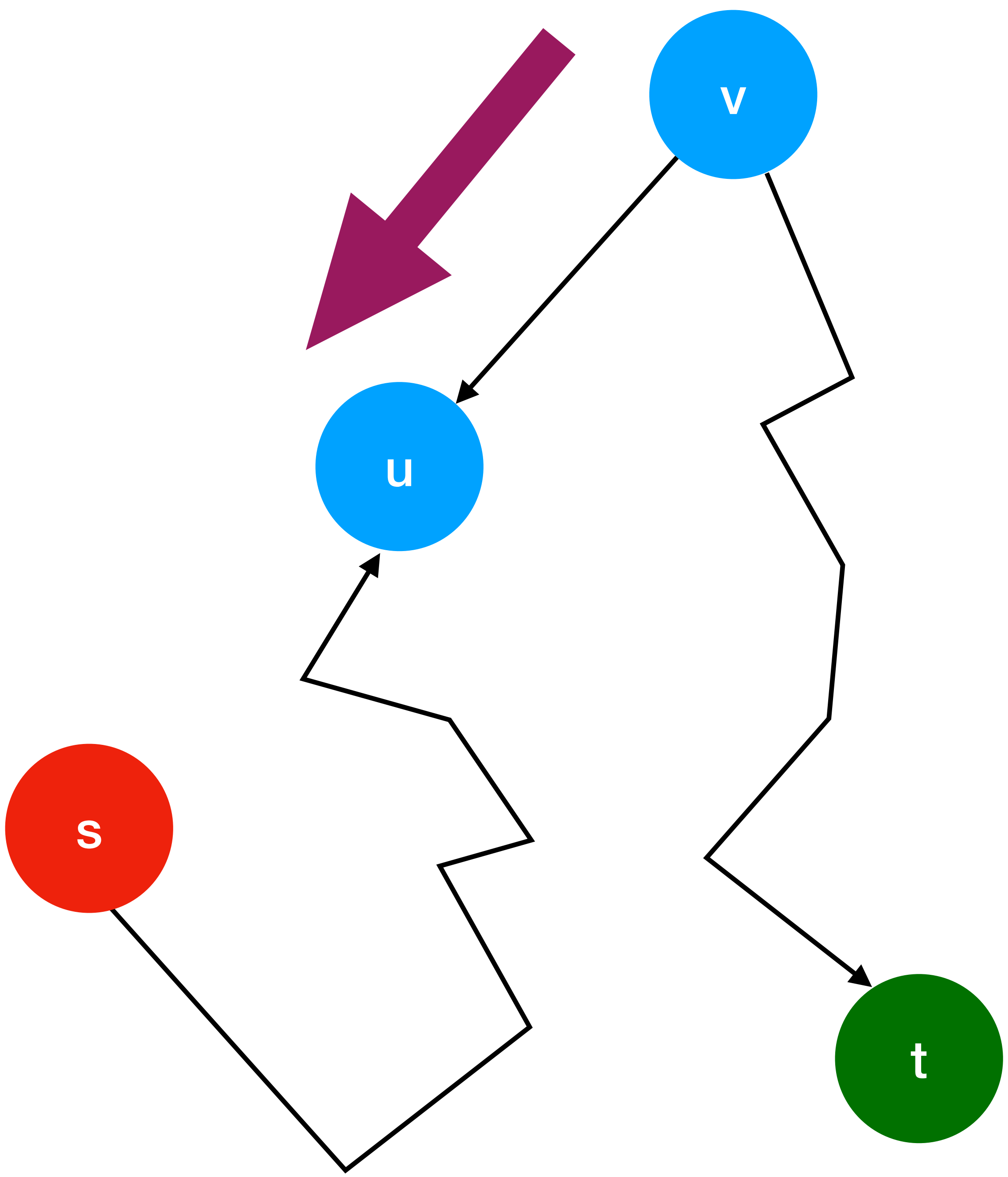

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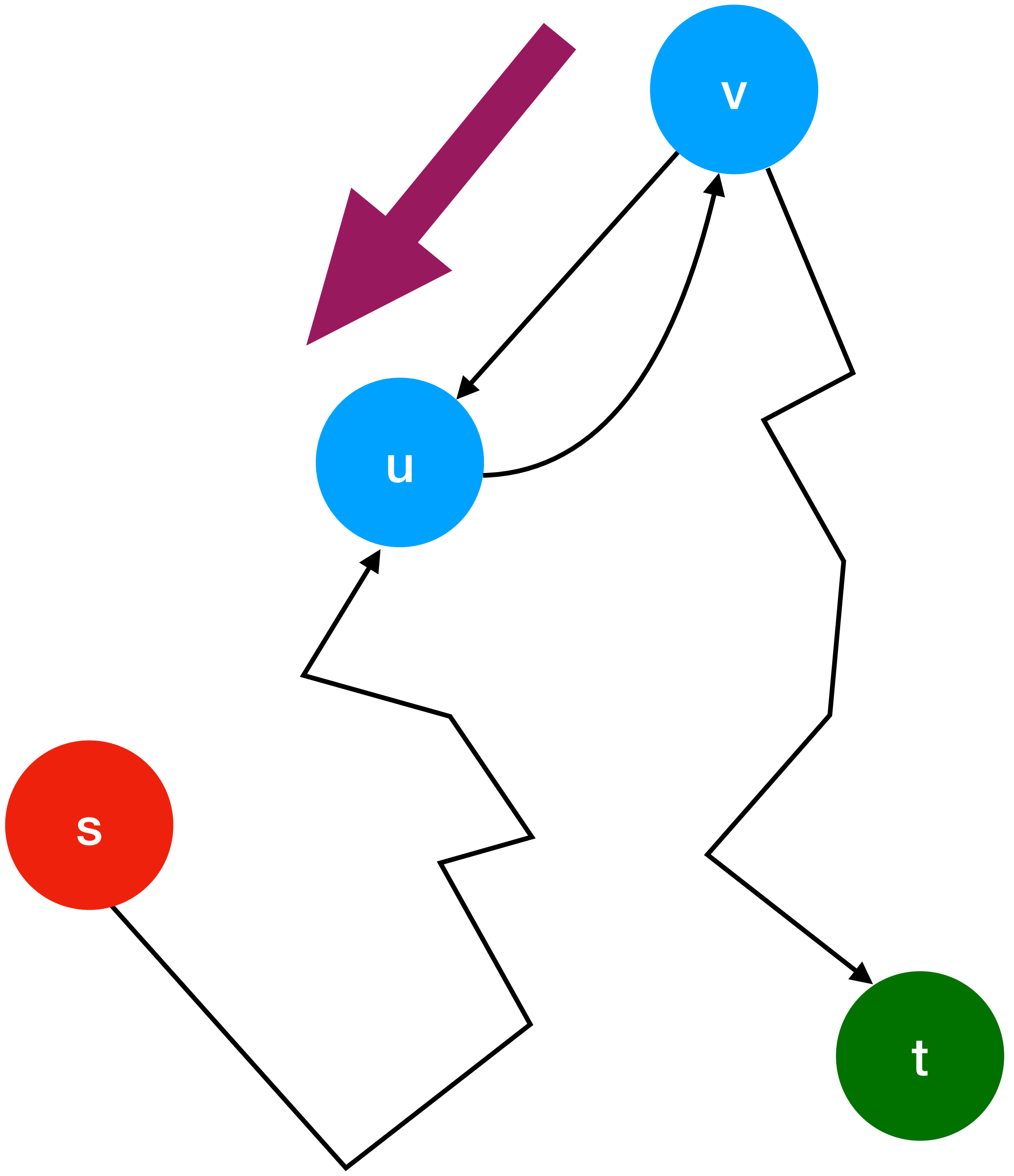

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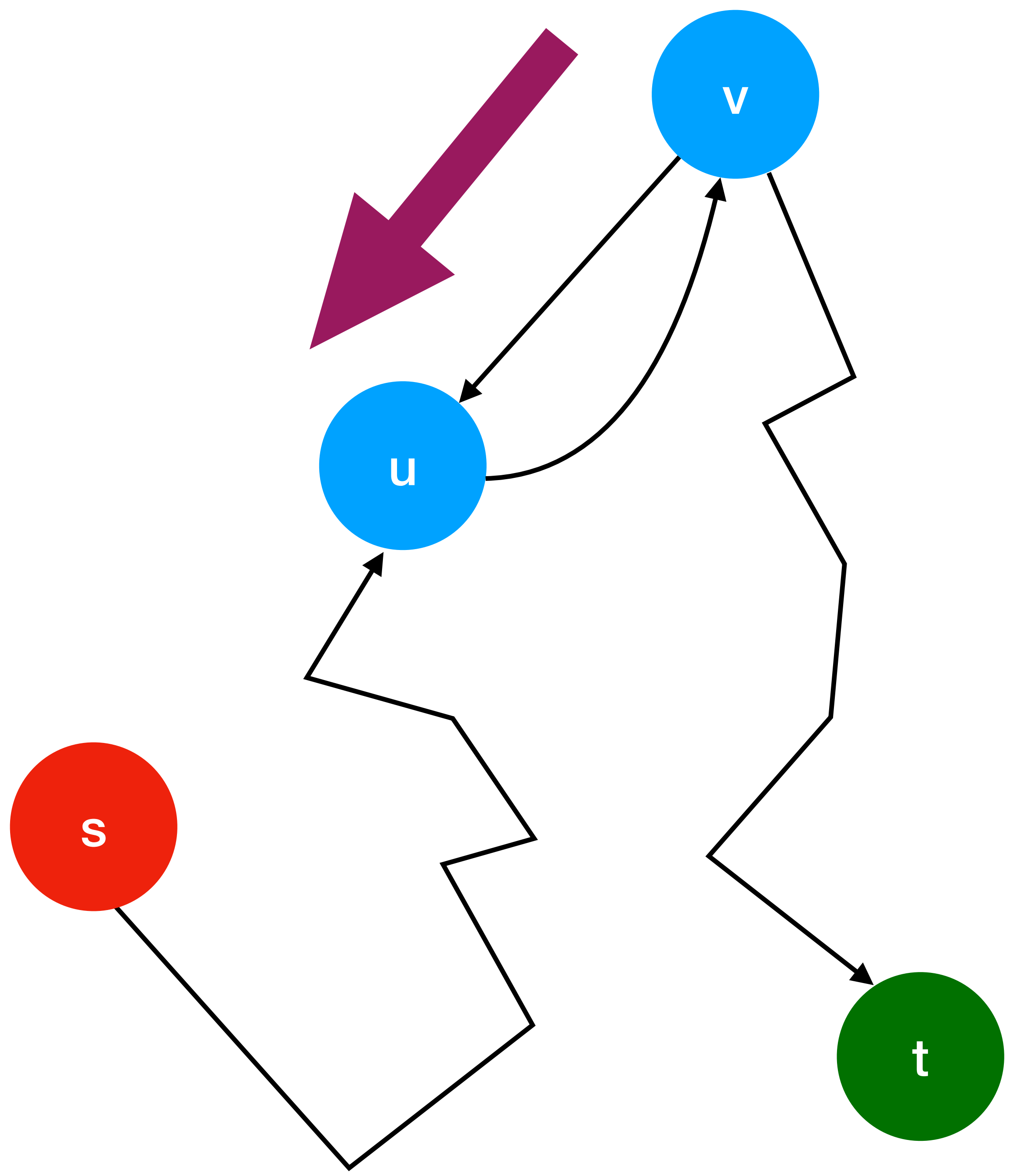

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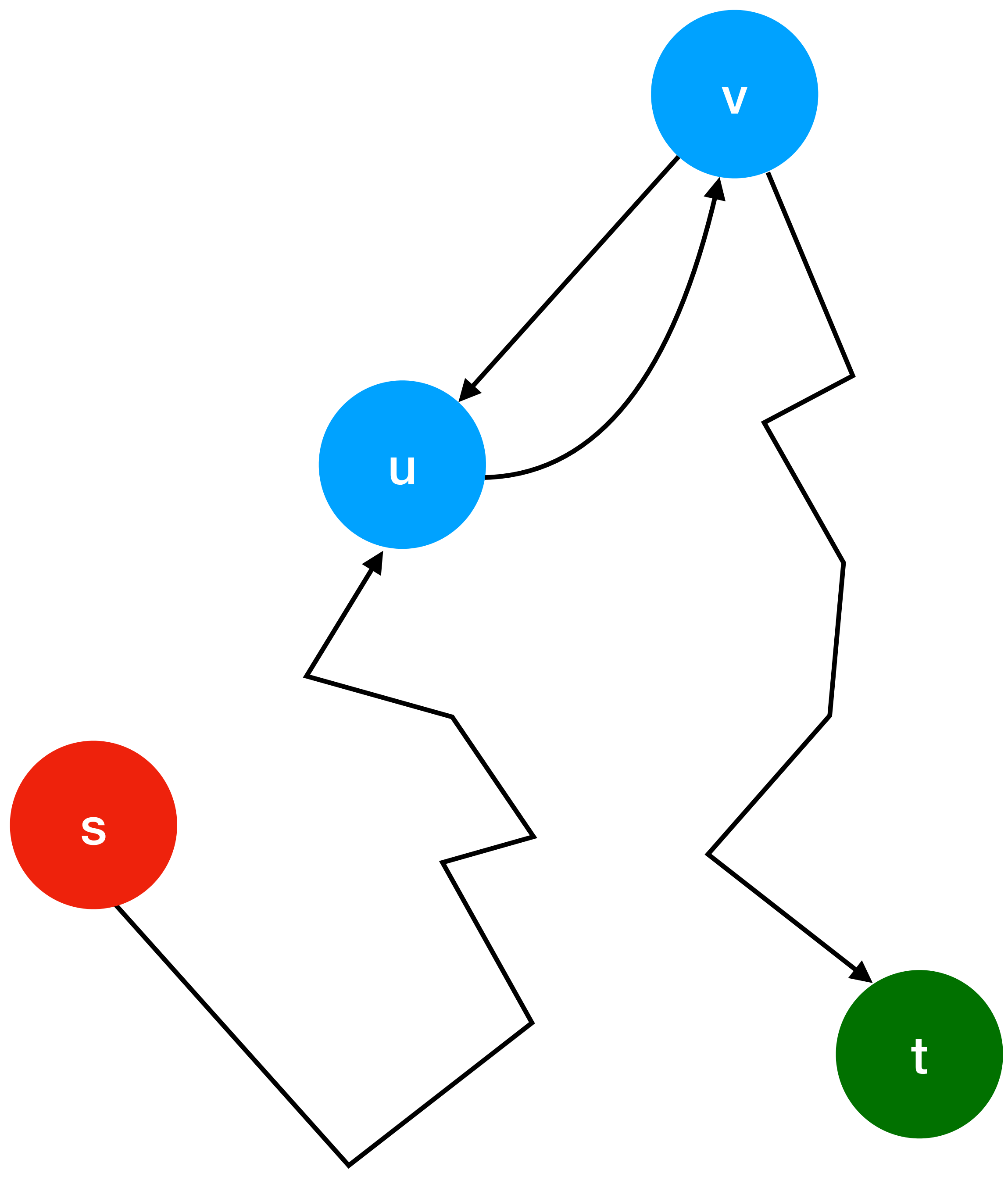

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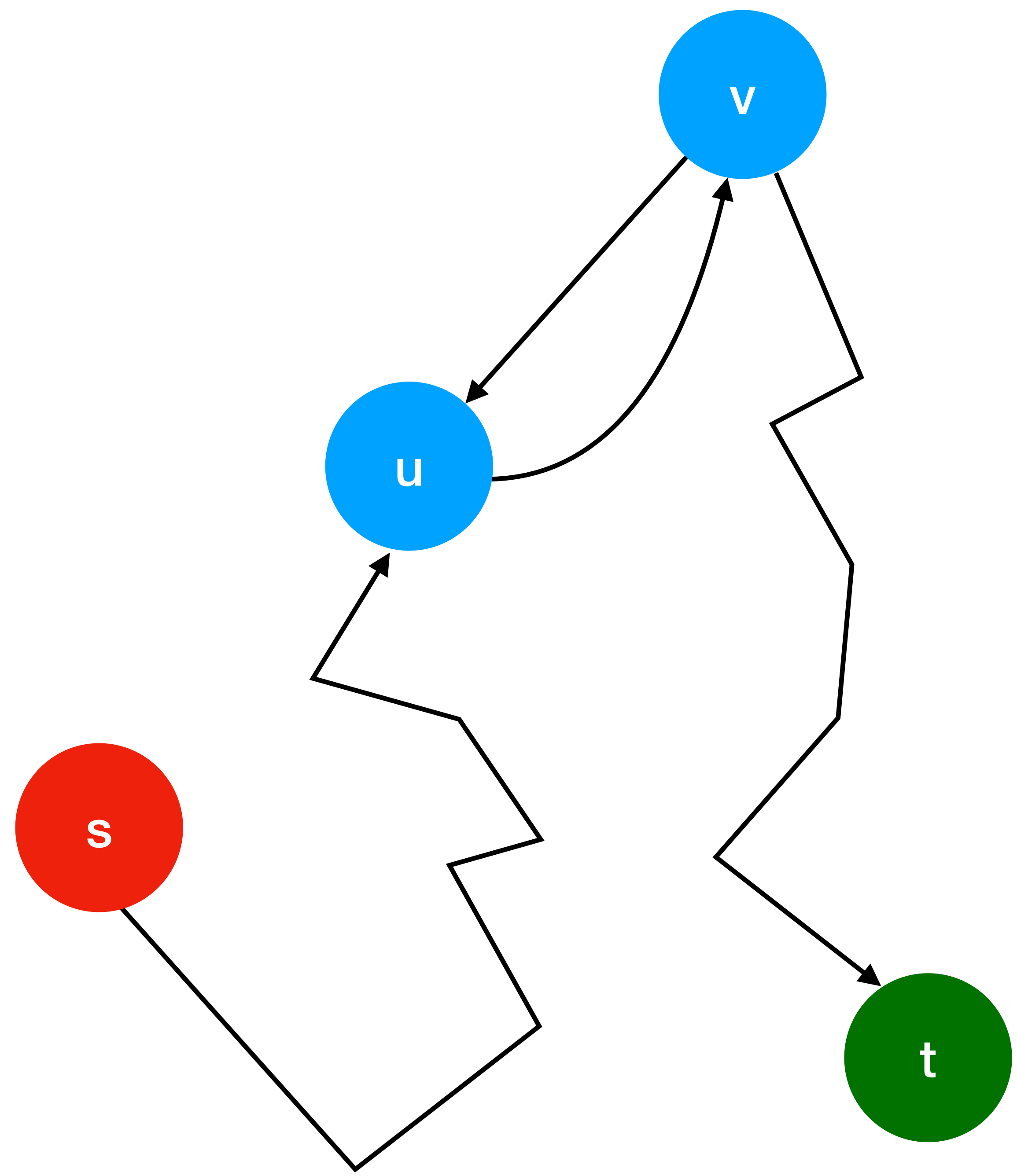

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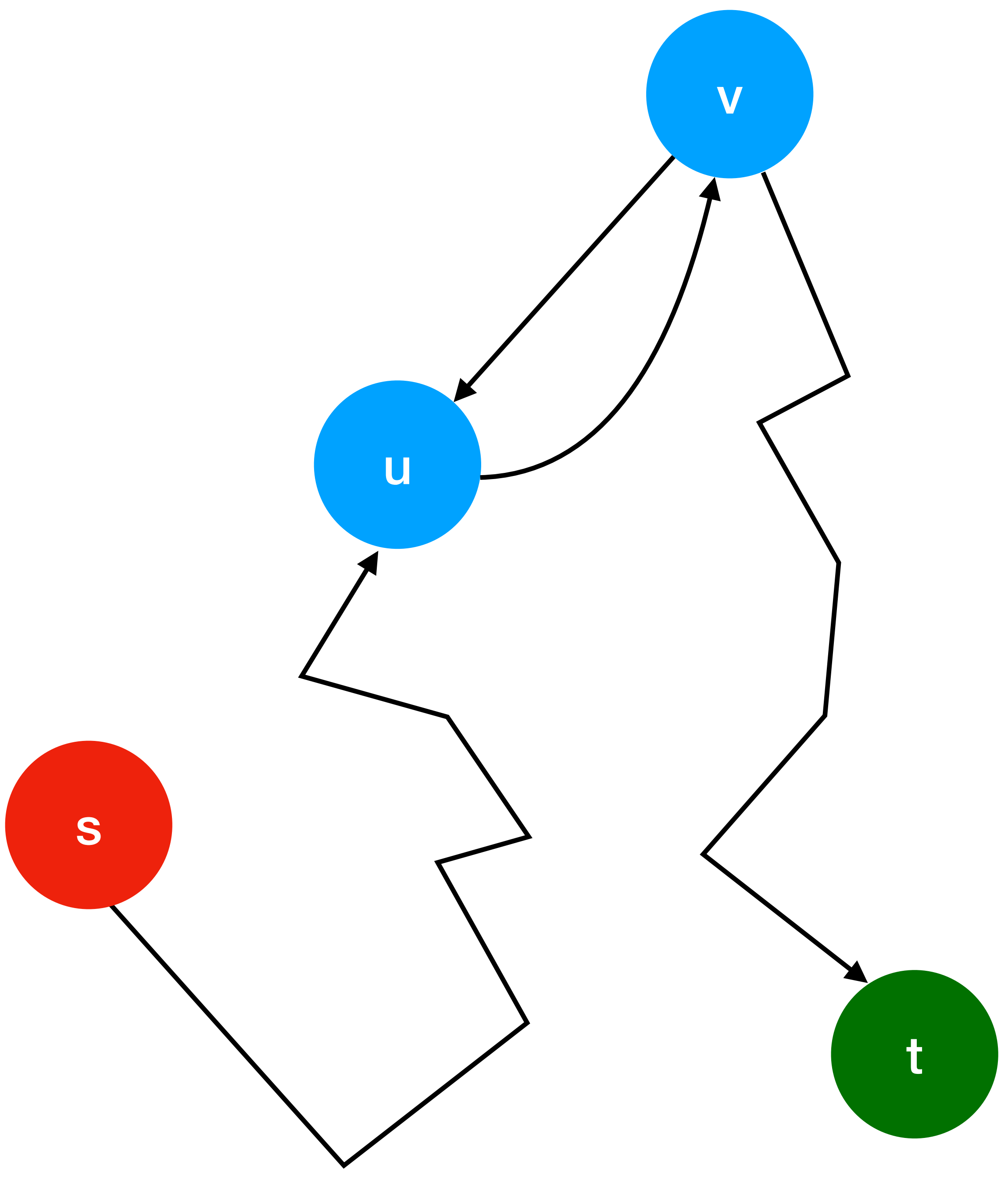

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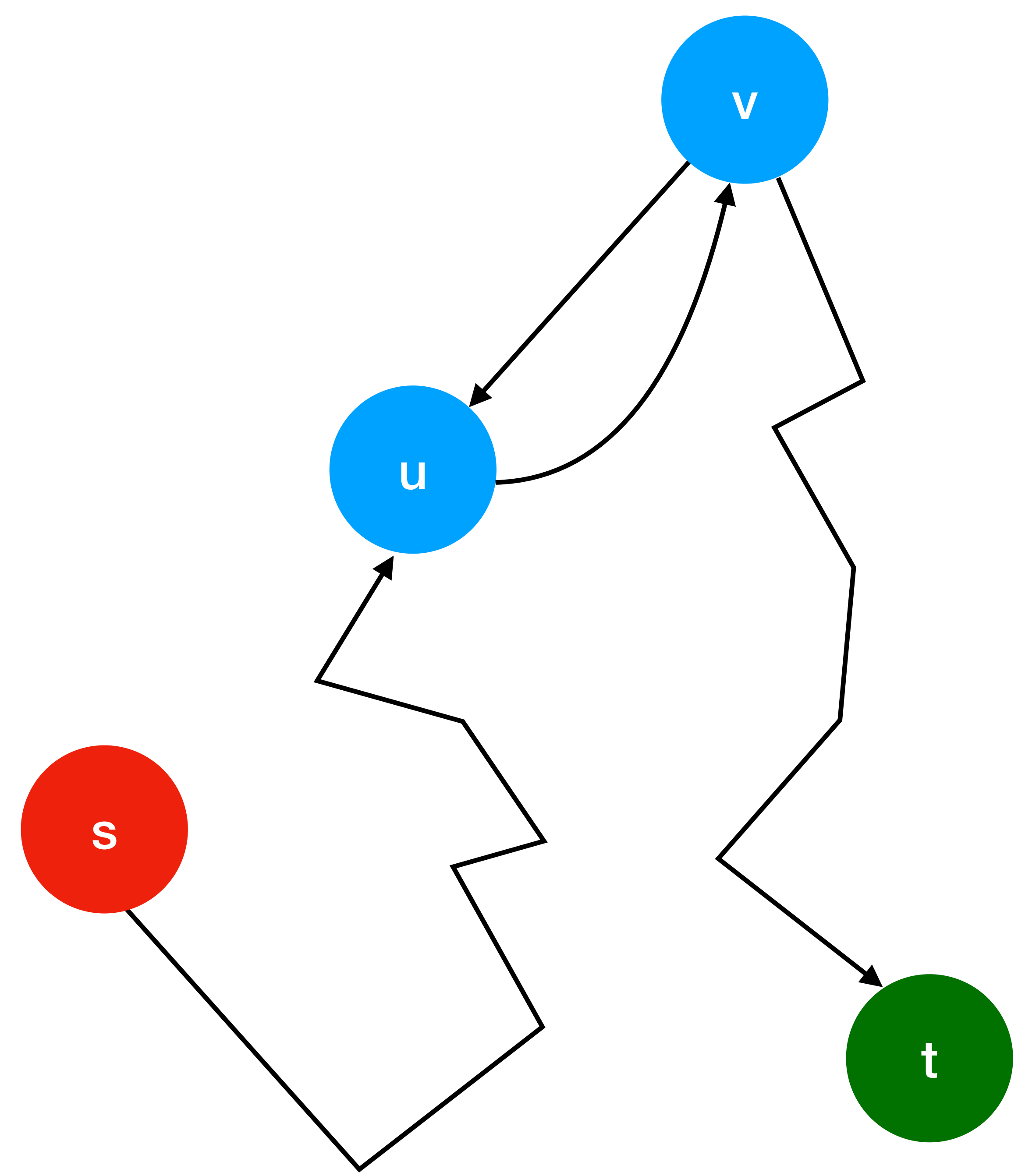

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How much time for each flow augmentation?

Edmonds-Karp Running Time

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How much time for each flow augmentation?

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Edmonds-Karp Running Time

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$O(nm^2)$.